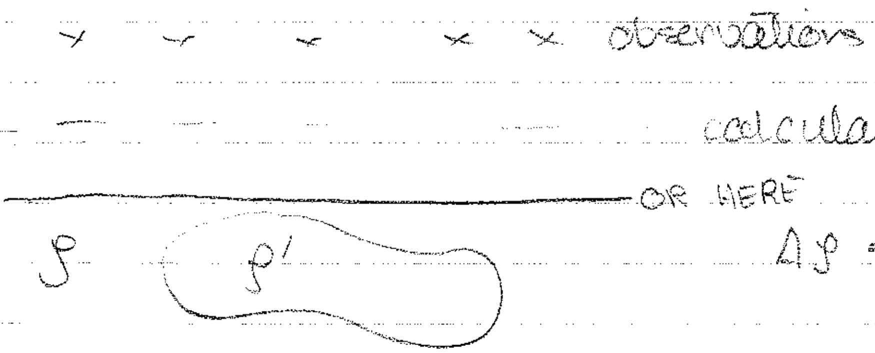
Solutions to Laplace's Eqⁿ

- We will look @ cartesian + spherical coordinates
- Approach: sepⁿ of variables
- Point: MATH Green's 3rd identity $\Longrightarrow$ analytical continuation

  If U, $\partial U/\partial n$ known over surface bounding a source-free region, R, then U can be calculated anywhere in the region.

  ie there is a unique solⁿ for U.

  but HOW?



$\Delta\rho = \rho' - \rho = 0$ elsewhere

$$\nabla^2 U = 0 \Rightarrow \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0$$

Choose a solⁿ of the form

$$U(u, v, x, y, z) = X(u, x)\, Y(v, y)\, Z(u, v, z)$$

this is only 1 possible solⁿ & is specified by choice of u, v

separation constants

Calculate $\frac{1}{u} \nabla^2 u \qquad (=0)$

substitute $u = XYZ$ & divide by XYZ

$$= \frac{1}{XYZ} \frac{\partial^2 (XYZ)}{\partial x^2} + \cdots\cdots$$

$$= \frac{1}{X} \frac{\partial^2 X}{\partial x^2} + \frac{1}{Y} \frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z} \frac{\partial^2 Z}{\partial z^2}$$

$$\Rightarrow \underbrace{\frac{1}{X} \frac{\partial^2 X}{\partial x^2}}_{f(x)} = -\underbrace{\frac{1}{Y} \frac{\partial^2 Y}{\partial y^2}}_{f(y)} - \underbrace{\frac{1}{Z} \frac{\partial^2 Z}{\partial z^2}}_{f(z)}$$

$\Rightarrow$ must be a constant

set the constant $= -u^2$

$$\text{LHS} \rightarrow \boxed{\frac{\partial^2 X}{\partial x^2} + u^2 X = 0} \quad --- (1)$$

similarly by substituting

$$\underbrace{\frac{1}{Y} \frac{\partial^2 Y}{\partial y^2}}_{f(y)} = u^2 - \underbrace{\frac{1}{Z} \frac{\partial^2 Z}{\partial z^2}}_{f(z)} = \text{const.} = -v^2$$

$$\Rightarrow \boxed{\frac{\partial^2 Y}{\partial y^2} + v^2 Y = 0} \quad --- (2)$$

and now $\frac{1}{Z} \frac{\partial^2 Z}{\partial z^2} = u^2 + v^2$

$$\Rightarrow \boxed{\frac{\partial^2 Z}{\partial z^2} - (u^2 + v^2) Z = 0} \quad --- (3)$$

equations ① - ③ have solutions

$$X(u,x) = a_1(u)\, e^{iux} + b_1(u) e^{-iux} \qquad u \geq 0$$
$$= c_1(u)\, e^{iux} \qquad -\infty < u < \infty$$
$$Y(v,y) = c_2(v)\, e^{ivy} \qquad -\infty < v < \infty$$
$$Z(u,v,z) = c_3(u,v)\, e^{\pm (u^2+v^2)^{1/2} z}$$

so $\underline{a}$ solution to Laplace's eq^n is

$$u = X(u,x)\, Y(v,y)\, Z(u,v,z)$$

④ $$\boxed{u(x,y,z) = \left[A(u,v)\, e^{-(u^2+v^2)^{1/2} z} + B(u,v)\, e^{(u^2+v^2)^{1/2} z}\right] \underbrace{e^{iux}\, e^{ivy}}_{e^{i(ux+vy)}}}$$

THIS IS ONE SOLUTION, specified by the values of u and v.

