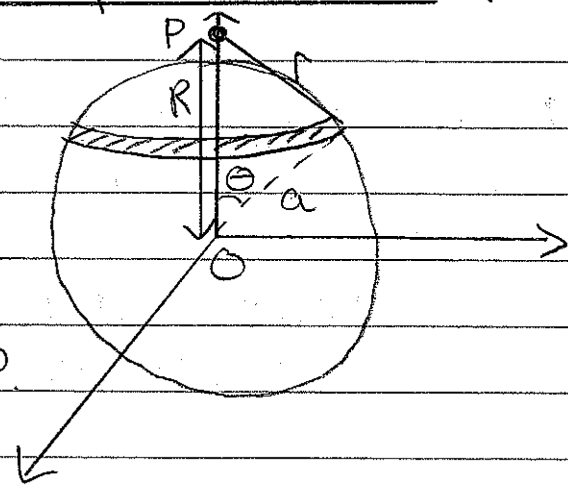
GRAVITY EOSC 450

FORWARD MODELING: Analytical Examples

See also Blakely pages 49-59

Example 1: Gravitational Potential + Gravity Field due to a spherical shell (Blakely p. 49-51)

1) Shell, radius = a
2) Pt. P outside spherical shell
3) Origin @ center of shell.
4) For convenience, orient coord system to go thru' P.



1. Procedure: a) calculate pot' ϕ from $\phi = G\int \frac{\rho(r)}{r}\,dv$

   b) calc. $\bar{g}$ from $\bar{g} = \bar{\nabla}\phi$

2. Assume mass concentrated on spherical shell so

$$\int \rho\, dv \longrightarrow \int \sigma\, ds$$

σ → surface mass density = mass per unit area.

So $$\phi(P) = G\int_s \frac{\sigma(s)}{r}\boxed{ds}$$

$ds = a^2 \sin\theta\, d\theta\, d\varphi$

$$\phi(P) = G\int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi} \frac{\sigma a^2 \sin\theta\, d\theta\, d\varphi}{r}$$

$$\Phi(P) = Ga^2\sigma \int_0^{2\pi}\int_0^{\pi} \frac{\sin\theta}{r}\, d\theta\, d\phi$$ ②

$$= Ga^2\sigma\, 2\pi \int_0^{\pi} \frac{\sin\theta}{r}\, dr$$

Issue: r is a function of θ in our picture.

$\boxed{3}$ Use Cosine Law

$$r^2 = R^2 + a^2 - 2aR\cos\theta$$

$$\Rightarrow 2r\frac{dr}{d\theta} = -2aR(-\sin\theta)$$

$$\Rightarrow \frac{\sin\theta}{r}\, d\theta = \frac{dr}{aR}$$

to allow

change of variables:

$$\theta = 0 \Rightarrow r_1^2 = R^2 + a^2 - 2aR = (R-a)^2$$

$$r_1 = \pm(R-a)$$

we'll use $r_1 = R - a$ $\boxed{\text{WHY}}$??

$$\theta = \pi \Rightarrow r_2^2 = R^2 + a^2 + 2aR = (R+a)^2$$

$$r_2 = \pm(R+a)$$

we'll use $r_2 = R + a$ $\boxed{\text{WHY}}$??

$\boxed{4}$ So $\Phi(P) = 2\pi a^2 G\sigma \int_{r_1}^{r_2} \frac{dr}{aR} = 2\pi a^2 G\sigma \int_{R-a}^{R+a} \frac{dr}{aR}$

$$= 2\pi a^2 G\sigma \left[\frac{(R+a)-(R-a)}{aR}\right]$$

$$= \frac{4\pi a^2 G\sigma}{R} = \frac{GM}{R}$$

$$\Rightarrow \boxed{\Phi(P) = \frac{GM}{r}}$$

- As there is rotational symmetry, we could have chosen P in any dirⁿ & arrived @ same answer.
- Note: Grav^l pot^l @ point, P, outside a spherical shell of surface mass density σ is the same as if the mass of the shell were all concentrated @ the center.

