Elasticity and Flexure

The introductory material here may overlap with what is covered in the 3rd year geophysics geodynamics / continuum mechanics class. Background material is provided below in case you have not yet taken that course. The math is mostly taken from Chapter 3 of Turcotte & Schubert's textbook "Geodynamics". We are interested in flexure as it is a way of supporting topography that may be evident in both topography and gravity fields.

- Section 1
- Sections 2-4 are background material on linear elasticity and abbreviated derivation of the flexure equation
- Section 5: The basic 1-D set-up of the flexure equation for planetary problems
- Section 6: how to estimate elastic thickness in practise
- Sections 7 & 8: the gravity signature associated with flexure

1. Introduction

Elastic materials deform when a force is applied and they return to their original state (shape) when the force is removed. In the simplest cases, the *strain (deformation) is linearly proportional to the applied stress* **and** the material has elastic properties that are independent of direction (*isotropic*). (This is Hooke's Law that you've seen in physics classes).

At stress levels that are high compared with the rock strength, deviations from elastic behavior are observed and the failure occurs. This can be either *brittle* failure or *ductile* failure. The mode of failure depends on whether the strength of the rock is governed by the confining pressure or by temperature, and whether the high stress levels are imposed over long or short time frames. Brittle failure results in faulting. Plastic or ductile failure or flow is an irreversible, continuous deformation without fracture. One example is long-term loading and unloading of Earth's lithosphere by ice caps – the viscosity of the mantle allows fluid-like behavior on long time scales and causes rebound of topography following deglaciation (*e.g.*, Hudson Bay). Also over long time scales, lower crustal rocks may deform plastically and so steep slopes *e.g.*, at the edge of mountain ranges can "flow away". We shall see later that the properties of crustal and mantle rocks can result in different regions of brittle and ductile failure as a function of depth.

For terrestrial planets in general, rock strength depends on temperature, pressure, and composition. At depths of a few hundred kilometers and less, temperature and composition (especially volatiles) are the main effects. Thus the terrestrial planets can be expected to have a cold, strong (rigid) outer layer, or lithosphere. Below this is mantle material that is weaker because of the strong decrease in strength with increasing temperature. Differences in lithospheric strength among the terrestrial planets reflect their different thermal environments and differing volatile contents of crustal and mantle rocks. A fundamental postulate of plate tectonics is that the Earth's lithosphere has sufficient strength to transmit stresses elastically over large horizontal distances, and so most terrestrial deformation occurs at plate boundaries (*i.e.*, is concentrated at weak zones). This is true to first order.

Investigating the long-term elastic behavior of the lithosphere

When subjected to loading (either surface loading through excess topography – *e.g.*, volcanoes; bending moments applied to the edges of plates; or loading from below the surface), a lithosphere with elastic strength will *bend (flex)*. The characteristics of the flexural response to loading, in particular the wavelength and amplitude of the flexure, provide information on the elastic properties of the lithosphere. Specifically, if we know something about the composition and rheology of the materials comprising the lithosphere we can use flexural reponses to loading to investigate the *lithospheric thickness*. Studies of the flexural response of the lithosphere to applied stresses also provide information on the stress state in the lithosphere.

