Here are some additional notes on Green's Functions to try to give a physical picture of how to use a Green's function (the response of a linear system to a delta function or impulse) to get the response of a system to a more complicated forcing function.

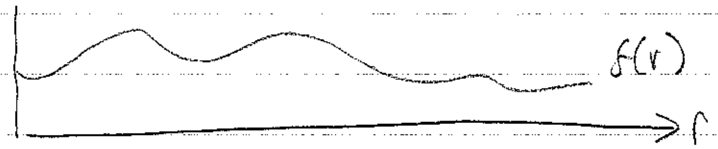
The first 3 pages are my notes on how to try to build up this picture physically: in the sketches the box labeled "Filter" is essentially the solution to the differential equation,

e.g., for the D.E. $\nabla^2 G(r,r_1) = \delta(r-r_1)$
a δ-function input results in an output $G(r, r_1)$

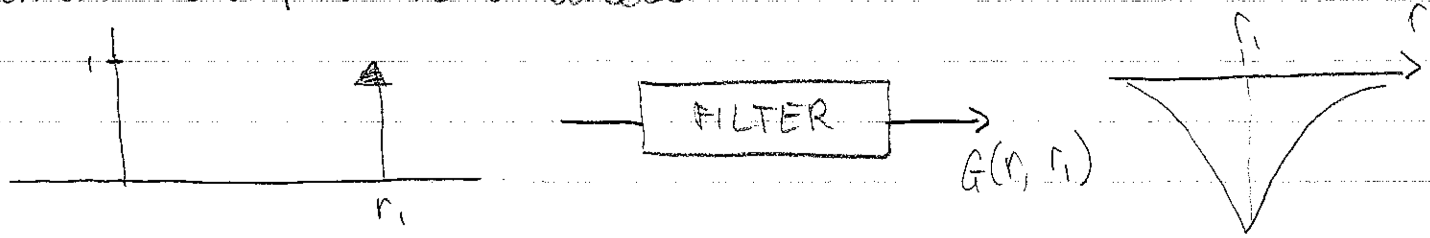
The second 3 pages are from Blakely, giving a specific linear systems example (linear damping).