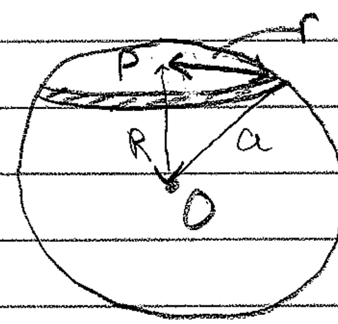
5 More generally (outside shell)

$$\Phi(\vec{r}) = \frac{GM}{r} \qquad \vec{g}(\vec{r}) = -\frac{GM}{r^2}\hat{r}$$

since $\vec{g} = \vec{\nabla}\Phi$

Remark: can obtain same result using Gauss's thm.

6 Inside the shell



- Point P inside shell dist. R from origin
- shell radius a.

$$\boxed{\Phi(P) = \frac{GM}{a}}$$

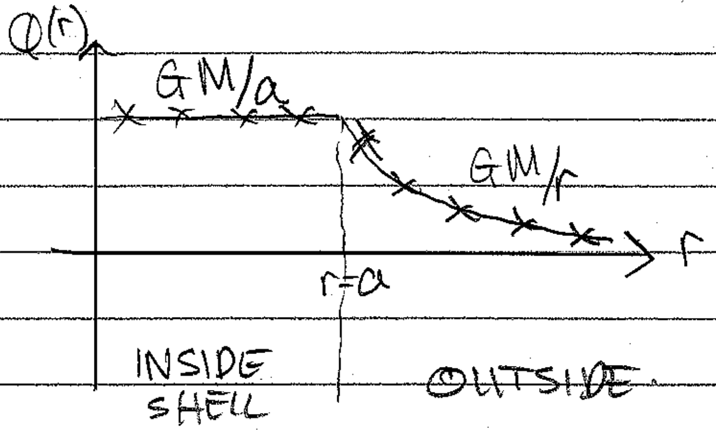
SHOW THIS

$$\boxed{\vec{g} = \vec{\nabla}\Phi = 0}$$

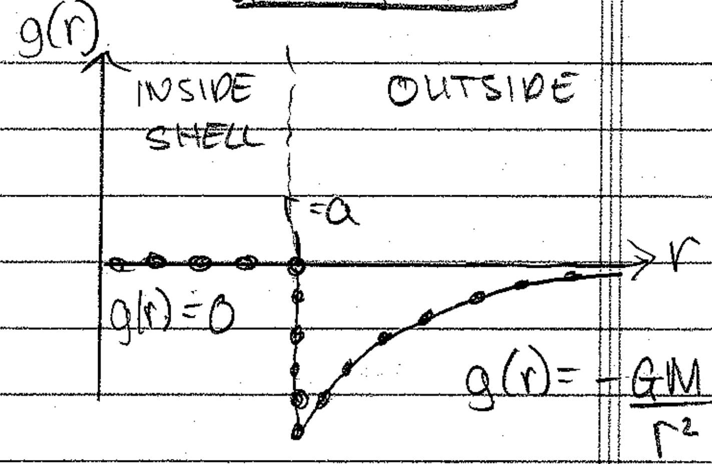
and so $\nabla^2\Phi = 0$

Spherical Shell

Potential



Gravity Field



Potential due to solid sphere
- see next 2 pages ; also Blakely p. 51-54

Example: Solid Sphere

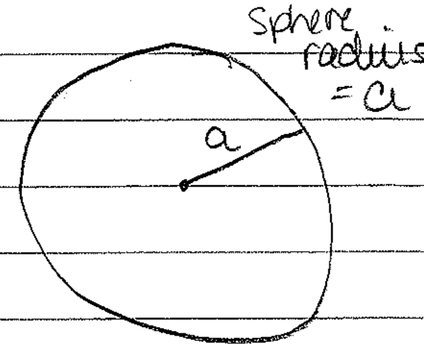
— follows from results for shell + superposition

