EOSC-450: Introductory Material

1 Concepts

1. relationships between physical and mathematical descriptions of fields
2. defining a vector field via its source density and circulation density
3. scalar fields and potential energy
4. equipotential surfaces
5. harmonic functions
6. δ-functions as a link between continuous and discrete representations

2 Fields

2.1 Physics Description: Material and Force Fields

Material Fields define material properties as a function of time and space. In geophysics they are usually scalar quantities, but may be vector quantities. *e.g.,*

- $\rho(x, y, z, t)$ – density
- $\chi(x, y, z, t)$ – susceptibility
- $C(x, y, z, t)$ – concentration
- $T(x, y, z, t)$ – temperature
- $\mathbf{m}(x, y, z, t)$ – magnetization.

In these notes I will denote vectors in bold face font, so *e.g.* $\mathbf{m} = \vec{m}$, and is a vector.

Force fields describe a condition in space such that a particle inserted into the region is subject to a force. *e.g.,* Newton's Law of gravitation gives the force $\mathbf{F_{12}}$ exerted by a mass, m_1 on a mass, m_2:

$$\mathbf{F_{12}} = \frac{-Gm_1m_2\hat{r}}{r^2}$$

r is the distance between the masses, G is the gravitational constant (= 6.67 x 10^{-11} m^3 kg^{-1} s^{-2}), $\hat{r}$ is the unit vector from m_1 to m_2, and the minus sign denotes that the force is attractive. See Figure 1.

Remember to always check dimensions / units.

We can think of a force field as "action at a distance". In the case of our simple example using Newton's Law we can say that the existence of m_1 alters the space around it, creating a gravitational field, $\mathbf{g}$:

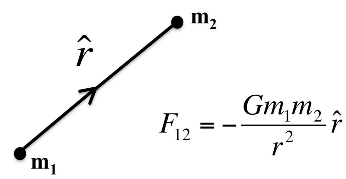

Figure 1:

$$\mathbf{g} = \lim_{m_2 \to 0} \frac{\mathbf{F}_{12}}{m_2} = -\frac{Gm_1\hat{r}}{r^2}$$

If we allow the 'test' mass (m_2) to move, it would map out a continuous trajectory:

- the tangent to m_2's trajectory defines the direction of the field line (*i.e.,* the field lines are tangent to every point in the vector field),
- the strength of the field is related to the density of the field lines.

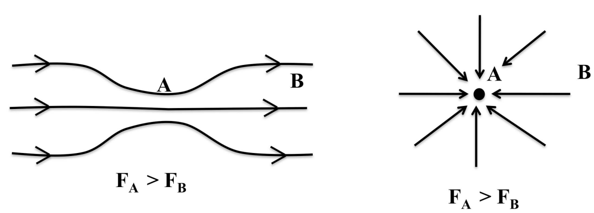

Figure 2:

2.2 Math Description: Scalar and Vector Fields

Scalar Field – described by its magnitude (a single #) at any point in space and time. *e.g.,* in Cartesian coordinates:

- $T(x, y, z, t)$ – temperature
- $g_z(x, y, z, t)$ – vertical component of gravity

Vector Field – described by a magnitude and direction:

- $\mathbf{g}(x, y, z, t)$ – gravity field
- $\mathbf{B}(x, y, z, t)$ – magnetic field
- $\mathbf{A}(x, y, z) = x^2\hat{x} + (1 - z)\hat{y} + x^2\hat{z}$ – arbitrary field

2.3 Important Vector Field Properties

Any vector field in R^3 is uniquely defined by:

1. its source density $\longleftrightarrow$ divergence, and
2. its circulation density $\longleftrightarrow$ curl

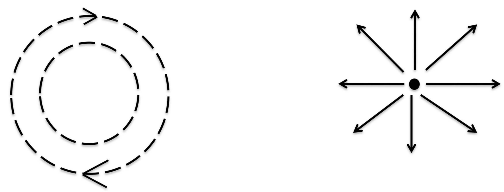

Figure 3:

3 Motivating the Helmholtz Theorem

This theorem states that we can decompose any vector field in R^3 as follows

$$\mathbf{f} = \nabla\Phi + \nabla \times \mathbf{A} \tag{1}$$

Why is this useful? At first it seems more complicated since we now need two fields (one scalar and one vector!) to describe our original vector field. The key is that

$$\Phi(\mathbf{r}) = \frac{-1}{4\pi} \int \frac{\nabla \cdot \mathbf{f}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d^3\mathbf{r}' \tag{2}$$

$$\mathbf{A} = \frac{1}{4\pi} \int \frac{\nabla \times \mathbf{f}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d^3\mathbf{r}' \tag{3}$$

So if we have theory for, or measurements of, the divergence and curl of our field of interest, $\nabla \cdot \mathbf{f}$, and $\nabla \times \mathbf{f}$ respectively, we can calculate $\mathbf{f}$. As in example in fluid dynamics problems, we may be interested in the fluid velocity, $\mathbf{v}$. If we have measurements of flux and vorticity we can calculate the velocity field $\mathbf{v}$. In gravity field problems our field of interest is $\mathbf{g}$, the gravitational field, and as we shall see we can write $\nabla \cdot \mathbf{g} = -4\pi G\rho$ and $\nabla \times \mathbf{g} = 0$. In problems in electromagnetics Maxwell's equations give us the divergence and curl of the electric and magnetic fields.

