Magnetics, Part I (see also Blakely, chapter 4)

1 Fundamental Equations

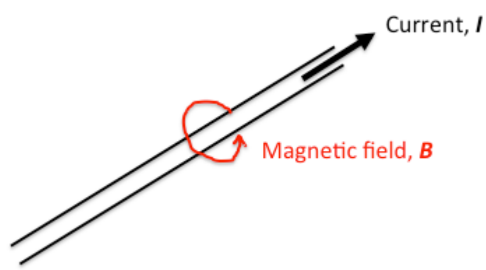
Recall that when introducing gravity, we built up the integral equation for the gravity field, $\mathbf{g}$ by the following approach (1) the force on a point mass due to another point mass, (2) the acceleration of one point mass due to another point mass, (3) the acceleration of a point mass due to many other point masses, and (4) the continuum representation of (3) - *i.e.,* the acceleration of a single point mass due to an arbitrary density distribution. We can take a similar approach in magnetics – here we consider the mutual attraction of two small loops of electric current loops, the magnetic analog of two point masses.

In gravity, we were then able to use the integral equation for $\mathbf{g}$, together with identities from vector calculus to derive the fundamental equations ($\nabla \cdot \mathbf{g} = -4\pi G\rho$ and $\nabla \times \mathbf{g} = 0$). The fundamental equations in magnetics are given by Maxwell's equations.

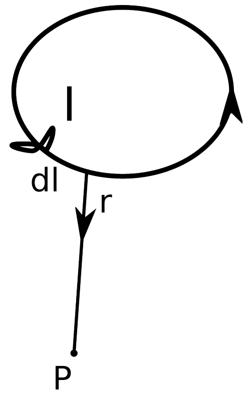
1.1 Background: Magnetic Induction

We're all familiar with magnetic fields; for example, bar magnets, wire carrying a current, etc. We can calculate the magnetic field produced by a current:

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r}\hat{\theta} \tag{1}$$



We can consider the magnetic field due to an element of a wire loop carrying a current:



$$\mathbf{dB}(P) = \frac{\mu_0 I}{4\pi} \frac{\mathbf{dl} \times \hat{r}}{r^2} \tag{2}$$

This comes from considering two loops of current with currents I_a and I_b respectively. The force acting on a small element $\mathbf{dl}_a$ of loop a caused by the electric current in element $\mathbf{dl}_b$ of the second loop is given by the *Lorentz force* ($\mathbf{f} = q\mathbf{v} \times \mathbf{B}$) – see Figure 4.1, p. 66 of Blakeley:

$$\mathbf{df}_a = I_a \mathbf{dl}_a \times \mathbf{dB}_b \tag{3}$$

In discussing gravity fields we then considered the force per unit mass - i.e. the force on a test particle of unit magnitude. Similarly, we now consider loop a to be a "test loop" and define a vector $\mathbf{B}$ such that

$$\mathbf{dB}_b = C_m I_b \frac{\mathbf{dl}_b \times \hat{r}}{r^2} \tag{4}$$

where the constant C_m in SI units is $C_m = \frac{\mu_0}{4\pi}$. So:

$$\mathbf{df}_a = C_m I_a I_b \frac{\mathbf{dl}_a \times \mathbf{dl}_b \times \hat{r}}{r^2} \tag{5}$$

and

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \oint_C \frac{\mathbf{dl} \times \hat{r}}{r^2} \tag{6}$$

where $\mathbf{B}$ is magnetic flux density in Tesla (= Weber/m^2). If the current flows in a volume, we can define the current density, $\mathbf{J}$ (Amp/m^2) and

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int_{V'} \frac{\mathbf{J} \times \hat{r}}{r^2} \, d\mathbf{r}' \tag{7}$$

This is the *Biot-Savart Law*, one of the fundamental equations in physics. We can now see that *all* magnetic fields arise due to currents.