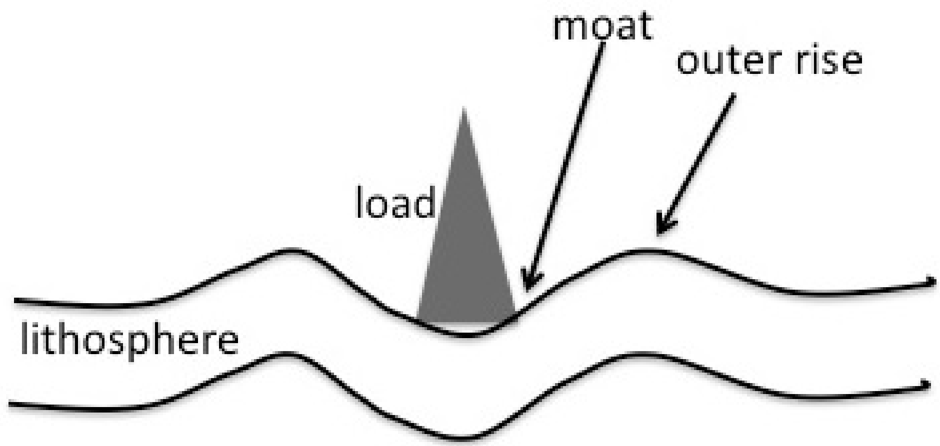

Figure 1: Characteristic topography associated with plate bending

Planetary examples of flexure

- Earth: Hawaiian islands, island chains and seamounts
- Earth: subduction zones
- Earth: folding in mountain belts
- Mars and Venus: loading by volcanoes
- Earth, Venus, Mars (and possibly Europa): rifting

Applications of flexural studies

- estimating effective elastic lithospheric thickness, h_e
- estimating stresses in the lithosphere: σ_{xx}
- from h_e we can get thermal gradient $\frac{dT}{dz}$
- from $\frac{dT}{dz}$ we get surface heat flow q_s
- variations in h_e temporally and spatially over a planet

Limitations of flexural modeling

(1) A purely elastic flexure model also assumes that the lithosphere can sustain infinite stresses; however, laboratory studies indicate that the strength of the upper lithosphere is limited by pressure-dependent brittle

failure and the strength of the lower lithosphere is limited by temperature and strain-rate dependent ductile flow. Thus estimates from an elastic plate model of the *effective elastic thickness (EET)* of the lithosphere need to be corrected for more realistic rheological behavior.

(2) While the lithosphere may behave elastically over short to intermediate time scales, over longer timescales the lower viscosity of the mantle compared with that of the lithosphere may cause topographic signatures associated with flexure to relax. The resulting topography may still look like flexure but is in fact the combination of an initial elastic response of the lithosphere followed by viscous relaxation.

(3) Our formulations will be *thin plate* flexure - *i.e.*, the wavelength of the flexure $>>$ plate thickness. This approximation is often not really valid; people use it anyway for simplicity and because the results aren't too different from the thick plate solution. Also our formulations assume the plate thickness is constant (*e.g.*, doesn't vary spatially over our features of interest).

2. Linear Elasticity

A linear, isotropic solid is one in which stresses are linearly proportional to strains and mechanical properties have no preferred orientation. Principal axes of stress and strain coincide in such a material.

The generalized form of Hooke's Law in this case is

$$\sigma_{ij} = \lambda \epsilon_{ii} \delta_{ij} + 2\mu \epsilon_{ij} \tag{1}$$

σ_{ij} is the stress applied in the j'th direction to a plane with normal in the i'th direction. ϵ denotes strain, δ_{ij} is the Kroenecker Delta function such that $\delta_{ij} = 1$, when $i = j$ and $\delta_{ij} = 0$ otherwise. λ and μ are collectively called the *Lamé* constants. μ is also called the shear modulus or the modulus of rigidity (note that G is used for μ in Turcotte & Schubert's text).

In our system then we can write:

$$\sigma_1 = (\lambda + 2\mu)\,\epsilon_1 + \lambda\epsilon_2 + \lambda\epsilon_3 \tag{2}$$

$$\sigma_2 = \lambda\epsilon_1 + (\lambda + 2\mu)\,\epsilon_2 + \lambda\epsilon_3 \tag{3}$$

$$\sigma_3 = \lambda\epsilon_1 + \lambda\epsilon_2 + (\lambda + 2\mu)\,\epsilon_3 \tag{4}$$

where we write σ_1 for σ_{11} and ϵ_1 for ϵ_{11} etc. In engineering and geology we often use E, Young's modulus and ν, Poisson's ratio. These are related to λ and μ as follows

$$E = \frac{\mu\,(3\lambda + 2\mu)}{(\lambda + \mu)} \tag{5}$$

$$\nu = \frac{\lambda}{2\,(\lambda + \mu)} \tag{6}$$

and so equations (2) – (4) become

$$\epsilon_1 = \frac{1}{E}\sigma_1 - \frac{\nu}{E}\sigma_2 - \frac{\nu}{E}\sigma_3 \tag{7}$$

$$\epsilon_2 = \frac{-\nu}{E}\sigma_1 + \frac{1}{E}\sigma_2 - \frac{\nu}{E}\sigma_3 \tag{8}$$

$$\epsilon_3 = \frac{-\nu}{E}\sigma_1 - \frac{\nu}{E}\sigma_2 - \frac{1}{E}\sigma_3 \tag{9}$$

A principal stress component σ produces a strain $\frac{\sigma}{E}$ in the same direction and strains $\frac{-\nu\sigma}{E}$ in the mutually orthogonal directions.