Furthermore, in gravity field problems, then because $\nabla \times \mathbf{g} = 0$, we see from (3) that $\mathbf{A} = 0$, and so from (1) we can write

$$\mathbf{g} = \nabla \Phi \tag{4}$$

and we see that

$$\nabla \cdot \mathbf{g} = \nabla \cdot \nabla \Phi = \nabla^2 \Phi = -4\pi G \rho \tag{5}$$

Thus we can work with a second order differential equation for a scalar field rather than working with a vector field. (5) is of course Poisson's equation, and it relates the gravitational potential energy to density sources. In a source-free region the non-homogeneous differential equation reduces to a homogeneous one and we get Laplace's equation:

$$\nabla^2 \Phi = 0 \tag{6}$$

Hence solving Poisson's and Laplace's equations is central to the topic of potential fields.

4 Work and Potential

Consider any force field, $\mathbf{f}$. If we move a particle from position P_0 to P work has to be done. *A conservative vector field* is one for which the work done is independent of the path and depends only on the end points and so,

$$W(P, P_0) = \int_{P_0}^{P} \mathbf{f} \cdot \mathbf{ds} = W(P) - W(P_0) \tag{7}$$

We can evaluate (7) in the limit that $\mathbf{ds} \to \mathbf{0}$ ($\mathbf{f}$ constant over $\mathbf{ds}$), to show that

$$\mathbf{f} = \nabla W. \tag{8}$$

In other words the force field is the gradient of the work function. So, the scalar potential, Φ, in the Helmholtz equation has a physical meaning and can be thought of as a work function. You'll have already seen this idea in elementary expositions of Newton's law of gravity (potential as potential energy per unit mass, and gravity as the gradient of potential).

Note:

- Sign conventions differ: sometimes (especially in gravity problems in Earth science) you'll see $\mathbf{f} = \nabla\Phi$, and other times $\mathbf{f} = -\nabla\Phi$. Physically of course we use different signs to denote whether particles of like sign attract (gravity) or repel (electrostatics), and whether the work is done by the field on the particle or whether work is required to move the particle in the presence of the field.
- Adding a scalar constant doesn't matter since $\nabla(\Phi + c) = \nabla\Phi + \nabla c = \nabla\Phi = \mathbf{f}$. For convenience, c is chosen such that $\Phi \to 0$ as $\mathbf{r} \to \infty$, so $\Phi(P) = \int_\infty^P \mathbf{f} \cdot \mathbf{ds}$.

5 Equipotential Surfaces

Unsurprisingly, these are surfaces on which $\Phi(\mathbf{r})$ is constant.
Let $\hat{s}$ be a unit vector on the surface. $\mathbf{f} = \nabla\Phi$ defines the field (force). Then:
from the definition of the equipotential surface we find:

$$F_s = \hat{s} \cdot \mathbf{f} = \hat{s} \cdot \nabla\Phi = \frac{\partial\Phi}{\partial s} = 0$$

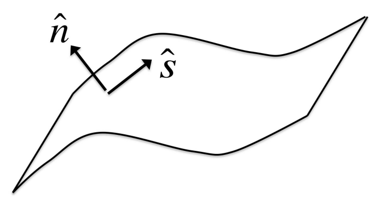

Figure 4:

Hence:

1. there is no component of force along the surface, so no work is required to move a particle parallel to the surface,
2. the only component of force is that which is normal to the surface: $F_n = \frac{\partial\Phi}{\partial n}$.

The distance between equipotential surfaces is a measure of the density of field lines. Greater force exists where the distance between two equipotential surfaces is smaller. An example of an equipotential surface in geophysics is the geoid.

6 Laplace's Equation and Harmonic Functions

Finding a solution to Laplace's equation, $\nabla^2\Phi = 0$, if one exists requires:

- need a "region" (finite or infinite), R, over which the D.E. is valid, and
- need R to have a boundary S (could be infinite) on which a boundary condition is applied. The boundary condition can either be Φ itself (Dirichlet B.C.) or $\frac{\partial \Phi}{\partial n}$ (Neumann B.C.)

