# Mathematical Background: Green's Functions, the Helmholtz Theorem and Green's Identities

## 1 Motivation

Problems in potential fields boil down to solving Poisson's Equation:

$$\nabla^2 \Phi(\mathbf{r}) = f(\mathbf{r}) \tag{1}$$

In other words given a function $f(\mathbf{r})$ (*e.g.* a source density), find the scalar potential $\Phi(\mathbf{r})$. We will show how this can be done, building up the approach with some of the math steps and with the physical reasoning behind these steps. The idea is to first solve a simpler problem, namely:

$$\nabla^2 G(\mathbf{r}) = \delta(\mathbf{r}) \tag{2}$$

*i.e.,* $G(\mathbf{r})$ is the solution for a simpler source density – that of a $\delta$-function. We can think of the problem in terms of physical system: $G(\mathbf{r})$ is the response of the system to a forcing function given by a $\delta$-function. The solution,$\Phi(\mathbf{r})$, to a more complex forcing function, $f(\mathbf{r})$ can be built up by noting first that from last time:

$$f(\mathbf{r}) = \int_{-\infty}^{\infty} f(\mathbf{r}')\, \delta(\mathbf{r} - \mathbf{r}')\, d\mathbf{r}' \tag{3}$$

and second, that because (1) is a linear system, we can write

$$\Phi(\mathbf{r}) = \int_{-\infty}^{\infty} G(\mathbf{r}, \mathbf{r}')\, f(\mathbf{r}')\, d\mathbf{r}', \tag{4}$$

# 2 Solution for Poisson's Equation

## 2.1 Preliminary Result

We first show the following result

$$\nabla^2 \frac{1}{|\mathbf{r}-\mathbf{r}'|} = -4\pi\delta(\mathbf{r}-\mathbf{r}') \tag{5}$$

Since the PDE is linear it suffices to show that

$$\nabla^2 \frac{1}{|\mathbf{r}|} = -4\pi\delta(\mathbf{r}) \tag{6}$$

We can use the first two properties of $\delta$-functions from last class, to see that the LHS of (6) should be zero everywhere except at $\mathbf{r} = 0$, and the integral over a sphere radius, $\epsilon$, should be equal to $-4\pi$. We can work in any coordinate system. Remember that:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \tag{7}$$

in cartesian coordinates, and

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial \theta}\left(sin\theta \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 sin^2\theta}\frac{\partial^2 f}{\partial \phi^2} \tag{8}$$

in spherical coordinates.

Since $f(r) = 1/r$ we use spherical coordinates, so that we need only use the first term on the RHS of (8) (*i.e.,* no dependence of the differential equation on $\theta$, $\phi$).

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\frac{1}{r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\left(\frac{-1}{r^2}\right)\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(-1) = 0, \qquad r \neq 0$$

So

$$\nabla^2 \frac{1}{|\mathbf{r}|} = 0, \qquad\qquad r \neq 0 \tag{9}$$

In other words, away from $\mathbf{r} = \mathbf{r}'$ (= origin here), there are no sources and Laplace's equation holds. Now we show that the integral of $\nabla^2 \left(\frac{1}{r}\right)$ over a sphere of radius $\epsilon$ is equal to $-4\pi$. Again use spherical coordinates:

$$\int_V \nabla^2 \left(\frac{1}{r}\right)\, dv = \int_V \nabla \cdot \nabla \left(\frac{1}{r}\right)\, dv = \int_S \nabla \left(\frac{1}{r}\right) \cdot \hat{n}\, ds \tag{10}$$

where we have used Gauss's theorem (the divergence theorem)

$$\int_V \nabla \cdot \mathbf{B}\, dv = \int_S \mathbf{B} \cdot \hat{n}\, ds.$$

Now

$$\nabla \frac{1}{r} = \hat{r} \frac{\partial}{\partial r} \frac{1}{r} = -\frac{\hat{r}}{r^2}$$

So (10) becomes

$$\int_S \frac{\hat{r} \cdot \hat{r}}{r^2}\, ds = \int_S \frac{-ds}{\epsilon^2} = -\frac{4\pi\epsilon^2}{\epsilon^2}$$

since $r = \epsilon$ on $S$.

$$\int_V \nabla^2 \left(\frac{1}{r}\right)\, dv = -4\pi$$

We have therefore shown that

$$\nabla^2 \frac{1}{|\mathbf{r}|} = -4\pi\delta(\mathbf{r}) \tag{11}$$

### 2.2 The Green's Function

From (11) it follows that

$$\nabla^2 \frac{1}{|\mathbf{r} - \mathbf{r}'|} = -4\pi\delta(\mathbf{r} - \mathbf{r}')$$

and so

$$\nabla^2 \left( \frac{-1}{4\pi|\mathbf{r} - \mathbf{r}'|} \right) = \delta(\mathbf{r} - \mathbf{r}')$$

Hence we have a solution for the response to an impulse function, at a location $\mathbf{r}'$

$$\nabla^2 G(\mathbf{r}) = \delta(\mathbf{r})$$

$$G(\mathbf{r}, \mathbf{r}') = \frac{-1}{4\pi|\mathbf{r} - \mathbf{r}'|}. \tag{12}$$

### 2.3 Building up the more General Solution

Earlier we noted that because the PDE is linear we can build up the solution to a more general forcing function quite easily, once we have the Green's function. The solution is given by (4), and we can now substitute for $G(\mathbf{r}, \mathbf{r}')$ using (12):

$$\Phi(\mathbf{r}) = -\frac{1}{4\pi} \int_R \frac{f(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}' \tag{13}$$

A Green's function is an integral kernel – see (4) – that can be used to solve an inhomogeneous differential equation with boundary conditions. A Green's function approach is used to solve many problems in geophysics. See also discussion in-class.

