# The Geoid, Free Air Gravity & Bouger Gravity

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Remember that an "equipotential surface" is a surface on which $U$ is a constant. The vector $\vec{g}$ will be normal to any such surface, thus $U$ defines the local horizontal. For Earth, the sea surface is an equipotential surface (apart from the effects of wind and currents), and the equipotential surface which defines sea level is called the *geoid*.

## 1 Review

We have defined reference states for the gravity field and the potential based on the approximate shape and rotation of a planet. We noted that we can set $U_R(pole) = U_R(equator))$ to calculate the actual surface, $r_0$ that corresponds to the potential for a rotating ellipsoid ($U_R$) that we have seen previously:

$$U_R(r_0, \lambda) = -\frac{GM}{r_0} + \frac{GMa^2 J_2}{2r_0^3}(3\sin^2\lambda - 1) - \frac{1}{2}\Omega^2 r_0^2 \cos^2\lambda \tag{1}$$

where

$$r_0 = a\left[1 + \frac{(2f - f^2)}{(1-f)^2}\sin^2\lambda\right]^{-1/2} \tag{2}$$

(See Turcotte and Schubert, pg. 202-203 for derivation). $f$ is the flattening and as we saw it is given by $f = \frac{a-c}{a}$ or alternatively by $f = \frac{3J_2}{2} + \frac{a^3\Omega^2}{2GM}$. Note that for most planets, $r_0$ is almost a spherical surface, and so $f << 1$; for Earth $f \approx 1/300$. As $f << 1$, $r_0$ is often approximated as

$$r_0 \approx a\left(1 - f sin^2\lambda\right) \tag{3}$$

This follows from equation (2) if one expands (2) in powers of $f$ and keeps only the linear terms.

## 2 The Geoid

Equation (2) is the expression for the reference ellipsoid surface, sometimes referred to as the reference spheroid or (confusingly!) even the reference geoid. Observed departures of the actual geoid from the reference geoid are called "geoid anomalies." The anomaly in the potential of the gravity field, measured on the reference geoid, $\Delta U$ can be directly related to the *geoid anomaly*, $\Delta N$ (i.e. the physical difference in height of the actual geoid and the reference geoid). $\Delta N$ is also called the *geoid height* and is measured in meters. The *potential anomaly* is defined by

$$\Delta U = U_{mo} - U_o \tag{4}$$

where $U_{mo}$ is the measured potential at the location of the reference geoid, and $U_o$ is the reference value of the potential (defined for an ellipsoid by equation (1) above). The potential $U_o$ is measured on the actual, or measured, geoid, as shown in the Figure below. It can be seen that $U_o$, $U_{mo}$ and $\Delta N$ are related by

$$U_0 = U(r_0 + \Delta N) \approx U_{mo} + \frac{\partial U_R}{\partial r_{r_0}} \Delta N \tag{5}$$

where we have used a Taylor Series expansion since $\Delta N << r_0$.

The radial derivative of the potential is the acceleration due to gravity so we can define a reference gravity field, $g_R$, (the acceleration due to gravity on the reference geoid) such that

$$g_R = \left( \frac{\partial U_R}{\partial r} \right)_{r=r_o} \tag{6}$$

Hence

$$\Delta U = U_{mo} - U_o = -g_R \Delta N, \tag{7}$$

and the geoid anomaly $\Delta N$ is

$$\Delta N = -\frac{\Delta U}{g_R} \tag{8}$$

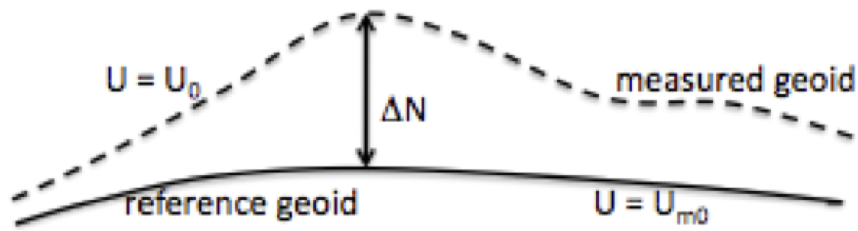


Figure 1:

Note that a local mass excess produces an outwarp of gravity equipotentials and therefore a positive $\Delta N$, and a negative $\Delta U$. (Remember that $U_o$ is negative from equation 1, so a negative $\Delta U$ means that $U_{mo}$ is larger in magnitude but negative in sign as expected for a positive mass anomaly.)