Note that second derivatives must exist throughout the region (otherwise $\nabla^2 \Phi$ cannot exist). This condition implies in turn that the first derivatives of Φ are continuous. Any function that satisfies these conditions (and is thus a solution to Laplace's equation) is a HARMONIC function. Harmonic functions have their maxima and minima at the boundaries of the region. A corollary is that any function that has maxima and minima in the region is not harmonic.

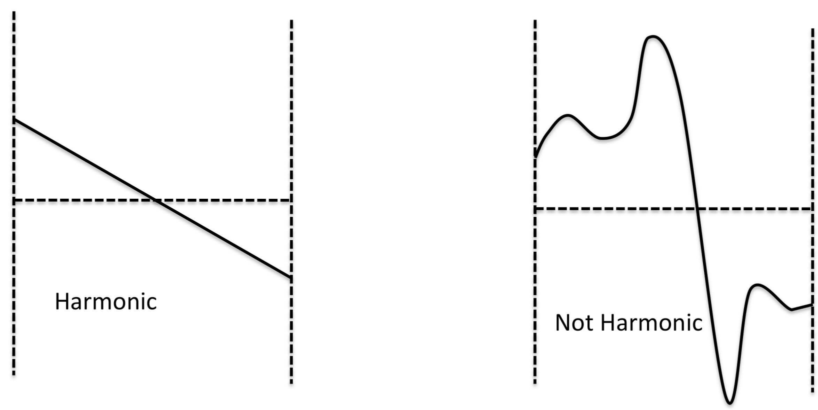

Figure 5:

7 Basic Tools for the Analyses of Potential Fields: Delta Functions

7.1 Properties of δ-functions

δ-functions are generalized functions, usually distributions, that greatly facilitate solving potential field problems. For example, they allow us to represent a point value as a function, and provide a means of generating an impulsive source for any differential equation. δ-functions can exist in R, R^2 and R^3.

In 1-D, $\delta(x)$ is defined on $-\infty < x < \infty$, and is the limit of a sequence of functions. It has the following properties:

$$\delta(x) = 0 \qquad x \neq 0 \tag{9}$$

$$\infty \qquad x = 0 \tag{10}$$

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1 \tag{11}$$

$$\int_{-\infty}^{\infty} f(x)\, \delta(x - x_0)\, dx = f(x_0) \tag{12}$$

$$\int_{-\infty}^{\infty} f(x)\, \delta'(x - x_0)\, dx = -f'(x_0) \tag{13}$$

Examples of sequences that can give rise to a δ-function in 1-D are a box-car function and a gaussian (see in-class). In R^2 and R^3 the properties all hold and are the same: only the volumes of integration change. We will often work in 2-D or 3-D. Note that

- $\delta(x, y, x) = \delta(x)\,\delta(y)\,\delta(z)$ is a product of δ-functions,
- we will often use the notation $\delta(\mathbf{r}) = \delta(x, y, z)$, where the δ-function is located at a position (x, y, z), or in spherical coordinates (r, θ, ϕ), with respect to the origin,
- $\int_V \delta(\mathbf{r})\, dV = 1$ is dimensionless. So a δ-function has units of $(1/l)^n$, where n is the dimension in which the δ-function exists.

7.2 Application of δ-functions: mapping between continuous and discrete representations

In geophysics, we often want to work with the fields of point masses / charges, but to consider quantities as functions. This is easily accomplished using δ-function notation. Consider the example of mass density.

Density, $\rho(x, y, z)$, is a continuous function of space and is of course mass per unit volume. We can think of the simple case of a sphere, density $\rho(x, y, z)$, total mass m, and its equivalent point mass representation. We can contract the sphere to be of smaller and smaller volume – its density increases, approaching infinity as the radius of the sphere tends to zero. In the limit the sphere becomes a point, and the density distribution is written as

$$\rho(x, y, z) = m\,\delta(x - x_0)\,\delta(y - y_0)\,\delta(z - z_0) = m\,\delta(\mathbf{r} - \mathbf{r_0})$$

We note that this form is dimensionally correct and keeps the correct relationship between mass and density:

$$m = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \rho(x, y, z)\,dx\,dy\,dz$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} m\,\delta(x - x_0)\,\delta(y - y_0)\,\delta(z - z_0)\,dx\,dy\,dz$$

$$= m\int_{-\infty}^{\infty}\delta(x - x_0)\,dx\int_{-\infty}^{\infty}\delta(y - y_0)\,dy\int_{-\infty}^{\infty}\delta(z - z_0)\,dz \qquad = m$$

suppose there are N masses, m_i, (i = 1,...,N) at locations $\mathbf{r}_i$ in space. Then a δ-function representation is

$$\rho(x, y, z) = \sum_{i=1}^{N} m_i\,\delta(\mathbf{r} - \mathbf{r_i})$$

A similar approach can be used to represent electric or magnetic charge density.