FOURIER TRANSFORMS: REVIEW MATERIAL

(DISTRIBUTED IN CLASS)

MOTIVATION:

- ANALYZING POTENTIAL FIELDS DATA IN FREQUENCY/WAVENUMBER DOMAINS → POWERFUL TOOLS:
  - → FFT
  - → FILTERING

- SOLVING ODEs AND PDEs ⟷ BUILDING FILTERS AND TRANSFER FUNCTIONS
  - LAPLACE AND POISSON'S EQNS
  - LINEAR ODEs + PDEs (FLEXURE)

  } • LINEAR, H/NH, C, VC ?
  } • BCs AT ∞; NO LENGTH/TIME SCALE A PRIORI

WHY USE FT?

PDE ⟶ ODE

ODE → ALGEBRAIC EQN

- WORKING IN FREQ DOMAIN ⟶ MATH IS MUCH SIMPLER
- F-T ARE THE MAIN TOOL USED TO SOLVE PROBLEMS IN POTENTIAL FIELDS IN GEODYNAMICS AND PLANETARY SCIENCE

1D FOURIER TRANSFORM

- LET $f(x)$ BE DEFINED ON $(-\infty, \infty)$ SUCH THAT $\int |f(x)|\, dx < \infty$
- $f(x)$ CAN BE PERIODIC OR APERIODIC

THE FOURIER TRANSFORM of $f(x)$:

$$F(k) = \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$

PROCEDURE:
- MULTIPLY FUNCTION BY e^{-ikx}
- INTEGRATE OVER DOMAIN TO OBTAIN $F(k)$ ($\int$ BY PARTS)
- NOTE: THE FOURIER TRANSFORM OF MANY FUNCTIONS ARE TABULATED OR CAN BE FOUND AT WOLFRAM WEBSITE

THE INVERSE FOURIER TRANSFORM of $F(k)$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k)\, e^{ikx}\, dk$$

PROCEDURE:
- INTEGRATE IN COMPLEX PLANE, RECOVER $f(x)$ (USE CAUCHY THM OR METHOD OF RESIDUES)
- NOTE: AS W/ F-T, MANY IFT ARE TABULATED.

HERE:

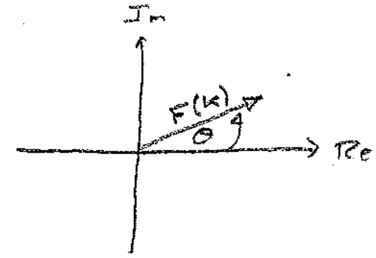
$$k = \frac{2\pi}{\lambda} \text{ IS THE (CIRCULAR WAVE NUMBER)} \qquad \left[\frac{1}{L}\right]$$

COMMENT: THESE ARE EACH EXAMPLES OF LINEAR INTEGRAL TRANSFORMS.

(1D FT (CONT'D))

THE QUANTITY $F(k)$ IS COMPLEX AND CAN BE WRITTEN AS:

