Magnetics, Part II (see also Blakely, Chapter 5)

1 Magnetization

Define the *magnetization* $\mathbf{M}$ as the magnetic moment per unit volume. So, if we have a series of dipoles in a volume V, the total dipole moment will be $= \sum_i m_i$ and

$$\mathbf{M} = \frac{1}{V}\sum_i m_i \tag{1}$$

Units: $\mathbf{M} = \frac{dipole\,moment}{volume} = \frac{(Amp)(m^2)}{m^3} = \frac{Amp}{m}$.

Consider an elementary volume dv with magnetization $\mathbf{M}$. The elementary dipole moment, $\mathbf{m}$, is given by $\mathbf{m} = \mathbf{M}dv$. Recall our expression for the magnetic potential:

$$\phi_m(P) = -\frac{\mu_0}{4\pi}\mathbf{m}\cdot\nabla_P(\frac{1}{r}) \tag{2}$$

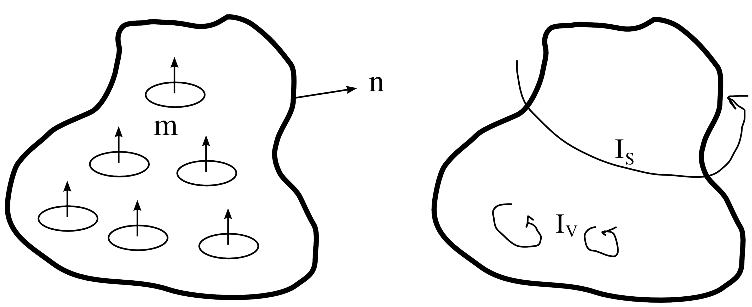
Setting $\mathbf{m} = \mathbf{M}dv$ and integrating over the volume, we get

$$\phi_m(P) = \frac{\mu_0}{4\pi}\int_V \mathbf{M}(Q)\cdot\nabla_Q(\frac{1}{r})dv \tag{3}$$

where $\nabla_P(\frac{1}{r}) = -\nabla_Q(\frac{1}{r})$.

1.1 Alternative representations for magnetic materials

1.1.1 Currents



Elementary current will tend to cancel on the inside. If the cancellation is complete, then there will only be a surface current (Units: Am^{-1}). If not complete, then there will be a volumetric current density (Units: Am^{-2}).

$$\mathbf{I_S} = \mathbf{M} \times \hat{n} \tag{4}$$
$$\mathbf{I_V} = \nabla \times \mathbf{M} \tag{5}$$

So, the field due to a magnetic body can be represented as the sum of fields due to surface and volume current. In particular, this is useful for working with the vector potential $\mathbf{A}$:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{r})}{|\mathbf{r} - \mathbf{r}'|} dv' = \frac{\mu_0}{4\pi} \int_S \frac{\mathbf{I_S}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dS + \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{I_V}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dv \tag{6}$$

1.1.2 Distribution of magnetic charges

$$\phi_m(P) = \frac{\mu_0}{4\pi} \int_V \mathbf{M}(Q) \cdot \nabla_Q (\frac{1}{r}) dv \tag{7}$$

Use the identity:

$$\nabla \cdot (\phi \mathbf{A}) = \nabla \phi \cdot \mathbf{A} + \phi \nabla \cdot \mathbf{A} \tag{8}$$

Substitute $\mathbf{A} = \mathbf{M}$ and $\phi = \frac{1}{r}$:

$$\mathbf{M} \cdot \nabla (\frac{1}{r}) = -\frac{1}{r} \nabla \cdot \mathbf{M} + \nabla \cdot (\frac{\mathbf{M}}{r}) \tag{9}$$

So,

$$\phi_m(P) = \frac{\mu_0}{4\pi} \left[\int_V \nabla \cdot \left(\frac{\mathbf{M}(Q)}{r} \right) dv - \int_V \frac{\nabla \cdot \mathbf{M}(Q)}{r} dv \right] \tag{10}$$