Obtain the general solution by adding the contributions for all possible values of u and v.

TWO CASES

A// u, v take discrete values

$$\left.\begin{aligned} u &= 2\pi n / L \\ v &= 2\pi m / L' \end{aligned}\right\} \quad \begin{aligned} &n, m \text{ are integers} \\ &L, L' \text{ period} \end{aligned}$$

corresponds to periodic sol^n + the add^n is a summation

$$\sum_m \sum_n$$

B// u, v vary continuously

Contribution to the general sol^n from param values btwn u and $u + \delta u$, and v and $v + \delta v$ is expressed by weighting f^{n}s $F(u,v)$, $G(u,v)$

where $\quad \frac{1}{2\pi} F(u,v)\, \delta u\, \delta v = A(u,v)$

$\quad \frac{1}{2\pi} G(u,v)\, \delta u\, \delta v = B(u,v)$ $\qquad 2\pi$ convenient

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addition of all possible sol^{ns} is now an integral over u & v rather than a discrete summation

**

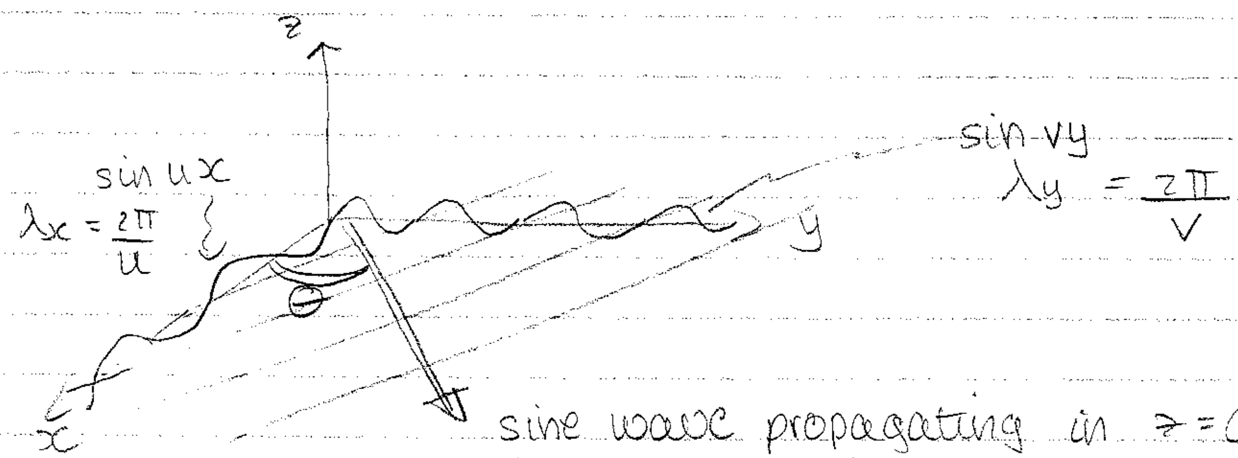
$$\boxed{U(x,y,z) = \frac{1}{2\pi}\int_{u=-\infty}^{\infty}\int_{v=-\infty}^{\infty} e^{i(ux+vy)}\left\{F(u,v)\,e^{-(u^2+v^2)^{1/2}z} + G(u,v)\,e^{(u^2+v^2)^{1/2}z}\right\} du\,dv}$$

PHYSICAL INTERPRETATION

when $z=0$

$$U(x,y,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\big(F(u,v)+G(u,v)\big)e^{i(ux+vy)}\,du\,dv$$

But we have seen this before : 2-D FT !