$$F(k) = F_R + i F_I$$

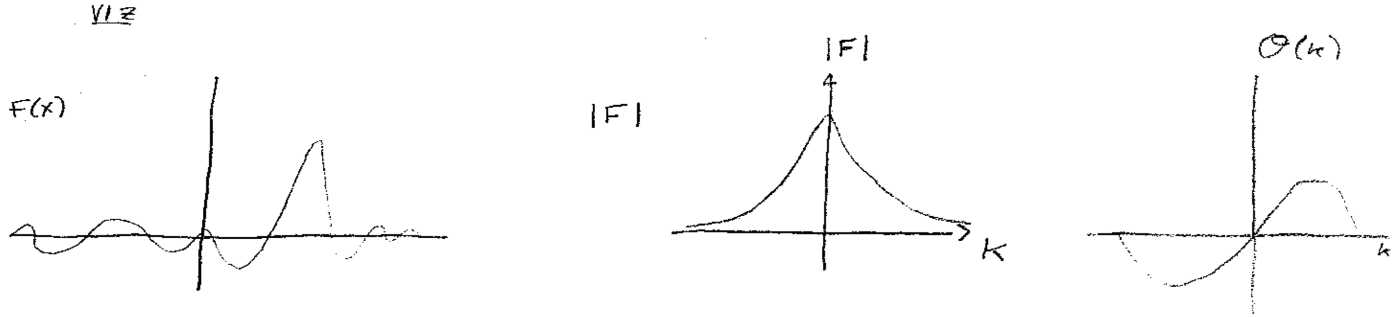


$$F(k) = |F| e^{i\theta(k)}$$

$$|F| = \sqrt{F_R^2 + F_I^2} \equiv \text{AMPLITUDE}$$

$$\theta(k) = \tan^{-1}\left(\frac{F_I}{F_R}\right) \equiv \text{PHASE}$$

VIZ



$F(x) \leftrightarrow F(k)$ $\equiv$ "FOURIER TRANSFORM PAIR"

SOME KEY PROPERTIES WE WILL RELY ON (THER ARE OTHERS...)

NOTE: $\mathcal{F}\{\ \} \equiv$ FOURIER TRANSFORM OPERATOR

1. LINEARITY

$$\mathcal{F}\{a_1 f_1(x) + a_2 f_2(x)\} = a_1 \mathcal{F}\{f_1(x)\} + a_2 \mathcal{F}\{f_2(x)\}$$
$$= a_1 F_1(k) + a_2 F_2(k)$$

2. SCALING OR "CHANGE OF TIME (t) OR LENGTH (x) SCALE"

$$\mathcal{F}\{f(ax)\} = \frac{1}{a} F\left(\frac{k}{a}\right)$$

3. SPACE OR TIME SHIFTING

$$\mathcal{F}\{f(x-x_0)\} = e^{-ikx_0} \mathcal{F}\{f(x)\} = e^{-ikx_0} F(k)$$

4. WAVENUMBER OR FREQUENCY SHIFTING

$$\mathcal{F}\{e^{ikx_0} f(x)\} = \mathcal{F}\{f(t)\}_{k \to (k-k_0)} = F(k-k_0)$$

5. DIFFERENTIATION IN SPACE OR TIME

$$\mathcal{F}\{f'(x)\} = ik\mathcal{F}\{f(x)\} = ik F(k)$$

FOR HIGHER (n^{th}) ORDER SITUATIONS (ASSUMING SUCCESSIVE DERIVATIVES ARE CONTINUOUS)

$$\mathcal{F}\{f^{(n)}(x)\} = (ik)^{(n)} F(k)$$

6. SYMMETRY

IF $f(x)$ IS REAL

$F(k) = F^*(-k)$; * INDICATES COMPLEX CONJUGATE

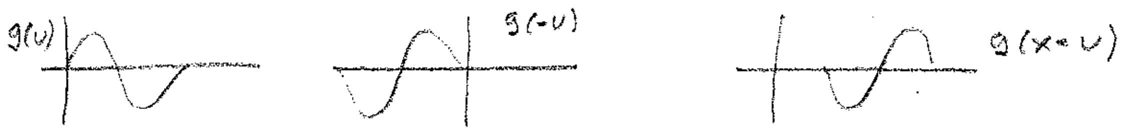
7. <u>CONVOLUTION</u>

$$h(x) = f(x) * g(x)$$

$$h(x) = \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du = \int_{-\infty}^{\infty} g(u)\, f(x-u)\, du$$

<u>VIZ</u>

PROCESS: 1. REVERSE
2. SHIFT
3. MULTIPLY



<u>THEOREM</u>

IF $h(x) = f(x) * g(x)$ (1)

$$\left.\begin{aligned} h(x) &\leftrightarrow H(k) \\ f(x) &\leftrightarrow F(k) \\ g(x) &\leftrightarrow G(k) \end{aligned}\right\} \text{F-T PAIRS}$$

THEN

$$H(k) = F(k)\, G(k) \qquad (2)$$

$$h(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(k)\, e^{jkx}\, dk$$

CONVOLUTION IN SPACE/TIME DOMAIN = MULTIPLICATION IN WAVENUMBER/FREQUENCY DOMAIN.

