Introduction to Gravity

1 Gravity: Introduction

1.1 Recap I: Newton's Law

Force on m_0 due to m at $\mathbf{r}'$ is

$$\mathbf{F}(\mathbf{r}) = -G\, m\, m_0 \frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \tag{1}$$

and the gravitational field is

$$\mathbf{g}(\mathbf{r}) = -G\, m \frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \tag{2}$$

Law of Superposition: for N point masses, each mass m_i

$$\mathbf{g}(\mathbf{r}) = -G \sum_{i=1}^{N} m_i \frac{(\mathbf{r} - \mathbf{r_i})}{|\mathbf{r} - \mathbf{r_i}|^3} \tag{3}$$

or for a distributed density distribution

$$\mathbf{g}(\mathbf{r}) = -G \int_V \rho(\mathbf{r}')\, \frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} dv' \tag{4}$$

Equation (4) is general and it holds for $\mathbf{r}$ inside or outside the volume containing the density distribution $\rho(\mathbf{r}')$.

1.2 Recap II: from problem set #1

$$\nabla \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) = -\frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \tag{5}$$

so we can write

$$\mathbf{g}(\mathbf{r}) = G \int_V \rho(\mathbf{r}')\, \nabla \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) dv' \tag{6}$$

1.3 Fundamental Equations for Gravitational fields

From (6) and our work solving for the Green's function for Poisson's equation, we can calculate:

$$\begin{aligned} \nabla \cdot \mathbf{g} &= G \int_V \rho(\mathbf{r}')\, \nabla \cdot \nabla \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) dv' \\ &= G \int_V \rho(\mathbf{r}')\, \nabla^2 \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) dv' \\ &= G \int_V \rho(\mathbf{r}')\, (-4\pi)\, \delta(\mathbf{r} - \mathbf{r}') dv' \\ &= -4\,\pi\, G\, \rho(\mathbf{r}) \end{aligned} \tag{7}$$

$$\nabla \times \mathbf{g} = G \int_V \rho(\mathbf{r}')\, \nabla \times \nabla \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) dv' = 0 \tag{8}$$

Note that (8) implies that the gravitational field is conservative.
The two fundamental equations for the gravitational field are:

$$\begin{aligned} \nabla \cdot \mathbf{g} &= -4\,\pi\, G\, \rho(\mathbf{r}) \\ \nabla \times \mathbf{g} &= 0 \end{aligned} \tag{9}$$

Since we have specified the divergence and curl of the vector field, then from the Helmholtz decomposition theorem, we can write

$$\mathbf{g} = \nabla\Phi + \nabla \times \mathbf{A} = \nabla\Phi \tag{10}$$

since from (8), $\mathbf{A} \equiv 0$.

1.4 Integral and Differential Forms for $\Phi(\mathbf{r})$ and $\mathbf{g}(\mathbf{r})$

The potential, Φ can be computed from various routes:

1. Use Helmholtz theorem to write down the result $\Longrightarrow$ I. E.
2. Use fundamental equations $\Longrightarrow$ D. E., Poisson's equation
3. Use Green's function approach $\Longrightarrow$ I. E.

(1) Helmholtz Decomposition Theorem:

$$\mathbf{g} = \nabla\Phi(\mathbf{r})$$

and since

$$\Phi(\mathbf{r}) = \frac{-1}{4\pi} \int_V \frac{\nabla \cdot \mathbf{g}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, dv'$$

and $\nabla \cdot \mathbf{g} = -4\pi G\rho$, then

$$\Phi(\mathbf{r}) = G \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dv' \tag{11}$$

(2) Poisson's Equation

$$\mathbf{g} = \nabla\Phi(\mathbf{r})$$

and

$$\nabla \cdot \mathbf{g} = \nabla \cdot \nabla\Phi = \nabla^2\Phi$$

Again using (7), we get

$$\nabla^2\Phi = -4\pi G\rho \tag{12}$$

(3) Green's function approach:
We can use a Green's function approach to solve Poisson's equation. Recall that

$$\nabla^2\Gamma(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}')$$

has the solution

$$\Gamma(\mathbf{r}, \mathbf{r}') = \frac{-1}{4\pi \, |\mathbf{r} - \mathbf{r}'|}$$

and that the solution to Poisson's equation $\nabla^2\Phi = f(r)$ is given by

$$\Phi = \int_V f(\mathbf{r}')\Gamma(\mathbf{r}, \mathbf{r}') \, dv'$$

So

$$\begin{aligned} \Phi(\mathbf{r}) &= \int_V (-4\pi G\rho) \left(\frac{-1}{4\pi \, |\mathbf{r} - \mathbf{r}'|} \right) dv' \\ &= G \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dv' \end{aligned} \tag{13}$$

as in (11) above.

2 Gauss's Law

Consider a closed surface, S, enclosing completely or partially a density distribution $\rho(\mathbf{r})$ in a volume, V. Using the fundamental equation (7) we can write

$$\int_V \nabla \cdot \mathbf{g}\, dv = -4\pi G \int_V \rho(\mathbf{r})\, dv$$

Now we can use the divergence theorem to write

$$\int_V \nabla \cdot \mathbf{g}\, dv = \int_S \mathbf{g} \cdot \hat{n}\, da$$

So we obtain Gauss's Law relating the flux of a field out of a closed surface to the mass contained within the surface.

$$\int_S \mathbf{g} \cdot \hat{n}\, da = -4\pi G M \tag{14}$$

Gauss' Law is useful for two reasons:

1. We can write

   $$M = \frac{-1}{4\pi G} \int_S \mathbf{g} \cdot \hat{n}\, da$$

   and calculate M given measurements of the normal component of gravity over a surface.

2. If certain symmetry conditions exist we can use the formula to calculate the gravity field.

We now have three formulations for gravity field modeling

1. Differential form: $\nabla^2 \Phi = -4\pi G \rho$, and $\mathbf{g} = \nabla \Phi$
2. Integral form for either Φ (equation 11) or $\mathbf{g}$ (equation 4)
3. Gauss's theorem for special cases (equation 14)