Global Gravity: J_2 and Moments of Inertia

1 Moments of inertia

1.1 Motivation and Measurement

Planetary moments of inertia provide important constraints on the internal density distribution of the planetary body. Estimates of moments of inertia require measurement of the J_2 contribution to the gravity field. Usually this is done by observing the rotational responses (precession) of a spacecraft or satellite orbiting a planet, or by measuring the shape (and hence the flattening) of the planet very accurately.

1.2 Spherically symmetrical planet

We consider the moment of inertia of a spherically symmetrical planet, *i.e.*, density is just a function of radius. Spherical symmetry means that the moment of inertia about any axis going through the center of the planet will be the same. For simplicity, we choose the rotation axis to compute the moment of inertia. *Note: For a general non-spherically symmetric density distribution we calculate the three principal moments of inertia of the planet. These are the moments of inertia about the x, y, and z axes. These are called A, B and C respectively. For a spherically symmetric planet it is clear that $A = B = C$. Here we compute C.*

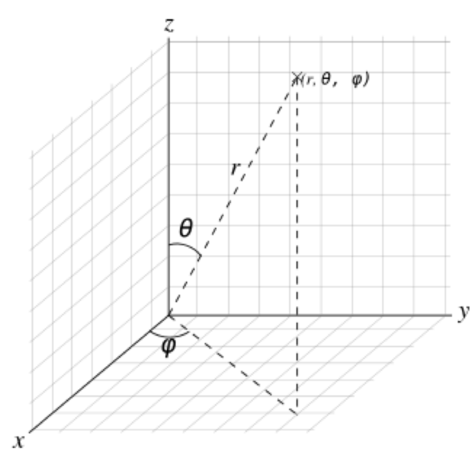

Figure 1:

Recall

$$\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta \end{aligned}$$

where ϕ is longitude and θ is colatitude. An element of volume is given by

$$\begin{aligned} dV \quad &= dxdydz && \text{in a cartesian coordinate system} \\ &= r^2 \sin\theta dr d\theta d\phi && \text{in a spherical coordinate system} \end{aligned}$$

Back to the planet: The mass element centered at point, P is a perpendicular distance s from the z axis. It has a moment of inertia about the z axis given by $s^2 dm$. The total moment of inertia of the planet about the z axis is the integral of all such elements over the entire mass distribution. So

$$C = \int_M s^2 \, dm \tag{1}$$

We have

$$dm = \rho(r) \, dV$$

and also

$$\begin{aligned} s^2 = x^2 + y^2 &= r^2 \sin^2\theta (\cos^2\phi + \sin^2\phi) \\ &= r^2 \sin^2\theta \end{aligned}$$

So now we can write

$$C = \int_V r^2 \sin^2\theta \, \rho(r) \, dV \tag{2}$$

where the integral is over the volume of the planet. This integral can be written as one over r (from 0 to a), over θ (from 0 to π) and over ϕ (from 0 to 2π) by substituting in the expression for dV:

$$C = \int_0^{2\pi} \int_0^{\pi} \int_0^{a} \rho(r) r^4 \sin^3\theta \, dr d\theta d\phi \tag{3}$$

We can do these integrals separately. You will note that nothing in the integrand ($\rho(r)\, r^4 \sin^3\theta$) depends upon ϕ, the longitude, so we can write

$$
\begin{aligned}
C &= \int_0^\pi \int_0^a \rho(r) r^4 \sin^3\theta \, dr d\theta \int_0^{2\pi} d\phi \\
&= 2\pi \int_0^\pi \int_0^a \rho(r) r^4 \sin^3\theta \, dr d\theta
\end{aligned}
\tag{4}
$$

This can also be written as

$$
C = 2\pi \int_0^a \rho(r) r^4 \, dr \int_0^\pi \sin^3\theta \, d\theta \tag{5}
$$

The last integral is simple to do if we let $f = \cos\theta$; we can write $df/d\theta = -\sin\theta$ and f varies from 1 to -1 as θ varies from 0 to π, *i.e.*, $df = -\sin\theta d\theta$ so

$$
\int_0^\pi \sin^3\theta \, d\theta = -\int_1^{-1} \sin^2\theta \, df = -\int_1^{-1} (1 - \cos^2\theta) \, df \tag{6}
$$

$$
= -\int_1^{-1} (1 - f^2) \, df \tag{7}
$$

$$
= -\left[f - \frac{f^3}{3} \right]_1^{-1} = \frac{4}{3} \tag{8}
$$

Finally, we have

$$
C = \frac{8\pi}{3} \int_0^a \rho(r) r^4 \, dr \tag{9}
$$

Thus the moment of inertia tells us something about the density distribution within the planet. However, unless we have other *a priori* information about the planet (such as the mean density, the planet's size, ..) there are many different density distributions that can give us the measured value of C. This is because C is proportional to the integral of the product $\rho(r)$ and r^4.

Suppose that a planet has a *uniform density*, $\overline{\rho}$, then

$$C = \frac{8\pi}{3}\overline{\rho}\int_0^a r^4\,dr = \frac{8\pi}{15}\overline{\rho}a^5 \tag{10}$$

The quantity that is used is usually not C but C/Ma^2 which is a dimensionless number (M is the mass of the planet), known as the "moment of inertia factor". Now

$$M = \frac{4}{3}\pi a^3\overline{\rho} \tag{11}$$

So for a uniform density planet

$$\frac{C}{Ma^2} = \frac{8\pi}{15}\overline{\rho}a^5\frac{3}{4\pi a^3\overline{\rho}}\frac{1}{a^2} = \frac{2}{5} = 0.4 \tag{12}$$

If the density is greater near the center of the planet we find that $C/Ma^2 < 0.4$. For example C/Ma^2 for the Earth is .3308.

1.3 *Principal moments of inertia for general planetary bodies*

On a rotationally distorted Earth we have $A = B$ (because of the symmetry about the rotation axis) but now $A \neq C$. For a more general planet $A \neq B \neq C$. Mars is a good example of such a planet – it is ellipsoidal due to rotation, but the mass distribution is not even symmetrical about the rotation axis due to the large excess mass associated with the Tharsis rise.

How might we measure C? A straightforward, but tedious, calculation allows J_2 to be cast in terms of the principal moments of inertia of the body (the details are on page 199–200 of Turcotte and Schubert - note how to calculate "A"). We find that

$$J_2 = \frac{C - A}{Ma^2} \tag{13}$$

This equation allows us to estimate $C - A$ because J_2 is measured. For the Earth the gravitational attractions of the Sun and the Moon, acting on the equatorial bulge, cause a precession of the axis of rotation. From the rate of precession we can find the "dynamical ellipticity", H, which is given by

$$H = \frac{C - A}{C} \tag{14}$$

H is estimated to be 1/305.51 and by combining equations we have, for the Earth:

$$C = \frac{J_2}{H} M a^2 = .3308 M a^2$$