8. <u>PARSEVAL'S (MODULUS) Thm</u>

REAL FCN $f(x) \longleftrightarrow F(k)$, WHERE $F(k) = r(k)\, e^{i\theta}$ (POLAR FORM)

$$\int_{-\infty}^{\infty} f^2(x)\, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} r^2(k)\, dk \; ; \quad r^2(k) = |F(k)|^2 \equiv \text{ENERGY SPECTRUM}$$

NOTE: $r^2(k)\,\Delta k$ = TOTAL ENERGY IN ANY Δk SEGMENT

$\int$ OVER $(-\infty, \infty)$ = TOTAL ENERGY IN SIGNAL.

2D FOURIER TRANSFORMS

CONSIDER A MAP... A FUNCTION $f(x,y)$

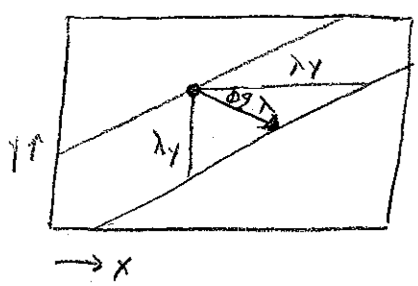
$$F(k_x, k_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-i(k_x x + k_y y)}\, dx\, dy$$

$$f(x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(k_x, k_y)\, e^{i(k_x x + k_y y)}\, dk_x\, dk_y$$

$\vec{k}$ = (CIRCULAR) WAVE VECTOR

$k_x = \dfrac{2\pi}{\lambda_x}$ $\qquad$ $k_y = \dfrac{2\pi}{\lambda_y}$

VIZ



$\lambda = \lambda_x \cos\phi \quad ; \quad \lambda_x = \lambda / \cos\phi$

$\lambda = \lambda_y \sin\phi \quad ; \quad \lambda_y = \lambda / \sin\phi$

$$\Rightarrow k_x = \frac{2\pi}{\lambda_x} = \frac{2\pi\cos\phi}{\lambda} = k\cos\phi$$

$$\Rightarrow k_y = \frac{2\pi}{\lambda_y} = \frac{2\pi\sin\phi}{\lambda} = k\sin\phi$$

FOURIER TRANSFORMS AND ODES

- LINEAR
- CONSTANT COEFFICIENT
- NONHOMOGENEOUS

GIVEN:

PERIODIC OR NOT

$$a_0 \frac{d^{(n)} y}{d x^{(n)}} + \dots + a_n y = f(x)$$

STRATEGY

1) FOURIER TRANSFORM BOTH SIDES; TAKE ADVANTAGE OF PROPERTY (5) ON P. 4 OF THESE NOTES.

LET $Y(k) = \mathcal{F}\{y(x)\}$ AND $F(k) = \mathcal{F}\{f(x)\}$

$$\underbrace{\left[a_0 (ik)^{(n)} + \dots + a_n\right]}\ \underbrace{Y(k)}_{\text{OUTPUT}} = \underbrace{F(k)}_{\text{INPUT}}$$

2) SOLVE FOR $Y(k)$; LET $H(ik) = \frac{1}{[\quad]}$ ≡ "TRANSFER FUNCTION"

$Y(k) = \underline{F}(k) H(k)$ ≡ "FREQUENCY RESPONSE TO INPUT $f(t)$"

$H(ik) \Rightarrow$ CHARACTERIZES THE RESPONSE OF THE SYSTEM TO <u>ANY</u> <u>FORCING</u>.