## 3 Helmholtz Decomposition Theorem

### 3.1 The Theorem – Words

A vector field vanishing at infinity is completely specified by its divergence and its curl if they are known throughout space. If both the divergence and the curl vanish at all points, then the field itself must vanish or be constant everywhere.

## 3.2 The Theorem - Math

Let $\mathbf{f}(x, y, z) = \mathbf{f}(\mathbf{r})$ be a vector field in $R^3$ such that

$$\nabla \cdot \mathbf{f} = s$$

$$\nabla \times \mathbf{f} = \mathbf{c}.$$

Then $\mathbf{f}$ has a unique decomposition:

$$\mathbf{f} = \nabla \Phi + \nabla \times \mathbf{A}, \tag{14}$$

where

$$\Phi(\mathbf{r}) = \frac{-1}{4\pi} \int \frac{\mathbf{s}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d^3\mathbf{r}' = \frac{-1}{4\pi} \int \frac{\nabla \cdot \mathbf{f}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d^3\mathbf{r}', \tag{15}$$

and

$$\mathbf{A} = \frac{1}{4\pi} \int \frac{\mathbf{c}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d^3\mathbf{r}' = \frac{1}{4\pi} \int \frac{\nabla \times \mathbf{f}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d^3\mathbf{r}' \tag{16}$$

## 3.3 Proof by Verification

We calculate the divergence and curl of $\mathbf{f}$ and demonstrate equations (15) and (16).

FIRST: Find $\Phi$. We calculate $\nabla \cdot \mathbf{f}$:

$$\nabla \cdot \mathbf{f} = \nabla \cdot \nabla \Phi + \nabla \cdot \nabla \times \mathbf{A}$$

so $\nabla \cdot \mathbf{f} = \nabla^2 \Phi$, and this $= \mathbf{s}$ from above. We solved $\nabla^2 \Phi = \mathbf{s}$ in the previous section, so we know that

$$\Phi(\mathbf{r}) = \frac{-1}{4\pi} \int \frac{\nabla \cdot \mathbf{f}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d^3\mathbf{r}'$$

SECOND: Find $\mathbf{A}$. We calculate $\nabla \times \mathbf{f}$:

$$\nabla \times \mathbf{f} = \nabla \times \nabla\Phi + \nabla \times \nabla \times \mathbf{A}$$

The first term on the RHS is zero for any scalar $\Phi$. The second term on the rhs can be re-written using the vector identity $\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$. So $\nabla \times \mathbf{f} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$.

The difficulty now is that in the Helmholtz equation (14) we have on the LHS, $f$ which is specified by 3 scalars (e.g., $f_x, f_y, f_z$). On the RHS we have $\mathbf{A}$ (specified by 3 scalars – e.g., $A_x, A_y, A_z$) and $\Phi$, a single scalar, so we have 3 scalars on the LHS and 4 unknowns on the RHS. Thus there is no unique solution for either $\mathbf{A}$ or $\Phi$. We can find a particular solution by implementing an additional constraint. We use $\nabla \cdot \mathbf{A} = 0$, so

$$\nabla \times \mathbf{f} = -\nabla^2\mathbf{A} \tag{17}$$

This a vector form of Poisson's equation. We can solve (17) on a component by component basis, such that we solve 3 scalar Poisson's equations. The solution for each component of $\mathbf{A}$ is thus:

$$A_i(\mathbf{r}) = \frac{1}{4\pi}\int \frac{(\nabla \times \mathbf{f})_i}{|\mathbf{r} - \mathbf{r}'|}dr',$$

and the vector field is given by

$$\mathbf{A}(\mathbf{r}) = \frac{1}{4\pi}\int \frac{\nabla \times \mathbf{f}}{|\mathbf{r} - \mathbf{r}'|}dr',$$

# 4 Consequences of the Helmholtz Theorem

## 4.1 Irrotational Fields: $\nabla \times \mathbf{f} = 0$

These are vector fields, $\mathbf{f}$ defined over a region $R$ for which the curl of the field, $\nabla \times \mathbf{f}$, vanishes at all points in $R$. These fields have no vorticity or eddies. Recall Stoke's theorem

$$\oint \mathbf{f} \cdot \hat{\mathbf{t}}\, ds = \int_S \hat{n} \cdot (\nabla \times \mathbf{f})\, ds$$

– "the line integral of the tangential component of a vector function over some closed path equals the surface integral of the normal component of the curl of that function integrated over any capping surface of the path". Thus if $\nabla \times \mathbf{f} = 0$, then $\oint \mathbf{f} \cdot \hat{\mathbf{t}}\, ds = 0$, so the work is path independent and hence $\mathbf{f}$ is conservative.