1.2 Maxwell's equations and Integral Equation for Vector Potential

Recall two of Maxwell's equations

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{d\mathbf{D}}{dt} \tag{8}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{9}$$

and the constitutive equation:

$$\mathbf{B} = \mu \mathbf{H} \tag{10}$$

where $\mathbf{H}$ is the magnetic field (Amp/m). In a steady state, displacement current $\frac{d\mathbf{D}}{dt} = 0$. Assuming that we're in free space, our equations become:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{11}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{12}$$

Consider $\nabla \cdot \mathbf{B} = 0$

$$\int_V \nabla \cdot \mathbf{B}\, dV = \int_S \mathbf{B} \cdot \hat{n}\, da = 0 \tag{13}$$

So, $\int_S \mathbf{B} \cdot \hat{n}\, da = 0$ for any closed volume. This says that the net flux of $\mathbf{B}$ out of any volume is zero. Equivalently, this says that there are no magnetic "charges". Contrast this with gravity, where we had

$$\nabla \cdot \mathbf{g} = -4\pi G \rho \tag{14}$$

$$\int \mathbf{g} \cdot \hat{n}\, da = -4\pi G \int \rho\, dV \tag{15}$$

Another consequence of $\nabla \cdot \mathbf{B} = 0$ is:

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{16}$$

where $\mathbf{A}$ is a vector potential.

We can use the Helmholtz Decomposition Theorem to get an expression for $\mathbf{A}$. We had, for an arbitrary vector field $\mathbf{F}$ with

$$\nabla \cdot \mathbf{F} = s$$

$$\nabla \times \mathbf{F} = \mathbf{c}$$

Then

$$\mathbf{F} = \nabla\phi + \nabla \times \mathbf{A} \tag{17}$$

with

$$\mathbf{A}\left(\mathbf{r}\right) = \frac{1}{4\pi}\int_V \frac{\mathbf{c}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dV' \tag{18}$$

$$\phi(\mathbf{r}) = -\frac{1}{4\pi}\int_V \frac{s(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dV' \tag{19}$$

So, for $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{B} = \mu_0\mathbf{J}$, we have $\mathbf{c} = \mu_0\mathbf{J}$. As a result,

$$\mathbf{A}\left(r\right) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(r')}{|\mathbf{r}-\mathbf{r}'|}dV' \tag{20}$$

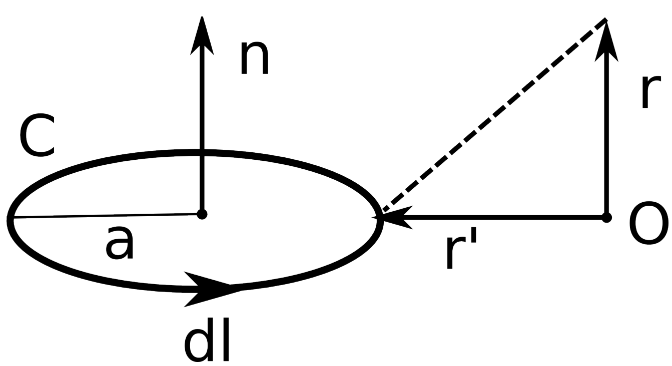
Or, if a wire is carrying the current:

$$\mathbf{A}\left(r\right) = \frac{\mu_0}{4\pi}I\int_C \frac{\mathbf{dl}'}{|\mathbf{r}-\mathbf{r}'|} \tag{21}$$

This expression is always valid. Given $\mathbf{J}$ or I, you can compute $\mathbf{A}$ from 20 or 21 and then $\mathbf{B} = \nabla \times \mathbf{A}$.

2 Vector Potential, Magnetic Induction and Scalar Potential due to Circular Current Loop

2.1 General form for A and B due to circular currrent loop



$$\mathbf{A}\left(r\right) = \frac{\mu_0}{4\pi}\oint_C \frac{I\,\mathbf{dl}'}{|\mathbf{r}-\mathbf{r}'|} \tag{22}$$

Vector identity: $\oint_C \psi \, \mathbf{dl} = \int_S \hat{n} \times \nabla\psi \, dS$

Let $\psi = \frac{\mu_0 I}{4\pi} \frac{1}{|\mathbf{r}-\mathbf{r}'|}$

$$\mathbf{A}(r) = \oint_C \psi \, \mathbf{dl} = \frac{\mu_0 I}{4\pi} \int_S \hat{n} \times \nabla \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} \right) dS = -\frac{\mu_0 I}{4\pi} \int_S \hat{n} \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \, dS \tag{23}$$

Let S be the plane inside the current loop, then $\hat{n}$ is constant.

$$\mathbf{A}(r) = -\frac{\mu_0 I}{4\pi} \hat{n} \times \int_S \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \, dS \tag{24}$$