sine wave propagating in $z=0$ plane direction defined by u, v

$\tan\theta = \frac{v}{u}$

check $v=0 \Rightarrow \theta = 0 \Rightarrow$ only in $\hat{x}$ dir^n

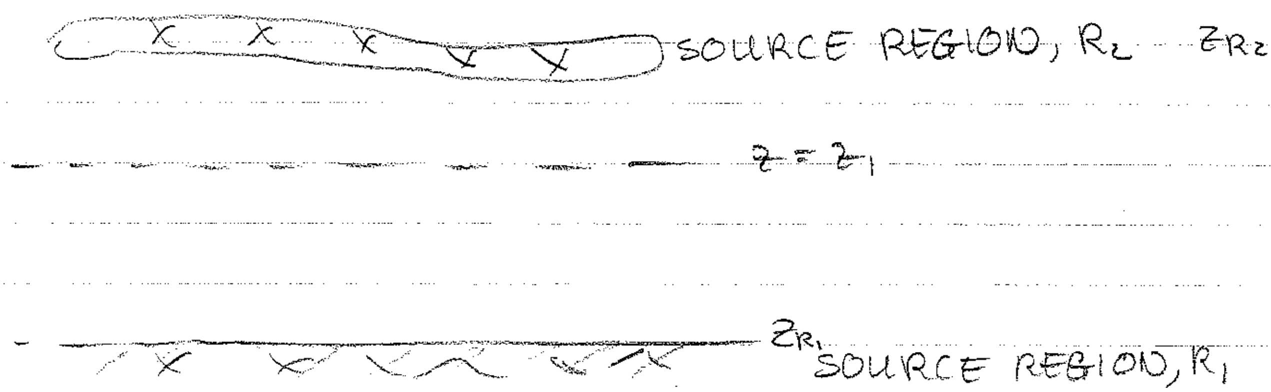
$u=0 \Rightarrow \theta = 90° \Rightarrow$ only in $\hat{y}$

$F(u,v)$ and $G(u,v)$ are the amplitude of the sinusoid on $z=0$

Consider $z \neq 0$

$F(u,v)$ term $\downarrow$ exponentially
$G(u,v)$ term $\uparrow$ exponentially
$\Big\}$ for $z > 0$

Consider obsv^ns on a plane $z = z_1$



$F(u,v)\, e^{-(u^2+v^2)^{1/2}(z_1 - z_{R1})}$ amplitude of sinusoidal var^{n}s $\left(e^{i(ux+vy)}\right)$ whose sources lie below z_1 (@ ~~R1~~ z_{R1})

$G(u,v)\, e^{(u^2+v^2)^{1/2}(z_1 - z_{R2})}$ - - - - - - sources above z_1 (@ z_{R2} here)

$\boxed{\text{This is the ESSENCE OF UPWARD / DOWNWARD CONTINUATION}}$

Partic in sph. geom.

(5) Φ_g $\qquad\qquad$ φ_B

$\tilde{g}$ (g_x, g_y, g_z) $\qquad\qquad$ $\bar{B}$

N $\qquad\qquad$ $\tilde{m}$

ρ, H, V_s, V_p $\qquad$ why?

because integrands orthog
so it's an expansion
in terms of basic fns

$$|k| = \sqrt{k_x^2 + k_y^2}$$

$$= \frac{1}{2\pi}\sqrt{u^2 + v^2}$$

$$\lambda = \frac{1}{|k|} = \frac{2\pi}{\sqrt{u^2+v^2}} = \frac{2\pi}{2\pi\sqrt{k_x^2+k_y^2}}$$

$$= \frac{1}{\sqrt{k_x^2 + k_y^2}}$$

FROM LAST TIME

$$\nabla^2 U = 0$$

orthogonal $4\pi^2 \delta(u_1+u_2)\delta(v_1+v_2)$

$$\Rightarrow U(x,y,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(ux+vy)}\left\{F(u,v)\,e^{-(u^2+v^2)^{1/2}z} + G(u,v)\,e^{(u^2+v^2)^{1/2}z}\right\} du\,dv \quad \text{--- (1)}$$

potential in (x,y,z) system

u, v spatial frequency

$u = 2\pi k_x$ $\quad k_x$ = wave# in x dir^n $= \frac{1}{\lambda_x}$ → wavelength

$v = 2\pi k_y$ $\quad k_y = 1/\lambda_y$

wave vol $\lambda = \frac{1}{|k|}$ $\quad \theta = \tan^{-1}\left(\frac{v}{u}\right)$

Important stuff to Remember

(1) spatial derivatives (and any linear combo of them) of U also satisfy Laplace's eq^n + can be represented by (1) — use for later + total field anomaly.