The elastic behavior of a linear isotropic material can thus be specificed via E (elastic modulus reflecting the relative change in length of a material) and ν (the ratio of lateral strain to longitudinal strain in a body stressed longitudinally within its elastic limit). For geological materials Young's modulus varies from about 10 to 100 GPa and Poisson's ratio varies between about 0.1 and 0.4. Poisson's ratio for a fluid is 0.5 (see T&S Appendix 2, Section E for typical values of E, μ, and ν for various rocks).

3. Various special cases

Uniaxial stress – only one of the principal stresses, say σ_1 is non-zero. Typically many laboratory experiments on rocks are unaxial stress experiments.

Uniaxial strain – only one non-zero component of principal strain, say ϵ_1. Changes in stress due to sedimentation and erosion may use this approximation – the horizontal components of strain are assumed to be negligible c.f. the vertical component.

Plane stress – only one zero component of principal stress. Applications include studies of thermal stresses (due to temperature changes) in the lithosphere, where the region under stress is confined.

Plane strain – only one zero component of principal strain.

Isotropic Stress – all principal stresses are equal. Principal strains are also equal. Use to define bulk modulus and its reciprocal, the compressibility (see T&S, page 112). Application: determination of density with depth in the Earth.

4. Bending or flexure of plates

Outline of Derivation of Flexure Equation for 2-D Cartesian Geometry

This is only a brief outline – see Turcotte and Schubert for the whole thing.

Plate, thickness h_e, width L, infinitely long in z-direction pinned at ends and subjected to a line force V_a (N m^{-1}) at its center.

Essentially one takes an element of the plate and balances the (1) applied forces ($q(x)$) with the net shear force ($V(x)$); and (2) the net bending moment ($M(x)$) with the torques due to horizontal forces ($P(x)$) and the shear force. From the moment and force balances one obtains:

$$\frac{d^2M}{dx^2} = -q + P\frac{d^2w}{dx^2} \tag{10}$$

This equation can be converted into a differential equation for the deflection, $w(x)$ if the bending moment M, can be related to w. It turns out that M is inversely proportional to the *local radius of curvature of the plate, R* and that R^{-1} is equal to $-\frac{d^2w}{dx^2}$ (see T&S for details).

......some intermediate steps lead to the (idealized, because elasticity assumed) relationship.....

$$M = \frac{-Eh_e^3}{12\left(1-\nu^2\right)}\frac{d^2w}{dx^2} \tag{11}$$

where E is Young's modulus, ν is Poisson's ratio, h is the elastic thickness of the plate. We often refer to the *flexural rigidity* of the plate, D, where

$$D = \frac{Eh_e^3}{12\left(1-\nu^2\right)} \tag{12}$$

Thus the bending moment is the flexural rigidity of the plate divided by its curvature, R:

$$M = -D\frac{d^2w}{dx^2} = \frac{D}{R} \tag{13}$$

Substituting equation (13) into (10) we obtain a general equation for the deflection of an elastic plate:

$$D\frac{d^4w}{dx^4} = q(x) - P\frac{d^2w}{dx^2} \tag{14}$$

Note that we have derived the flexure equation assuming a 2-dimensional (x,z) cartesian geometry. You should be aware that more general formulations for deflection of a spherical shell due to an imposed load are often required in planetary problems, in particular for small planets.

5. BOTTOM LINE STUFF TO KNOW: Application to planetary lithospheres

Review – lithosphere: strong outer layer; mantle beneath lithosphere is significantly weaker (lower viscosity) due to higher temperature and possibly the presence of volatiles. On Earth this region of the mantle is sometimes referred to as the asthenosphere.

$$D\frac{d^4w}{dx^4} = q(x) - P\frac{d^2w}{dx^2} \tag{15}$$

• Often, $P(x)$, the horizontal force is zero in plate bending problems. In other words the lithosphere is loaded only from above or below by $q(x)$. This will be the case in our examples.