Assume that the radius of the loop $a << |\mathbf{r}-\mathbf{r}'|$. Then:

$$\int_S \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \, dS \simeq \pi a^2 \frac{\mathbf{r}-\mathbf{r_c}}{|\mathbf{r}-\mathbf{r_c}|^3} \tag{25}$$

where $\mathbf{r_c}$ is the position vector of the center of the loop. Define $\mathbf{m} = \pi a^2 I \, \hat{n}$, the *dipole moment*. Then,

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0}{4\pi} \mathbf{m} \times \frac{\mathbf{r}-\mathbf{r_c}}{|\mathbf{r}-\mathbf{r_c}|^3} \tag{26}$$

The magnetic field from a small loop is

$$\mathbf{B} = \nabla \times \mathbf{A} = -\frac{\mu_0}{4\pi} \nabla \times \mathbf{m} \times \nabla \left(\frac{1}{|\mathbf{r}-\mathbf{r_c}|} \right) \tag{27}$$

Use the identity:

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a}) + (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b}$$

Put $\mathbf{a} = \mathbf{m}$ and $\mathbf{b} = \nabla \left(\frac{1}{|\mathbf{r}-\mathbf{r_c}|} \right)$

$$\begin{aligned} \nabla \times \mathbf{m} \times \nabla \left(\frac{1}{|\mathbf{r}-\mathbf{r_c}|} \right) &= \mathbf{m} \left(\nabla \cdot \nabla \left(\frac{1}{|\mathbf{r}-\mathbf{r_c}|} \right) \right) - \nabla \left(\frac{1}{|\mathbf{r}-\mathbf{r_c}|} \right) \nabla \cdot \mathbf{m} \\ &\quad + \left(\nabla \left(\frac{1}{|\mathbf{r}-\mathbf{r_c}|} \right) \cdot \nabla \right) \mathbf{m} - (\mathbf{m} \cdot \nabla) \nabla \left(\frac{1}{|\mathbf{r}-\mathbf{r_c}|} \right) \end{aligned} \tag{28}$$

But $\mathbf{m}$ is constant so $\nabla \cdot \mathbf{m} = 0$ and so is $\nabla \mathbf{m}$. Thus,

$$\nabla \times \mathbf{m} \times \nabla \left(\frac{1}{|\mathbf{r}-\mathbf{r_c}|} \right) = \mathbf{m} \left(\nabla \cdot \nabla \left(\frac{1}{|\mathbf{r}-\mathbf{r_c}|} \right) \right) - (\mathbf{m} \cdot \nabla) \nabla \left(\frac{1}{|\mathbf{r}-\mathbf{r_c}|} \right) \tag{29}$$

But

$$\nabla \cdot \nabla \left(\frac{1}{|\mathbf{r} - \mathbf{r_c}|} \right) = \nabla^2 \left(\frac{1}{|\mathbf{r} - \mathbf{r_c}|} \right) = -4\pi\delta(\mathbf{r} - \mathbf{r_c}) \tag{30}$$

We are assuming that the observer is far from the loop, so this first term is 0. Thus,

$$\nabla \times \mathbf{m} \times \nabla \left(\frac{1}{|\mathbf{r} - \mathbf{r_c}|} \right) = -(\mathbf{m} \cdot \nabla)\nabla \left(\frac{1}{|\mathbf{r} - \mathbf{r_c}|} \right) \tag{31}$$

and

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}(\mathbf{m} \cdot \nabla)\nabla \left(\frac{1}{|\mathbf{r} - \mathbf{r_c}|} \right) \tag{32}$$

2.2 Specific form for A and B due to circular currrent loop

Now consider the current loop to be *centered at the origin*, oriented so that $\hat{n} = \hat{z}$. The dipole moment is $\mathbf{m} = \pi a^2 I \hat{z}$.