(2) If sources for U, $\vec{g}$, $\vec{B}$.... are below $z=0$ then $G(u,v) = 0$ (if not $U \to \infty$ as $z \to \infty$)
Conversely $F(u,v) = 0$ for sources above $z = 0$.

(3) Eqn (1) on the plane $z = 0$ (or on any const z. plane) looks like a 2-D FFT

(4) Surface harmonic (var^n of U on const z) ; solid harmonic

$\boxed{\text{eq on } z = 0}$

$$U(x,y,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(ux+vy)} F(u,v)\,du\,dv$$

sources $z < 0$

SO, now obvious how to calculate $F(u,v)$!

with BC

$$\text{Take } \mathcal{F}[U(x,y,0)] \rightarrow F(u,v) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U(x,y,0)\,e^{-i(ux+vy)}\,dx\,dy$$

could have used z_0 — — — — → extra term in here $e^{-(u^2+v^2)^{1/2} z_0}$

(5) see ←

Can calculate u at any z

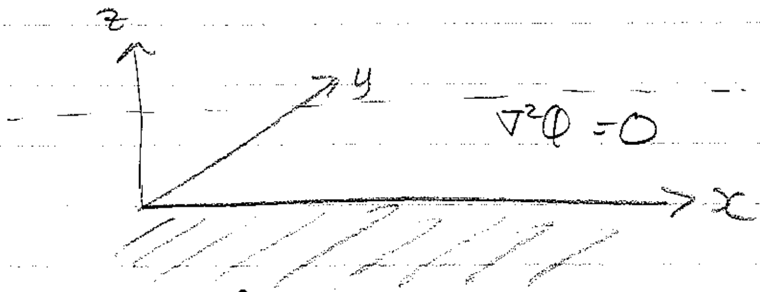
an alternative approach (why FTs are so slick ....)

consider satellite measuring 'Φ' (the 'disturbing' or 'anomalous' potential)

or something that gives us Φ

$$\underset{\text{disturbing pot}^{\text{l}}}{\Phi} = \underset{\text{total pot}^{\text{l}}}{U} - \underbrace{\underset{\text{reference pot}^{\text{l}}}{U_0}}$$

⟵ going to talk ~ how to calc. 'ref' pot's



$\nabla^2 \Phi = 0$ ‒ ‒ $z = z_{satellite} = z_s$

sources below $z = 0$

$\nabla^2 \Phi = -4\pi G \rho$

Assume we can use flat earth approxm^n

$$\nabla^2 \Phi = 0 \quad \Rightarrow \quad \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \quad \text{---} \; (A)$$

Need 6 B.C.s to solve (A)

$$\lim_{|x| \to \infty} \Phi = 0 \qquad \lim_{|y| \to \infty} \Phi = 0 \qquad \lim_{z \to \infty} \Phi = 0 \quad \Big\} \; \underline{5 \text{ BCs}}$$

The 6th one comes from $\Phi(x, y, z_0)$ or $\partial\Phi/\partial z \, (= -\Delta g)$ on z_0

Before we used sep^n of variables

BUT

we can use FTs (to get an algebraic eq^n in k_x, k_y)

$$F(\bar{k}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\bar{x})\, e^{-2\pi i(\bar{k},\bar{x})}\, d^2\bar{x}$$

$$f(\bar{x}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\bar{k})\, e^{2\pi i(\bar{k},\bar{x})}\, d^2\bar{k}$$

where $\bar{x} = (x, y)$ is pos^n vector
$\bar{k} = (1/\lambda_x, 1/\lambda_y)$ is wavenumber vector
and $\bar{k}.\bar{x} = k_x x + k_y y$

<u>Now F.T the D.E.</u>

Recall $\mathcal{F}[\partial\Phi/\partial x] = -i2\pi k_x \mathcal{F}[\Phi]$
$\mathcal{F}[\partial^2\Phi/\partial x^2] = -4\pi^2 k_x^2 \mathcal{F}[\Phi]$
and likewise for y

