ATSC 404: Vector Calculus Assignment

Due January 08 (in class hand in)

1. Compute the gradient $\vec{\nabla} f$ for each of the following functions.
   (a) $f(x, y, z) = (x^2 + y^2 + z^2)^{1/2}$
   (b) $f(x, y, z) = xy + yz + xz$
   (c) $f(x, y, z) = (x^2 + y^2 + z^2)^{-1}$

2. Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}|$. Prove that $\vec{\nabla}(1/r) = -\vec{r}/r^3$

3. Captain Ralph finds himself on the sunny side of Mercury and notices his space suit is melting. The temperature in a rectangular coordinate system in his vicinity is $T(x, y, z) = \exp^{-x} + \exp^{-2y} + \exp^{-3z}$. He is at (1,1,1), in what direction should he start to move in order to cool down the fastest?

4. Sketch the gradient field $-\vec{\nabla} V$ for $V(x, y) = (x+y)/(x^2+y^2)$. Sketch the equipotential surface $V = 1$.

5. Sketch a few streamlines of the two-dimensional flow fields
   (a) $\vec{u}(x, y) = (y, -x)$
   (b) $\vec{u}(x, y) = (x, -y)$
   (c) $\vec{u}(x, y) = (x, x^2)$

6. Suppose that the isotherms in a region are all concentric spheres centered at the origin. Prove that the heat flux vector field points either toward or away from the origin.

7. Compute the divergence and the curl of the following vector fields.
   (a) $\vec{F}(x, y, z) = x\hat{i} + y\hat{j} + z\hat{k}$
   (b) $\vec{F}(x, y, z) = yz\hat{i} + xz\hat{j} + xy\hat{k}$
   (c) $\vec{F}(x, y, z) = (x^2 + y^2 + z^2)(3\hat{i} + 4\hat{j} + 5\hat{k})$

8. Verify that $\vec{F} = y\hat{i} + x\hat{j}$ is incompressible.

9. Let $\vec{F}(x, y, z) = 3x^2 y\hat{i} + (x^3 + y^3)\hat{j}$.
   (a) Verify that $\vec{\nabla} \times \vec{F} = 0$.
   (b) Find a function f such that $\vec{F} = \vec{\nabla} f$.

10. Let $f(x, y, z) = x^2 y^2 + y^2 z^2$. Verify directly that $\vec{\nabla} \times \vec{\nabla} f = 0$. Find $\nabla^2 f$.