For $|\mathbf{r}| >> a$,

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0}{4\pi}\mathbf{m} \times \frac{\hat{r}}{r^2} \tag{33}$$

To evaluate $\mathbf{m} \times \hat{r} = |m||\hat{r}|sin\theta\hat{\phi} = |m|sin\theta\hat{\phi}$

So there is only a $\hat{\phi}$ component of the vector potential. Next, we evaluate $\mathbf{B} = \nabla \times \mathbf{A}$:

$$\nabla \times \mathbf{A} = \hat{r}\frac{1}{rsin\theta}\left(\frac{\partial}{\partial\theta}(sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right) + \hat{\theta}\frac{1}{rsin\theta}\left(\frac{\partial A_r}{\partial\phi} - sin\theta\frac{\partial}{\partial r}(rA_\phi)\right) + \hat{\phi}\frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right) \tag{34}$$

$$\mathbf{A} = \frac{\mu_0}{4\pi}\frac{msin\theta}{r^2}\hat{\phi} \tag{35}$$

$$\nabla \times \mathbf{A} = \hat{r}\frac{1}{rsin\theta}\left(\frac{\partial}{\partial\theta}(\frac{\mu_0}{4\pi}m\frac{sin^2\theta}{r^2})\right) + \hat{\theta}\frac{1}{rsin\theta}\left(-sin\theta\frac{\partial}{\partial r}(\frac{\mu_0}{4\pi}m\frac{sin\theta}{r})\right) \tag{36}$$

$$= \hat{r}\frac{1}{rsin\theta}(\frac{\mu_0}{4\pi}m\frac{2sin\theta}{r^2})cos\theta + \hat{\theta}\left(-\frac{1}{r}\right)\frac{\mu_0}{4\pi}msin\theta\left(\frac{-1}{r^2}\right) \tag{37}$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 m}{4\pi r^3}(2cos\theta\,\hat{r} + sin\theta\,\hat{\theta}) \tag{38}$$

The magnetic field looks like that of a bar magnet.

2.3 Scalar potential for the magnetic field

The vector potential representation of $\mathbf{B}$ is completely general and holds everywhere. We also have

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{39}$$

Where $\mathbf{J} = 0$, there are no current sources, so $\nabla \times \mathbf{B} = 0$. Since $\nabla \times \nabla\phi = 0$ for any scalar ϕ we can write

$$\mathbf{B} = -\nabla\phi_m \tag{40}$$

The choice of sign is arbitrary, but generally (-) is used. Let's return to our small loop.

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\mathbf{m} \cdot \nabla\nabla\left(\frac{1}{|\mathbf{r} - \mathbf{r_c}|}\right) \tag{41}$$

We want to write this as $\mathbf{B} = -\nabla\phi$. We'll use the vector identity

$$\nabla(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a} + \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a}) \tag{42}$$

Let $\mathbf{a} = \mathbf{m}$, $\mathbf{b} = \frac{\mu_0}{4\pi}\nabla\left(\frac{1}{|\mathbf{r}-\mathbf{r_c}|}\right)$. So, $\mathbf{B} = \mathbf{a} \cdot \nabla\mathbf{b}$

$$\begin{aligned}
(\mathbf{a} \cdot \nabla)\mathbf{b} &= \nabla(\mathbf{a} \cdot \mathbf{b}) - (\mathbf{b} \cdot \nabla)\mathbf{a} - \mathbf{a} \times (\nabla \times \mathbf{b}) - \mathbf{b} \times (\nabla \times \mathbf{a}) &(43)\\
&= \nabla\left(\frac{\mu_0}{4\pi}\mathbf{m} \cdot \nabla\left(\frac{1}{|\mathbf{r} - \mathbf{r_c}|}\right)\right) - 0 - \mathbf{m} \times \left(\frac{\mu_0}{4\pi}\nabla \times \nabla\left(\frac{1}{|\mathbf{r} - \mathbf{r_c}|}\right)\right) - 0 &(44)
\end{aligned}$$

But $\nabla \times \nabla\left(\frac{1}{|\mathbf{r}-\mathbf{r_c}|}\right) = 0$. So,

$$\mathbf{B} = \nabla\left(\frac{\mu_0}{4\pi}\mathbf{m} \cdot \nabla\left(\frac{1}{|\mathbf{r} - \mathbf{r_c}|}\right)\right) \tag{45}$$

$$\phi_M(\mathbf{r}) = -\frac{\mu_0}{4\pi}\mathbf{m} \cdot \nabla\left(\frac{1}{|\mathbf{r} - \mathbf{r_c}|}\right) \tag{46}$$

For a loop centered at the origin, $\phi_m(\mathbf{r}) = \frac{\mu_0}{4\pi}\mathbf{m} \cdot \frac{\hat{r}}{r^2}$. (Remember that $\nabla(\frac{1}{\mathbf{r}}) = -\frac{\hat{r}}{r^2}$.)