A Outside sphere

$$\Phi(r) = \frac{GM}{r}, \quad r > a$$

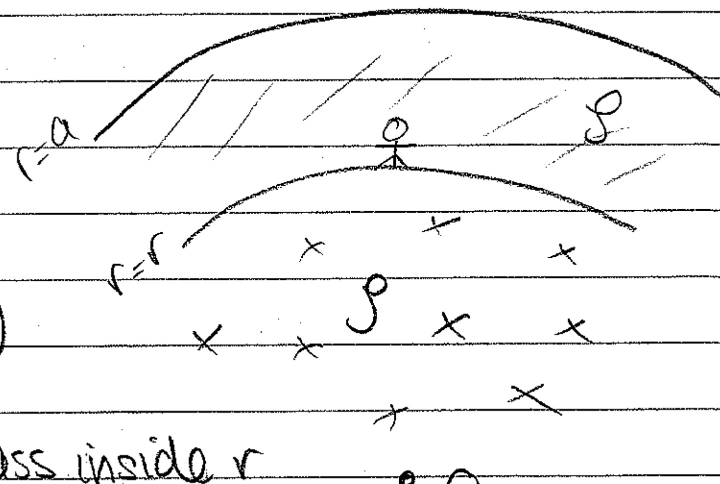
and $\nabla^2 \Phi = 0$

$$M = \sum_{i=1}^{N} M_i^{shell} = \frac{4\pi}{3} \bar{\rho} a^3$$



B Inside sphere

Imagine an observer @ radius 'r' inside the planet



The pot' @ r can be written

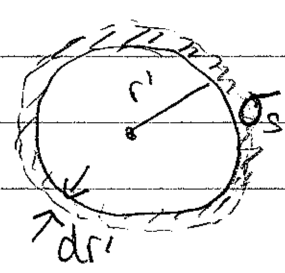
$$\Phi(r) = \Phi_O(r) + \Phi_I(r)$$

$\Phi_I(r)$ pot' @ r due to mass inside r

$\Phi_O(r)$ " " " " " outside r

$$\Phi_I(r) = \frac{GM}{r} = \frac{G}{r} \frac{4\pi r^3}{3} \rho = \frac{4}{3} \pi r^2 G \rho$$

$$\Phi_O(r) = \int_r^a \left(\begin{array}{c} \text{pot' due to a} \\ \text{spherical shell @ } r' \end{array} \right) dr' \qquad r \le r' \le a$$



$\sigma_s = \rho dr'$

$$\Phi(r < r') = \frac{GM}{r'} = \frac{G\, 4\pi \sigma_s r'^2}{r'} = 4\pi G \sigma_s r' = 4\pi G \rho r' dr'$$

so $\Phi_o(r) = \int_r^a 4\pi G \rho r' \, dr' = 2\pi G \rho (a^2 - r^2)$

So $\Phi(r) = \Phi_i(r) + \Phi_o(r)$
$= 4/3 \pi r^2 G \rho + 2\pi G \rho (a^2 - r^2)$

$$\Rightarrow \boxed{\Phi(r) = \frac{2\pi G \rho}{3} \left(3a^2 - r^2\right)}$$

Potential inside solid sphere of uniform density

$$\bar{g} = \bar{\nabla} \Phi(\bar{r}) = \hat{r} \frac{\partial}{\partial r} \Phi(r) = -\frac{4}{3} \pi G \rho r \hat{r}$$

$$\Rightarrow \boxed{\bar{g} = -\frac{4\pi G \rho r \hat{r}}{3}}$$

i.e. gravity varies linearly w/ r inside the sphere.

SHOW THAT POISSON'S EQ^N HOLDS INSIDE the sphere

i.e. $\boxed{\nabla^2 \Phi = -4\pi G \rho}$