• The equation that relates D to plate thickness is:

$$D = \frac{Eh_e^3}{12\left(1-\nu^2\right)} \tag{16}$$

Always use SI units when using this equation.....

• If you push down with an applied load $q_a(x)$ and displace material with density ρ_{below} with material ρ_{above}, the displacement will result in a restoring force (assuming $\rho_{below} > \rho_{above}$)

$$(\rho_{below} - \rho_{above})gw(x) = \Delta\rho g w(x)$$

The effective load is then

$$q(x) = q_a(x) - \Delta\rho g w(x)$$

So

$$D\frac{d^4w(x)}{dx^4} + P\frac{d^2w(x)}{dx^2} + \Delta\rho g w(x) = q_a(x) \tag{17}$$

1) For the oceanic case

$$\Delta\rho = \rho_{mantle} - \rho_{water}$$

2) For the continental case

$$\Delta\rho = \rho_{mantle} - \rho_{crust}$$

3) For loading by an isolated volcano

$$\Delta\rho = \rho_{mantle} - \rho_{air} \approx \rho_{mantle}$$

6. Elastic Plate Loading: General Solution in 1-D

The elastic response of the lithosphere to a load produces a characteristic flexural profile in topography. The load is surrounded by a depression or *moat*. Outboard of this depression is a *flexural bulge* or upwarp. The wavelength of the flexural signature and the amplitude of the flexural bulge are related to the elastic plate thickness and the magnitude of the applied load.

One can clearly solve equation (17) for a range of different loading scenarios. Scenarios covered in Turcotte and Schubert include deflection of the lithosphere due to (1) a load on a continuous or (2) a broken plate and (3) a bending moment applied to a plate.

We consider the behavior of the plate under a line load V_0 applied at $x = 0$. $q_a(x) = 0$ everywhere except $x = 0$. The horizontal load $P = 0$. Equation (16) reduces to

$$D\frac{d^4w(x)}{dx^4} + \Delta\rho g w(x) = 0 \tag{18}$$

i.e.,

$$\frac{d^4w(x)}{dx^4} + \frac{\cancel{1}\,4}{\alpha^4}w(x) = 0 \tag{19}$$

which has a general solution

$$w = e^{x/\alpha}\left(c_1 cos\frac{x}{\alpha} + c_2 sin\frac{x}{\alpha}\right) + e^{-x/\alpha}\left(c_3 cos\frac{x}{\alpha} + c_4 sin\frac{x}{\alpha}\right) \qquad (20).$$

α is known as the flexural parameter, or flexural wavelength and is given by

$$\alpha = \left(\frac{4D}{\Delta\rho g}\right)^{(1/4)} \tag{21}$$

Since there is symmetry about $x = 0$, we need only determine $w(x)$ for $x \geq 0$. $w(x) \to 0$ as $x \to \infty$. Thus c_1 and c_2 must be zero. We get

$$w = e^{-x/\alpha}\left(c_3 cos\frac{x}{\alpha} + c_4 sin\frac{x}{\alpha}\right) \tag{22}$$

In references like Turcotte & Schubert a lot of energy is expended in solving for the coefficients c_3 and c_4 in equation (22), given specific loading situations. The solutions are all different from each other and some are rather cumbersome. In practice they turn out to not be very useful anyway because one does not necessarily know, *a priori* what *e.g.*, the magnitude of the load V_0 or the applied bending moment M_0 is.

In practice we are usually interested in estimating the lithospheric (plate) thickness because we can see a flexural signature in a topographic and/or gravity profile. This can be done in at least 3 ways:

(1) SEE SKETCH in class. For a 1-D problem in cartesian geometry, the distance to the first zero-crossing (x_0) and to the flexural bulge (x_b) are related by

$$x_b - x_0 = \pi\alpha/4 \tag{23}$$

irrespective of the details of how the plate is loaded. So one could measure this distance, get α, hence the flexural rigidity D from (21) and the plate thickness h_e from (16). This can be difficult to do accurately in practice because of noise in the topographic signal, and the small magnitude of the flexural bulge.