By the Divergence Theorem,

$$\int_V \nabla \cdot \left(\frac{\mathbf{M}(Q)}{r} \right) dv = \int_S \left(\frac{\mathbf{M}(Q)}{r} \right) \cdot \hat{n} \, dS \tag{11}$$

So,

$$\phi_m(P) = \frac{\mu_0}{4\pi}\int_S \frac{\mathbf{M}(Q)\cdot\hat{n}}{r}dS - \frac{\mu_0}{4\pi}\int_V \frac{\nabla\cdot\mathbf{M}(Q)}{r}dv \tag{12}$$

The quantities $\nabla\cdot\mathbf{M}$ are $\mathbf{M}\cdot\hat{n}$ are referred to as the *volumetric magnetostatic charge* and *surface charge density* respectively. This formulation is very helpful in computing the magnetic field arising from uniform bodies, since we only need to compute the effects of the magnetic charges at the ends of the body.

2 Magnetic susceptibility

The magnetization $\mathbf{M}$ is composed of *induced* and *remanent* contributions. The induced part depends upon the strength of an inducing field:

$$\mathbf{M_i} = \kappa\mathbf{H} \tag{13}$$

where κ is the constant of proportionality, known as the *susceptibility*. This linear relationship is generally valid for the materials that we deal with in geophysics, but it is not universal. This also yields some insight into the relationship between $\mathbf{B}$ and $\mathbf{H}$. Maxwell's equations for $\mathbf{B}$ in a general medium (again neglecting displacement currents and electric polarization) are:

$$\begin{aligned} \nabla\times\mathbf{B} &= \mu_0\left(\mathbf{J_{total}}\right) && (14)\\ \nabla\times\mathbf{B} &= \mu_0\left(\mathbf{J}+\nabla\times\mathbf{M}\right) && (15)\\ \nabla\cdot\mathbf{B} &= 0 && (16) \end{aligned}$$

where $\mathbf{J_{total}}$ is the total current, and $\mathbf{J}$ is the free current. We can rewrite equation (15) as:

$$\nabla\times(\mathbf{B}-\mu_{\mathbf{0}}\mathbf{M}) = \mu_0\mathbf{J} \tag{17}$$

So:

$$\begin{aligned} \mathbf{B}-\mu_0\mathbf{M} &= \mu_0\mathbf{H} && (18)\\ \mathbf{B} &= \mu_0(\mathbf{H}+\mathbf{M}) && (19)\\ \mathbf{B} &= \mu_0(\mathbf{H}+\kappa\mathbf{H}) && (20)\\ \mathbf{B} &= \mu_0(1+\kappa)\mathbf{H} && (21)\\ \mathbf{B} &= \mu\mathbf{H} && (22) \end{aligned}$$

So, $\mu = \mu_0(1 + \kappa)$. Remember that $\mathbf{H}$ is the magnetic field intensity and $\mathbf{B}$ is the magnetic flux density. The relationship between $\mathbf{M}$ and $\mathbf{H}$ is complicated and depends upon what is happening at the atomic level. There are four main types of induced magnetism:

Type	Sign	Amplitude
diamagnetic	$\kappa < 0$	small
paramagnetic	$\kappa > 0$	small
ferrimagnetic	$\kappa > 0$	moderate
ferromagnetic	$\kappa > 0$	large

For most rocks, the susceptibility is controlled by the amount of ferrimagnetic minerals: magnetite (Fe_3O_4), ilmenite ($FeTiO_3$), and pyrrhotite (FeS). κ is also controlled by grain size.

The final magnetization is controlled by induced $\mathbf{M_i}$ and remanent $\mathbf{M_r}$ contributions. $\mathbf{M} = \mathbf{M_i} + \mathbf{M_r}$. These can have different directions depending on when and how the remanent magnetization was acquired. This can create difficulties in isolating either the remanent (planetary, global scale applications) or the induced (local exploration applications) components.