Write $\mathcal{F}[\Phi(x,y,z)] = \Phi(\bar{k}, z)$

So $\dfrac{\partial^2\Phi}{\partial x^2} + \dfrac{\partial^2\Phi}{\partial y^2} + \dfrac{\partial^2\Phi}{\partial z^2} = 0$

becomes $-4\pi^2(k_x^2 + k_y^2)\,\Phi(\bar{k}, z) + \dfrac{\partial^2\Phi(\bar{k}, z)}{\partial z^2} = 0$

General sol^n is

$$\Phi(\bar{k}, z) = A(\bar{k})\, e^{-2\pi|k|z} + B(\bar{k})\, e^{2\pi|k|z}$$

but $B(\bar{k}) \equiv 0$ since $\Phi(\bar{k}, z)$ vanishes @ ∞

$$\Phi(\bar{k}, z) = \Phi_0(\bar{k}, z_0) = A(\bar{k})\, e^{-2\pi|k|z_0}$$

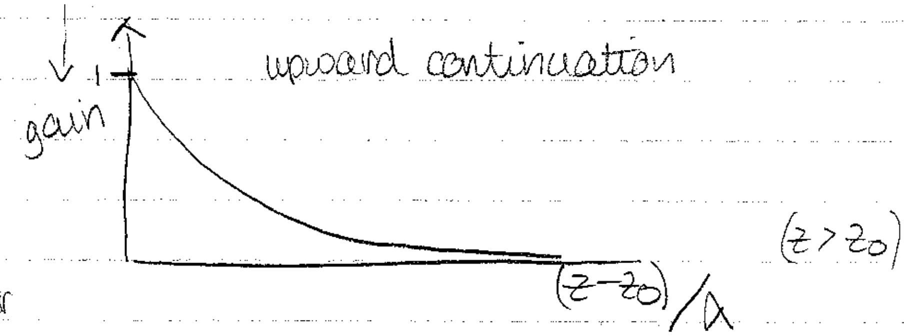
$|k| = (k_x^2 + k_y^2)^{1/2}$, with $k_x \to \dfrac{u}{2\pi}$, $k_y \to \dfrac{v}{2\pi}$

so $\Phi(\bar{k}, z) = \Phi_0(k, z_0)\, e^{-2\pi|k|(z-z_0)}$

pot' @ any $z \geq 0$ = pot' @ $z = z_0$ × upward (downward) continuation kernel (factor)

$z = z_0$ → usually measurement altitude

shorthand $\Phi(z) = \Phi_0\, e^{-2\pi\left(\frac{z-z_0}{\lambda}\right)}$



Note short λs ↓ faster than long λs

SO If you measure the gravity anomaly @ surface of Earth $\Delta g(\bar{x}, 0)$ then can calc @ any height z by

i) FT $\Delta g(\bar{x}, 0) \rightarrow \Delta g(\bar{k}, 0)$

ii) multiply by $e^{-2\pi|k|z} \rightarrow \Delta g(\bar{k}, z) = \Delta g(\bar{k}, 0)\, e^{-2\pi|k|z}$

iii) Inverse FT result

↓

$\Delta g(x, y, z)$

The upward continuation kernel is WHY it is hard or impossible to recover small-scale features from measurements @ altitude.

From before

$$u(x,y,z) = \frac{1}{2\pi}\iint e^{i(ux+vy)} F(u,v)\, e^{-(u^2+v^2)^{1/2} z}\, du\, dv \quad \text{----} ①$$

$$u(x,y,0) = \frac{1}{2\pi}\iint e^{i(ux+vy)} F(u,v)\, du\, dv \quad \text{----} ②$$

$$② \Rightarrow \quad F(u,v) = \mathfrak{F}[u(x,y,0)] = u(\overset{(u,v)}{\tilde{k}}, 0)$$

$$① \Rightarrow \quad F(u,v)\, e^{-(u^2+v^2)^{1/2} z} = \mathfrak{F}[u(x,y,z)] = u(\overset{(u,v)}{\tilde{k}}, z)$$

$$\Rightarrow u(\overset{(u,v)}{\tilde{k}}, z) = u(\overset{(u,v)}{\tilde{k}}, 0)\, e^{-(u^2+v^2)^{1/2} z}$$

Here $\quad \tilde{\Phi}(\tilde{k}, z) = \tilde{\Phi}(\tilde{k}, z_0)\, e^{-2\pi|k|(z-z_0)}$