(2) Fitting equation (22) to a real profile. The algorithm is as follows. Allow the real topography to be modeled by flexure + linear slope + mean value, so

$$w = c_1 + c_2 x + e^{-x/\alpha}\left(c_3 cos\frac{x}{\alpha} + c_4 sin\frac{x}{\alpha}\right) \tag{24}$$

The algorithm would be as follows: (a) pick a value for h_e, calculate α (eqns 16 and 21), (b) do a least squares fit of eqn (24) to your topographic profile, (c) keep the root-mean-square misfit, along with c_1, c_2, c_3, c_4 for this value of h_e, (d) step h_e through a range of values, repeating steps (a) – (c). (e) Plot the rms misfit versus h_e. (f) Keep the model (ie the value of h_e and corresponding estimates of c_1, c_2, c_3, c_4 for the minimum in your misfit curve.

(3) If you have gravity AND topography you can use how they vary in the spectral domain to estimate h_e – we'll do this......

7. Limiting Cases of Flexure: Isostacy and Uncompensated Topography

Here we look at flexural loading in the limit of long-wavelength and short-wavelength topography. We will also investigate how the compensation of topography by flexure of the lithosphere is manifested in the free air and Bouguer gravity anomalies.

7:1 Periodic loading

Assume elevation h, of topography (crustal density ρ_c) is periodic (sinusoidal), with wavelength λ. Positive / negative h correspond to ridges / valleys respectively.

$$h = h_0 sin\left(2\pi\frac{x}{\lambda}\right) \tag{25}$$

The load on the lithosphere is

$$q_a(x) = \rho_c g h = \rho_c g h_0 sin\left(2\pi\frac{x}{\lambda}\right)$$

The crustal material displaces mantle material below and so $\Delta\rho = \rho_m - \rho_c$ (see below eqn 17). Thus

$$D\frac{d^4w(x)}{dx^4} + (\rho_m - \rho_c)\, g w(x) = \rho_c g h_0 sin\left(2\pi\frac{x}{\lambda}\right) \tag{26}$$

We assume a solution of the form

$$w = w_0 sin\left(2\pi\frac{x}{\lambda}\right) \tag{27}$$

substituting (27) into (26) we get the maximum deflection of the lithosphere:

$$w_o = \frac{h_0}{\frac{\rho_m}{\rho_c} - 1 + \frac{D}{\rho_c g}\left(\frac{2\pi}{\lambda}\right)^4} \tag{28}$$

We can use this equation to investigate the response of the lithosphere to long and short-wavelength loads. If the wavelength of the topography is sufficiently short, *i.e.*

$$\lambda << 2\pi\left(\frac{D}{\rho_c g}\right)^{1/4} \tag{29}$$

then $w_0 << h_0$ and thus short-wavelength topography causes virtually no deformation of the lithosphere. The lithosphere is effectively infinitely rigid for such loads.

If the wavelength of the topography is sufficiently long *i.e.*

$$\lambda >> 2\pi \left(\frac{D}{\rho_c g} \right)^{1/4} \tag{30}$$

then

$$w_{0\infty} = \frac{\rho_c h_0}{(\rho_m - \rho_c)} \tag{31}$$

This is the result we obtained for isostatic compensation of topography. Thus for long-wavelength topography the lithosphere effectively has zero rigidity and the topography is fully compensated.

Sometimes in geophysics we wish to have a measure of the degree to which topography is compensated. We define the degree of compensation as the ratio of the deflection of the lithosphere to the maximum or hydrostatic deflection

$$C = \frac{w_0}{w_{0\infty}} \tag{32}$$

From (28), (31), and (32) we get

$$C = \frac{(\rho_m - \rho_c)}{\rho_m - \rho_c + \frac{D}{g} \left(\frac{2\pi}{\lambda} \right)^4} \qquad (33).$$

8. Compensation (support of topography) due to lithospheric flexure

Here we will look at the free air gravity anomaly associated with flexure of the lithosphere. We continue to use the simplified case of the previous section. There are two contributions to the surface free-air gravity – the contribution due to the mass anomaly associated with the topography (Δg_t) and the contribution due to the mass anomaly associated with the deflection of the Moho (Δg_m) .