Building the solⁿ to a complicated forcing function

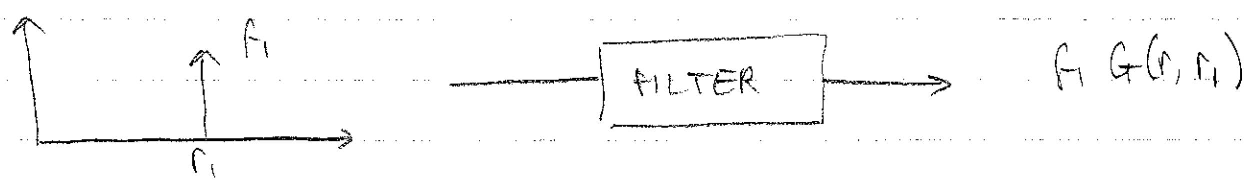
eg in 1-D



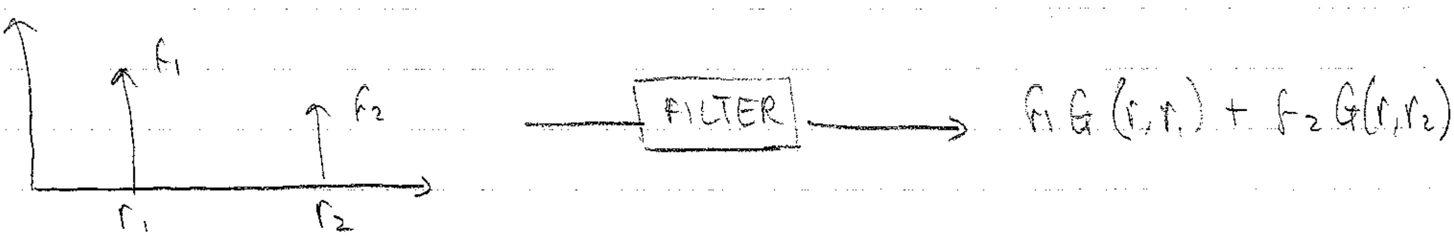
Remark: our problem is linear



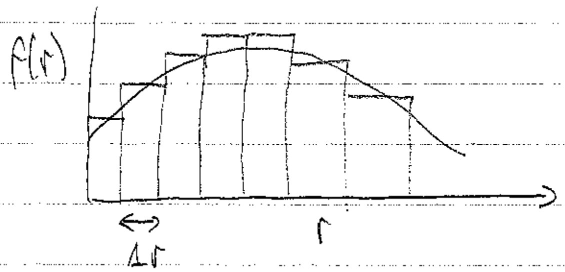
a) scale the I/P → scale the O/P



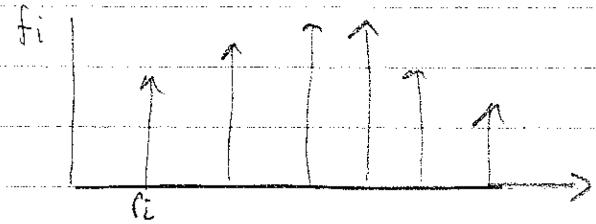
b) Superposition holds
O/P from two sources = sum of the individual outputs



Consider the forcing f^n again



equiv to
=



$f_i = f(r_i)\,\Delta r$ so that sum of f_i s $\equiv$ area under $f(r)$

<u>Input</u> in picture above $\qquad\qquad$ <u>Output</u>

$$\sum_{i=1}^{N} f_i\,\delta(r-r_i) \longrightarrow \sum_{i=1}^{N} f_i\,G(r,r_i) \equiv \phi(r)$$

OR

$$\sum_{i=1}^{N} f(r_i)\,\Delta r_i\,\delta(r-r_i) \dashrightarrow \sum f(r_i)\,G(r,r_i)\,\Delta r_i \equiv \phi(r)$$

let $\Delta r_i \longrightarrow \delta r'$

$r_i \longrightarrow r$

$\Sigma \longrightarrow \int$

$$\int f(r')\,\delta(r-r')\,dr' \longrightarrow \int f(r')\,G(r,r')\,dr' \equiv \phi(r)$$

↓ THIS IS THE TOTAL I/P $f(r)$ $\qquad$ THIS IS THE TOTAL O/P $\phi(r)$

<u>so</u> $\quad f(r) \longrightarrow \int f(r')\,G(r,r')\,dr' \equiv \phi(r)$

So $\quad \nabla^2 \phi(r) = f(r)$

has the solution $\quad \phi(r) = \int f(r')\,G(r,r')\,dr'$

where $G(r,r')$ is the Green's f^n satisfying $\nabla^2 G(r,r') = \delta(r-r')$

Substituting from before $\quad G(\bar{r},\bar{r}') = \dfrac{-1}{4\pi(\bar{r}-\bar{r}')}$

$$\boxed{\phi(r) = \frac{-1}{4\pi}\int_R \frac{f(r')}{|\bar{r}-\bar{r}'|}\,dr'}$$

2.3 Green's Functions

We now turn to *Green's functions*, important tools for solving certain classes of problems in potential theory. A heuristic approach will be used, first considering a mechanical system and then extending this result to Laplace's equation.

2.3.1 Analogy with Linear Systems

We begin with the differential equation describing motion of a particle subject to both a resistance R and an external force $f(t)$,

$$m\,\frac{d}{dt}v(t) = -R\,v(t) + f(t)\,, \tag{2.25}$$

where $v(t)$ is the velocity and m is the mass of the particle, respectively. One conceptual way to solve equation 2.25 is to abruptly strike the particle and observe its response; that is, we let the force be zero except over a short time interval $\Delta\tau$,

$$f(t) = \begin{cases} \frac{I}{\Delta\tau}, & \text{if } \tau < t < \tau + \Delta\tau\,; \\ 0, & \text{otherwise}\,. \end{cases} \tag{2.26}$$

As soon as the force returns to zero, the velocity of the particle behaves like a decaying exponential, and the solution has the form

$$v(t) = A\exp\left[-\frac{R}{m}\left(t - (\tau + \Delta\tau)\right)\right], \quad t > \tau + \Delta\tau\,. \tag{2.27}$$

The coefficient A can be found if the velocity of the particle is known at the moment that the force returns to zero; that is, $v(\tau + \Delta\tau) = A$. To find this velocity, we integrate both sides of equation 2.25 over the duration of the force

$$m[v(\tau + \Delta\tau) - v(\tau)] = -R\int_{\tau}^{\tau+\Delta\tau} v(t)\,dt\; + \frac{I}{\Delta\tau}\int_{\tau}^{\tau+\Delta\tau} dt\,.$$