Hence $\nabla \times \mathbf{f} = 0$ is a necessary and sufficient condition to be able to write $\mathbf{f} = \nabla\Phi$. The corollary holds: A field that can be written in terms of the gradient of a scalar potential has no curl, since $\nabla \times \nabla\Phi \equiv 0$.

### 4.2 Solenoidal Fields: $\nabla \cdot \mathbf{f} = 0$

These are vector fields, $\mathbf{f}$ defined over a region $R$ for which the divergence of the field, $\nabla \cdot \mathbf{f}$, vanishes at all points in $R$. From the divergence theorem, then the normal component of $\mathbf{f}$, $f_n$, vanishes when integrated over any closed surface within the region. Hence the number of field lines entering the region equals the number leaving, *i.e.,* there is no net flux or equivalently there are no sources or sinks within the region.

(A) If $\mathbf{f} = \nabla\Phi$, then $\nabla \cdot \mathbf{f} = \nabla^2\Phi$, If $\mathbf{f}$ is also solenoidal, then from the Helmholtz equation $\nabla \cdot \mathbf{f} = \nabla^2\Phi = 0$, which is Laplace's equation. Hence if the divergence of a conservative field vanishes in a region, the potential of the field is harmonic in the region, *i.e.,* it satisfies Laplace's equation.

(B) If $\mathbf{f} = \nabla \times \mathbf{A}$, then $\nabla \cdot \mathbf{f} = 0$ by definition. Hence if the divergence of a vector field, $\mathbf{f}$, vanishes at all points in a region, $R$, then $\mathbf{f}$ can be written as $\mathbf{f} = \nabla \times \mathbf{A}$.

## 5 Green's Identities

Green's identities are 3 identities that can be derived from vector calculus, with particular cases when Laplace's equation holds. They lead to theorems and to useful algorithms for practical analyses of potential fields. We will not derive the identities since the derivation is purely an exercise in math. We will write them down and focus on their importance for potential fields studies. Abbreviated derivations can be found in Blakely, chapter 2 pages 19–22.

### 5.1 Green's First Identity

Let U, V be continuous functions with continuous partial derivatives throughout a closed region, R. Let U have second order partial derivatives. The $1^{st}$ identity is:

$$\int_R \left(\nabla V \cdot \nabla U + V\nabla^2 U\right) dv = \int_S V\frac{\partial U}{\partial n}dS \tag{18}$$

For our purposes, this has some interesting consequences. First, when V = 1, and when U is harmonic *i.e.*, $\nabla^2 U = 0$:

$$\int_S \frac{\partial U}{\partial n}dS = 0 \tag{19}$$

Hence the normal derivative of a function must average to zero on any closed boundary surrounding a region throughout which the function is harmonic. Note of course that this is just solenoidal fields again. This is known as Gauss's Law - we'll see it again in gravity field studies.

Second: It can be shown that if $U$ is harmonic and continuously differentiable in $R$ and if $U$ vanishes at all points of $S$, then $U$ must also vanish at all points in $R$. Think about steady-state heat flow – if the temperature is zero at all points of a region's boundary and there are no sources or sinks within the region then the temperature must be equal to zero throughout the region once equilibrium is achieved.

### 5.2 Green's Second Identity

This is mostly of use in deriving the $3^{rd}$ identity so we just write it down here:

$$\int_R \left(U\nabla^2 V - V\nabla^2 U\right) dv = \int_S \left(U\frac{\partial V}{\partial n} - V\frac{\partial U}{\partial n}\right) ds \tag{20}$$

## 5.3 Green's Third Identity

Given a point $P$ inside a region $R$ we can write the potential $U$ at P as follows:

$$U(P) = -\frac{1}{4\pi}\int_V \frac{\nabla^2 U}{r}\, dv + \frac{1}{4\pi}\int_S \frac{1}{r}\frac{\partial U}{\partial n}\, ds - \frac{1}{4\pi}\int_S U\frac{\partial}{\partial n}\left(\frac{1}{r}\right) ds \tag{21}$$

Recall the form of our solution to Poisson's equation from last time. We can use that to think of the terms on the RHS of (21) as being potentials as follows: The first term can be thought of as the potential due to a volume distribution with source density proportional to $\nabla^2 U$. The second term can be thought of as the potential due to a surface distribution with source density proportional to $\frac{\partial U}{\partial n}$. The third term can be thought of as the potential due to a surface distribution with source density proportional to $U$. If $U$ is harmonic, the first term $= 0$ and so we can write

$$U(P) = \frac{1}{4\pi}\int_S \left(\frac{1}{r}\frac{\partial U}{\partial n} - U\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right) ds \tag{22}$$

Hence a harmonic function can be calculated at any point of a region in which it is harmonic, simply from the the values of the function and its normal derivatives on the boundary. This is of great practical importance in potential fields problems because it allows us to calculate the potential everywhere given measurements only on a boundary. It allows us to do upward and downward continuation.