The Bouguer correction gives us

$$\Delta g_t = 2\pi G \rho_c h = 2\pi G \rho_c h_0 sin \left(2\pi \frac{x}{\lambda} \right) \tag{34}$$

The contribution due to the Moho reflects the deflection of the Moho which is equal to the deflection of the lithosphere (since the Moho is a compositional boundary embedded in the lithosphere). The anomalous surface mass density associated with the deflection of the Moho is:

$$\sigma = -(\rho_m - \rho_c) \, w \tag{35}$$

Substituting (26) and (27) into (34) we get:

$$\sigma = -(\rho_m - \rho_c) \frac{h_0}{\frac{\rho_m}{\rho_c} - 1 + \frac{D}{\rho_c g} \left(\frac{2\pi}{\lambda} \right)^4} sin \left(2\pi \frac{x}{\lambda} \right) \tag{36}$$

The gravity anomaly close to the Moho is also given by the Bouguer formula:

$$\Delta g = 2\pi G\sigma$$

However the Moho is buried at a mean depth b_m. Gravity anomalies due to the density contrasts decay with increasing distance away from the density contrast according to

$$\Delta g(z) = 2\pi G\sigma e^{\frac{-2\pi z}{\lambda}} \tag{37}$$

so at a height equal to one wavelength above the density constrast, the gravity anomaly will be $e^{-2\pi} = 0.002$ of its value just above the density contrast. Thus for our case, at an observation level of $z = -b_m$:

$$\Delta g_m = -2\pi\left(\rho_m - \rho_c\right)G\left[\frac{e^{\frac{-2\pi b_m}{\lambda}}}{\frac{\rho_m}{\rho_c} - 1 + \frac{D}{\rho_c g}\left(\frac{2\pi}{\lambda}\right)^4}\right]h_0 sin\left(2\pi\frac{x}{\lambda}\right) \tag{38}$$

The surface free air anomaly is found by summing equations (34) and (38):

$$\Delta g_{fa} = \Delta g_t + \Delta g_m$$

$$\Delta g_{fa} = 2\pi\rho_c G\left[1 - \frac{e^{\frac{-2\pi b_m}{\lambda}}}{1 + \frac{D}{(\rho_m-\rho_c)g}\left(\frac{2\pi}{\lambda}\right)^4}\right]h_0 sin\left(2\pi\frac{x}{\lambda}\right) \tag{39}$$

Remember that the surface Bouguer anomaly is given by $\Delta g_{fa} - 2\pi\rho_c Gh$ so

$$\Delta g_B = -2\pi\rho_c G\left[\frac{e^{\frac{-2\pi b_m}{\lambda}}}{1 + \frac{D}{(\rho_m-\rho_c)g}\left(\frac{2\pi}{\lambda}\right)^4}\right]h_0 sin\left(2\pi\frac{x}{\lambda}\right) \tag{40}$$

As in the previous section we can look at the limits of short and long-wavelength topography. When λ is small:

$$\Delta g_{FA} = 2\pi G\rho_c h_0 sin\left(2\pi\frac{x}{\lambda}\right) \tag{41}$$

$$\Delta g_B = 0 \tag{42}$$

The mass of the local topography is uncompensated and the Bouguer gravity anomaly is thus zero.

For long wavelength topography

$$\Delta g_{FA} = 0 \tag{43}$$

$$\Delta g_B = -2\pi G \rho_c h_0 sin\left(2\pi \frac{x}{\lambda}\right) \tag{44}$$

Thus for long-wavelength topography, we have isostatic compensation.

We can see that the general equation for flexurally compensated topography reduces to the completely compensated or uncompensated cases in the limits of long and short wavelength topography respectively.

From equations (40) and (41) we can see that the ratio of either the Free Air gravity anomaly or the Bouguer gravity anomaly to topography (known as the *admittance*) as a function of wavelength depends on the flexural rigidity of the lithosphere. We often use such admittance curves to investigate the elastic plate thickness (from D) beneath *e.g.*, large volcanoes or other loads on planetary surfaces.