The first integral can be ignored if $\Delta\tau$ is small and the particle has some mass. Also $v(\tau) = 0$. Hence,

$$m\,v(\tau + \Delta\tau) = I\,,$$

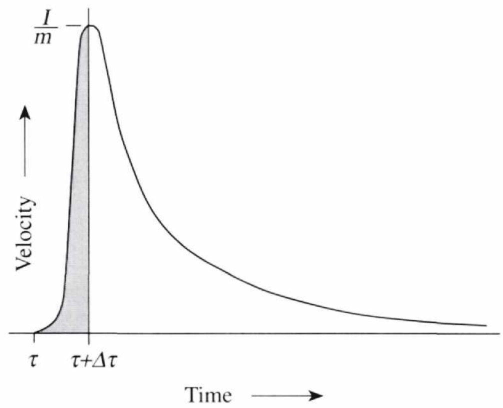


Fig. 2.4. Velocity of a particle of mass m resulting from an impulsive force of magnitude I.

and $A = I/m$ for small $\Delta\tau$. Combining this result with equation 2.27 provides

$$v(t) = \begin{cases} \frac{I}{m}\, e^{-\frac{R}{m}(t-\tau)}, & \text{if } t > \tau\,; \\ 0, & \text{if } t \leq \tau\,. \end{cases} \tag{2.28}$$

Equation 2.28 represents the response of the particle to a single abrupt blow (Figure 2.4). Now suppose that the particle suffers a series of blows I_k at time τ_k, $k = 1, 2, \ldots, N$. The response of the particle to each blow should be independent of all other blows, and the velocity becomes

$$v(t) = \sum_{k=1}^{N} \frac{I_k}{m}\, e^{-\frac{R}{m}(t-\tau_k)}, \quad t > \tau_N\,. \tag{2.29}$$

If the blows become sufficiently rapid, the particle is subjected to a continuous force. Then $I_k \to f(\tau)d\tau$ and

$$v(t) = \frac{1}{m}\int_{\tau_0}^{t} f(\tau)\, e^{-\frac{R}{m}(t-\tau)} d\tau, \quad t > \tau_0\,,$$

which can be rewritten as

$$v(t) = \int_{-\infty}^{t} \psi(t,\tau) f(\tau)\, d\tau\,, \tag{2.30}$$

where

$$\psi(t,\tau) = \begin{cases} 0, & \text{if } t < \tau; \\ \frac{1}{m} e^{-\frac{R}{m}(t-\tau)}, & \text{if } t \geq \tau. \end{cases}$$

Equation 2.30 is the solution to the differential equation 2.25. It presumes that the response of the particle at each instant of impact is independent of all other times. Given this property, the response of the particle to $f(t)$ is simply the sum of all the instantaneous forces, and the particle is said to be a *linear system.* Many mechanical and electrical systems (and, as it turns out, many potential-field problems) have this property.

The function $\psi(t,\tau)$ is the response of the particle at time t due to an impulse at time τ; it is called the *impulse response* or *Green's function* of the linear system. The Green's function, therefore, satisfies the initial conditions and is the solution to the differential equation 2.25 subject to the initial conditions when the forcing function is an impulse.

Equation 2.26 is a heuristic description of an impulse. In the limit as $\Delta\tau$ approaches zero, the impulse of equation 2.26 becomes arbitrarily large in amplitude and short in duration while its integral over time remains the same. The limiting case is called a Dirac delta function $\delta(t)$, which has the properties

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 ;$$

$$\delta(t) = 0 ,\ t \neq 0 ;$$

$$\int_{-\infty}^{\infty} f(t)\delta(t)\, dt = f(0) ;$$

$$\int_{-\infty}^{\infty} f(t)\delta(\tau - t)\, dt = f(\tau) . \tag{2.31}$$

These definitions and properties are meaningless if $\delta(t)$ is viewed as an ordinary function. It should be considered rather as a "generalized function" characterized by the foregoing properties.

Green's functions are very useful tools; equation 2.30 shows that if the Green's function ψ is known for a particular linear system, then the state of the linear system due to any forcing function can be derived for any time.