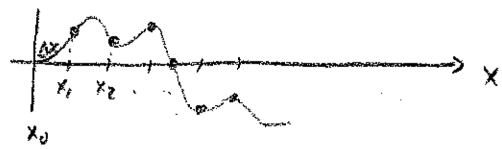
NOTE: IN SPACE DOMAIN FROM CONVOLUTION THEOREM:

$$y(x) = f(x) * h(x)$$

THE DISCRETE F-T (1D) (WHAT THE FFT DOES)

SUPPOSE WE HAVE A SEQUENCE OF DATA IN TIME OR SPACE THAT IS SAMPLED ON A CONSTANT INTERVAL Δx.

$$f_r = f(x_r) = f(r\Delta x) \qquad r = 0 \ldots (N_x - 1) \qquad (N_x = \#\ \text{SAMPLES})$$



DISCRETE F-T PAIR:

$$F_j = \sum_{r=0}^{N_x-1} f_r\, e^{-i2\pi rj/N_x} \qquad j = 0 \to N_x - 1$$

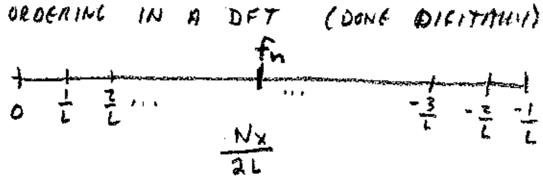
$$f_r = \frac{1}{N_x} \sum_{j=0}^{N_x-1} F_j\, e^{i2\pi rj/N_x} \qquad r = 0 \to N_x - 1$$

- ASSUMING THE DATA ARE PERIODIC WITH PERIOD $L_x = \Delta_x N_x$
- THE <u>LINEAR</u> WAVENUMBERS $\left(\tilde{k}_j = k_j/2\pi\right)$:

$$\tilde{k}_j = \frac{j}{N_x \Delta x} = \frac{j}{L_x} \qquad j = 0, \ldots N_x - 1$$

- THE WAVE NUMBERS CORRESPONDING TO $j = 0 \ldots N_x - 1$ PERTAIN TO POSITIVE <u>AND</u> NEGATIVE WAVENUMBERS (FREQUENCIES).

* THE ORDERING IN A DFT (DONE DIGITALLY):



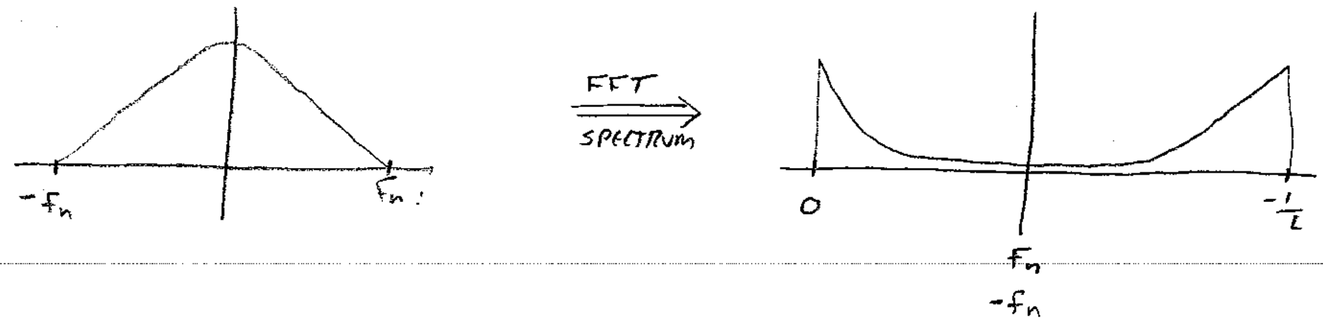
} TOTAL LENGTH IS 2L BECAUSE OF (+) AND (−) VALUES.