# 3 Free Air Correction

If we make a gravity measurement at various elevations, we are at different distances from the center of mass of the Earth and so $g$ changes. We can correct for this effect. Suppose we make a gravity measurement at a particular position on the surface of the Earth. To calculate the gravity anomaly we would first subtract out the reference gravity field, $g_R$, (which has a latitude dependence) evaluated on the reference geoid which is at $r_0$ (equation (2)). We can calculate the effect of elevation in the following way. To zeroth order we have

$$g = \frac{GM}{r^2} \tag{9}$$

so the effect of changing distance from the center of a spherical earth is:

$$\frac{dg}{dr} = -2\frac{GM}{r^3} = -2\frac{g}{r} \tag{10}$$

For small changes in $r$ (*c.f.*, the radius of the planet) the anomaly caused by elevation is approximated by:

$$\Delta g_h = -\frac{2h}{r_0} g_R \tag{11}$$

where we have used $\frac{dg}{dr} \approx \frac{\Delta g_h}{h}$, and $g = g_R$ at $r = r_o$. Gravity is reduced by an amount $|\Delta g_h|$, for positive elevation (i.e., above sea level). If we add $\Delta g_h$ to our measurement we have corrected for the effect of elevation. The correction, $\Delta g_h$, is called the *free-air correction* and a gravity anomaly defined by

$$\Delta g_{fa} = g_{obs} - g_R + \Delta g_h \tag{12}$$

is called a *free-air gravity anomaly.*

Free air corrections on Earth are typically very small, *e.g.,* Mount Everest has a height of 8848 m, and so the free air correction at the top of Mt. Everest is $\approx 0.2\%$ of gravity on the reference ellipsoid. However, even though they are small, it is important to take account of them since we are interested in deviations from the reference gravity field.

## 4 The effect of topography on gravity – Bouguer anomaly

Mass anomalies associated with topography give rise to surface gravity anomalies. The effect of general topography must be treated numerically (sometimes called a terrain correction) by performing the integral equation we saw earlier in the course:

$$\vec{g} = G \int \frac{\rho(r')(\vec{r} - \vec{r'})}{|\vec{r} - \vec{r'}|^3} dv' \tag{13}$$

If the topography is slowly varying (*i.e.*, it has a shallow slope) we can derive an approximate expression for the gravitational effect due to topography.

Consider a cylindrical disc of material of radius $R$ and thickness $h$. An observer is located a distance $b$ above the upper surface of the disc. The density in the disc is assumed to be a function of depth, so $\rho = \rho(z)$

Because of the symmetry of the disc we know that the net gravitational attraction will be vertically downwards ($g_z$). We consider the contribution $\delta g_z$ to $g_z$ due to a cylindrical ring of

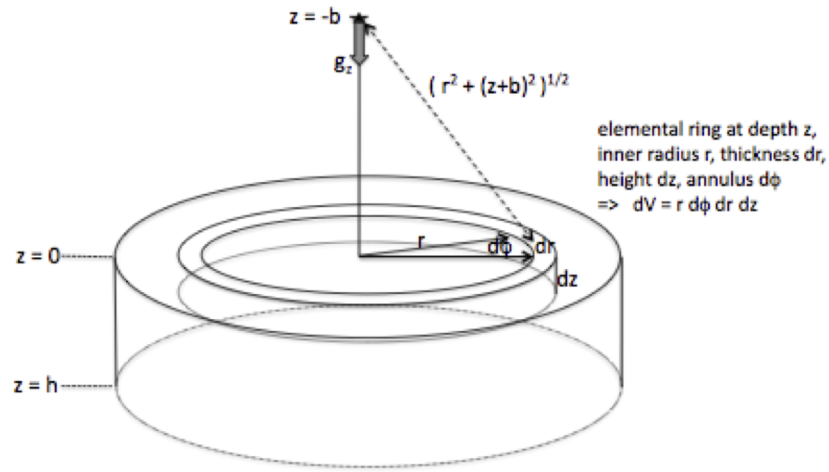


Figure 2:

radius $r$ and thickness $dr$ inside the disc at a depth $z$. The volume of the ring is $2\pi r dr dz$ so the mass of the ring is $2\pi r dr dz \rho(z)$. The observer is a distance $b+z$ from the center of the ring and so is a distance of $(r^2 + (z+b)^2)^{1/2}$ from a segment of the ring. To get the mathematical expression for $\delta g_z$ we divide the ring into segments.

The volume of a segment, $\delta V$, is $r d\phi dr dz$. The volume of the whole ring is therefore $\int_0^{2\pi} r dr dz d\phi = 2\pi r dr dz$. The gravitational attraction of the segment at the observer a distance $(r^2+(z+b)^2)^{1/2}$ away is

$$\frac{G}{r^2+(z+b)^2}\rho(z)\delta V = \frac{G}{r^2+(z+b)^2} r\, d\phi dr dz \rho(z)$$

This points directly towards the segment from the observer. The contribution to the vertical component of $g$ (*i.e.*, $\delta g_z$) is found by multiplying by $\cos\theta$, where

$$\cos\theta = \frac{z+b}{[r^2+(z+b)^2]^{1/2}}$$

The total contribution of the ring, $\delta g_z$, is found by summing up all the contributions of the segments in the ring which is equivalent to integration over $\phi$ from 0 to $2\pi$. Putting all these bits together gives

$$\delta g_z = \int_0^{2\pi} \frac{G}{r^2 + (z+b)^2} r dr dz \rho(z) \cos\theta d\phi$$
$$= \frac{2\pi G r \rho(z) dr dz}{r^2 + (z+b)^2} \frac{z+b}{[r^2 + (z+b)^2]^{1/2}}$$

(Note that the angle $\theta$ is the same for all segments of the ring so $\theta$ doesn't depend upon $\phi$.)

To get the total gravitational attraction of the disc we sum up the contributions of all the rings – this is accomplished by integrating the expression for $\delta g_z$ over $r$ (from $r = 0$ to $R$) and over $z$ (from $z = 0$ to $h$):

$$g_z = 2\pi G \int_0^h \int_0^R \frac{\rho(z)\ r\ (z+b)}{[r^2 + (z+b)^2]^{3/2}} dr dz$$
$$= 2\pi G \int_0^h \rho(z)(z+b) \left\{ \int_0^R \frac{r}{[r^2 + (z+b)^2]^{3/2}} dr \right\} dz$$

Integrating with respect to $r$ gives

$$g_z = 2\pi G \int_0^h \rho(z) \left[ 1 - \frac{z+b}{[R^2 + (z+b)^2]^{1/2}} \right] dz$$

In the limit that the disc is very broad (*i.e.*, $R \to \infty$) we have a slab of topography of thickness $h$. This is a good approximation if the topography has a gentle (low) gradient. In this case,

$$g_z \simeq 2\pi G \int_0^h \rho(z)\, dz \qquad (14)$$

If topography has a height $h$ and a (constant) density $\rho_c$ its contribution to $g$ is

$$\Delta g = 2\pi G \rho_c h \tag{15}$$

This is called the *Bouguer correction.* It corrects gravity measurements for the effect of mass excess or deficit due to topography.

We can now make a further correction to our *free air anomaly* ($\Delta g_{fa}$) for the effect of topography using the Bouguer correction.

$$\Delta g_B = \Delta g_{fa} - 2\pi G \rho_c h \tag{16}$$

$\Delta g_B$ is called a *Bouguer gravity anomaly.*