Note that the magnetic potential due to a current loop has the same form as the electric potential (ϕ_E) due to a dipole charge:

$$\mathbf{E} = -\nabla\phi_E \tag{47}$$
$$\phi_E(\mathbf{r}) = -\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{r}}{r^2} \tag{48}$$

where $\mathbf{p}$ is the electric dipole moment. This correspondance between static electric and magnetic fields has led many people to work with magnetic charges, though none exist.

2.3.1 Derivation of the scalar potential using magnetic monopoles

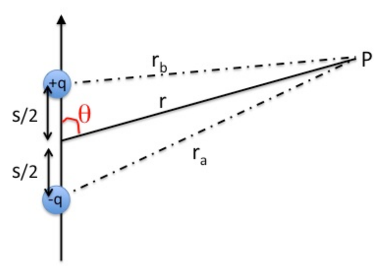
We can introduce a magnetic mass or monopole (a mathematical construct only). The magnetic potential due to this monopole is

$$\phi(P) = +\frac{\mu_0 q}{4\pi}\frac{1}{r} \tag{49}$$

q (units of Weber) is often referred to as the pole strength. Now consider a dipole:

$$\phi(P) = \phi_1(P) + \phi_2(P) \tag{50}$$
$$\phi(P) = +\frac{\mu_0 q}{4\pi}\left(\frac{1}{r_b} - \frac{1}{r_a}\right) \tag{51}$$

where ϕ_1 and ϕ_2 are the potentials due to q and -q, respectively. *See sketch below*



Proceed exactly as in electrostatics.

$$r_a^2 = r^2 + \left(\frac{s}{2}\right)^2 + rscos\theta \tag{52}$$
$$\frac{1}{r_a} = \frac{1}{(r^2 + rscos\theta + \left(\frac{s}{2}\right)^2)^{\frac{1}{2}}} = \frac{1}{r}\frac{1}{(1 + \frac{s}{r}cos\theta + \left(\frac{s}{2r}\right)^2)^{\frac{1}{2}}} \tag{53}$$
$$\frac{1}{r_a} \simeq \frac{1}{r}\left(1 - \frac{s}{2r}cos\theta\right) \tag{54}$$

Similarly, $\frac{1}{r_b} \simeq \frac{1}{r}\left(1 + \frac{s}{2r}cos\theta\right)$. Put together,

$$\phi(P) = +\frac{\mu_0 q}{4\pi}\left(\frac{1}{r_b} - \frac{1}{r_a}\right) = +\frac{\mu_0 q}{4\pi r}\left(1 + \frac{s}{2rcos\theta} - 1 + \frac{s}{2rcos\theta}\right) \tag{55}$$

$$\phi(P) = \frac{\mu_0 q}{4\pi r^2} scos\theta \tag{56}$$

$m = qs$ is the dipole moment, so

$$\phi(P) = \frac{\mu_0}{4\pi}\frac{mcos\theta}{r^2} = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\cdot\hat{r}}{r^2} = -\frac{\mu_0}{4\pi}\mathbf{m}\cdot\nabla(\frac{1}{r}) \tag{57}$$

The magnetic field due to a dipole can be found by taking the gradient.

$$\mathbf{B} = -\nabla\phi \tag{58}$$

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{m}{r^3}(3(\hat{m}\cdot\hat{r})\hat{r} - \hat{m}) \tag{59}$$

The magnetic field is symmetric about the dipole axis, and the potential can be written as

$$\phi(P) = \frac{\mu_0}{4\pi}\frac{mcos\theta}{r^2} \tag{60}$$

$$\mathbf{B} = -\nabla\phi \tag{61}$$

$$\mathbf{B} = -\left(\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta} + \frac{\hat{\phi}}{rsin\theta}\frac{\partial}{\partial\phi}\right)\phi \tag{62}$$

$$\mathbf{B} = \frac{\mu_0 m}{4\pi r^3}(2cos\theta\,\hat{r} + sin\theta\,\hat{\theta}) \tag{63}$$

Why did we bother with all of this? In considering gravity fields we realized that we can build up the gravity field due to an arbitrary density distribution as the sum of the contributions from small point masses. The same is true in magnetics – we can build up the solution for the magnetic induction, $\mathbf{B}$, due to an arbitrary distribution of magnetization in terms of the sum of contributions from elemental dipoles, each carrying a magnetic moment m.