~~where again M_T is the total mass. In other words, the potential of any bounded mass distribution appears as a point source when viewed sufficiently far away. Hence, as $a \to \infty$, $r^2 \frac{\partial U}{\partial r}$ can be moved outside the last integral of equation 3.24, and~~

$$\begin{aligned}\int_S \mathbf{g} \cdot \hat{\mathbf{n}}\, dS &= -\int_{S_\mathrm{P}} g_z\, dS - 2\pi\gamma M_\mathrm{T} \int_{\frac{\pi}{2}}^{\pi} \sin\theta\, d\theta \\ &= -\int_{S_\mathrm{P}} g_z\, dS - 2\pi\gamma M_\mathrm{T}\,.\end{aligned}$$

~~Combining with equation 3.23 provides~~

$$\int_{S_\mathrm{P}} g_z\, dS = 2\pi\gamma M_\mathrm{T}\,, \tag{3.25}$$

~~where S_P now includes the entire horizontal plane.~~

~~Hence, the vertical component of gravity integrated over an infinite plane is proportional to the total mass below the plane, so long as the mass is bounded in volume. In principle, equation 3.25 provides a way to estimate the total excess mass causing an anomaly in measured gravity if we can successfully isolate the field of the anomalous mass from all other gravitational sources. No assumptions are required about the shape of the source or how the density is distributed, so long as it is small with respect to the dimensions of the survey.~~

~~This may seem simple enough, but Gauss's law has many limitations in such applications. Gravity surveys are never available over infinite planes. The best that we can hope for is that the survey extends well beyond the localized sources of interest. Unfortunately, isolated sources never exist in nature, and it is often difficult to separate the gravitational anomaly caused by the masses of interest from anomalies caused by all other local and regional sources. We'll have more to say about this problem of "regional-residual" separation in a later chapter.~~

3.5 Green's Equivalent Layer

An argument was presented in Section 2.1.3 on the basis of Green's third identity that any given potential has an infinite variety of consistent boundary conditions. Here we carry that point a little further and show that a gravitational potential caused by a three-dimensional density distribution is identical to the potential caused by a surface density spread over any of its equipotential surfaces (Ramsey [235]).

Let S_e be a closed equipotential surface resulting from a distribution of mass with density ρ, and let R represent the region inside S_e. The gravitational potential is observed at point P outside of S_e. Green's second identity (Section 2.1.2) is given by

$$\int_R [U\nabla^2 V - V\nabla^2 U]\, dv = \int_{S_e} \left[U\frac{\partial V}{\partial n} - V\frac{\partial U}{\partial n}\right] dS\,,$$

where U and V are any functions with partial derivatives of first and second order. Now let U be the potential of the mass and let $V = 1/r$, where r represents the distance away from P. Because P is located outside the region, the second identity reduces to

$$-\int_R \frac{\nabla^2 U}{r}\, dv = U_s \int_{S_e} \frac{\partial}{\partial n}\frac{1}{r}\, dS - \int_{S_e} \frac{1}{r}\frac{\partial U}{\partial n}\, dS\,,$$

where U_s is the constant potential of the equipotential surface. The first integral on the right-hand side vanishes according to equation 2.2, and substituting Poisson's equation into the integral on the left-hand side provides

$$\gamma \int_R \frac{\rho}{r}\, dv = -\frac{1}{4\pi} \int_{S_e} \frac{1}{r}\frac{\partial U}{\partial n}\, dS\,. \tag{3.26}$$

The left-hand side of equation 3.26 is the potential of the density distribution observed at P. The right-hand side is the potential at P of a surface distribution σ spread over S_e, where $\sigma = -\frac{1}{4\pi\gamma}\frac{\partial U}{\partial n}$. Hence, from the perspective of point P, *the potential caused by a three-dimensional density distribution is indistinguishable from a thin layer of mass spread over any of its equipotential surfaces.* This relationship is called *Green's equivalent layer.*

Furthermore, the total mass of the body is equivalent to the total mass of the equivalent layer. This can be seen by integrating the surface density over the entire surface and applying the divergence theorem (Appendix A), that is,

$$\int_{S_e} \sigma \, dS = -\frac{1}{4\pi\gamma} \int_S \frac{\partial U}{\partial n} \, dS$$
$$= -\frac{1}{4\pi\gamma} \int_R \nabla^2 U \, dv$$
$$= \int_R \rho \, dv \, .$$

Green's equivalent layer is of more than just academic interest. It shows that a potential can be caused by an infinite variety of sources, thus demonstrating the nonuniqueness of causative mass distributions. In later chapters, we will discuss applications of equivalent layers to the interpretation of gravity and magnetic data. The fact that the equivalent layer may have no resemblance to the true source will be of no importance in those applications. These hypothetical sources simply prove to be handy tools in manipulating the potential field.

~~3.6 Problem Set~~

1. ~~Starting with the equation for gravitational attraction outside a uniform sphere, derive the "infinite slab formula"~~

$$\cancel{\mathbf{g} = 2\pi\gamma\rho t\hat{\mathbf{k}} \, ,} \qquad \cancel{(3.27)}$$

~~where ρ and t are the density and thickness of the slab, respectively, and $\hat{\mathbf{k}}$ is a unit vector directed vertically down. (Hint: Use superposition of two spheres and let their radii $\to \infty$.)~~

2. ~~A nonzero density distribution that produces no external field for a particular source geometry is called an *annihilator* (Parker [207]). The annihilator quantitatively describes the *nonuniqueness* of potential field data because any amount of the annihilator can be added to a possible solution without affecting the field of the source. Find a simple annihilator ρ for a spherical mass of radius a as viewed from outside the sphere. (Hint: Let ρ represent *density contrast* so that ρ can reach negative values.)~~

3. ~~Let the radius and density of the earth be represented by a and ρ, respectively.~~