!! $f_n = \frac{N_x}{2L} = \frac{1}{2\Delta x} \equiv$ NYQUIST FREQUENCY

$\Rightarrow$ THIS IS THE HIGHEST FREQUENCY THAT YOU CAN RESOLVE (MORE, LATER)

KEY ISSUE: UNDERSTANDING THE ORDERING OF FOURIER COMPONENTS F_j

<u>CONTINUOUS F-T</u>   <u>DIGITAL FFT</u> (FOLDED)



* THE SPECTRUM FROM A DIGITAL FFT COMES OUT FOLDED.

** TO UNFOLD (RIGHT) AND OBTAIN "CORRECT" SPECTRUM (LEFT) IN MATLAB, YOU MUST USE THE <FFTSHIFT> FUNCTION

i.e. $F(k) = \text{FFTSHIFT}(\text{FFT}(F(x)))$

2D DFT

$F(x,y)$ SAMPLED AT INCREMENTS $\Delta x, \Delta y$

N_x : # POINTS IN X-DIRECTION

N_y : # POINTS IN Y-DIRECTION

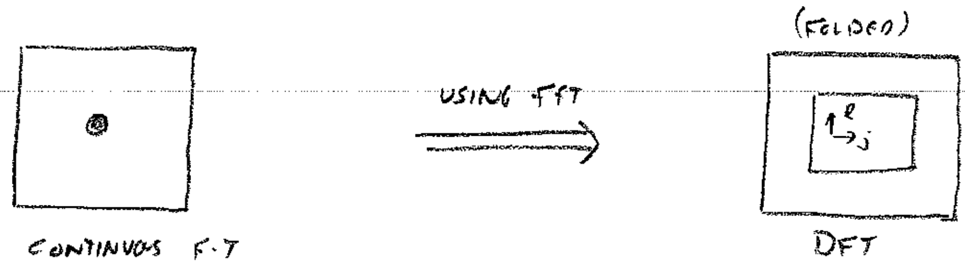
$$F_{jl} = \sum_{r=0}^{N_x-1} \sum_{s=0}^{N_y-1} f_{rs}\, e^{-i2\pi(rj/N_x + sl/N_y)}$$

$$f_{rs} = \frac{1}{N_x N_y} \sum_{j=0}^{N_x-1} \sum_{l=0}^{N_y-1} F_{jl}\, e^{i2\pi(rj/N_x + sl/N_y)}$$

$$\tilde{k}_{xj} = \frac{j}{N_x \Delta x} = \frac{j}{L_x} \qquad j = 0, \dots N_x - 1 \qquad \tilde{k}_{Nx} = 1/2\Delta x \quad \text{NYQUIST} - x$$

$$\tilde{k}_{y\ell} = \frac{\ell}{N_y \Delta y} = \frac{\ell}{L_y} \qquad \ell = 0, \dots N_y - 1 \qquad \tilde{k}_{Ny} = 1/2\Delta y \quad \text{NYQUIST} - y$$

WHAT COMES OUT:



IN MATLAB:

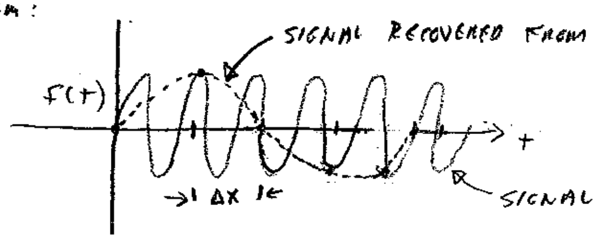
TO UNFOLD

$$F(\vec{k}) = \text{fftshift}\left(\text{fft2}\left(f(r)\right)\right)$$

<u>ALIASING</u>

THE HIGHEST FREQUENCY (OR WAVENUMBER) THAT WE CAN CAPTURE WITH A SAMPLING INTERVAL Δ IS $k_N = 1/2\Delta x$ (NYQUIST)

PROBLEM:



<u>KEY:</u> IF DATA CONTAINS FREQUENCIES HIGHER THAN NYQUIST, THIS INFORMATION WILL APPEAR AS A LONG-PERIOD SIGNAL

<u>RESULT</u>: ENHANCED ENERGY AT LOW FREQUENCIES