Spherical Harmonic Descriptions of Global Fields

1 Motivation

Here we will look at functions defined on a sphere. In Cartesian coordinates we saw that the solution

$$V(x, y, z) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} A(\vec{k})\ e^{-2\pi|k|(z-z_0)}\ e^{2\pi i\vec{k}\cdot\vec{x}}\ d^2\vec{k} \tag{1}$$

is not only the solution to Laplace's equation (physics and math) due to internal sources, but is a general way in which other spatially varying quantities can be represented (*i.e.*, ones that don't necessarily satisfy Laplace's equation; math, but not physics). This because for any constant z, the solution is just a Fourier transform, so it is like representing the variable in terms of a sum (integral in the limit) of linearly independent functions – the $e^{im\phi}$ (in linear algebra these are referred to as basis functions).

The spherical geometry equivalent of equation (1) / Fourier transforms / Fourier series that we used for cartesian problems are spherical harmonic expansions. Before looking at a more general solution, let's first consider a field V, that varies only as a function of latitude, λ, (or equivalently colatitude, $\theta = 90° - \lambda$). On a given spherical surface we can write

$$V(\theta) = \sum_{l=0}^{\infty} A_l P_l(cos\theta) \tag{2}$$

Note the similar form of this equation to a Fourier series, except that the sines and cosines have been replaced by the functions P_l. These are known as Legendre polynomials, and they are functions of $\cos\theta$. We'll see what they look like later.

2 More general spherical harmonic description

We'll now show that equation (2) is in fact a solution to Laplace's equation in spherical coordinates. In fact, it's a solution in a special case: one in which there is no variation of the field with longitude. We'll look at the most general form of the solution.

$$\nabla^2\Psi = 0 \tag{3}$$

In spherical polar coordinates

$$\nabla^2\Psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2} \tag{4}$$

(r, θ, ϕ) are radius, colatitude and longitude. As in the cartesian case we use the technique of separation of variables to come up with 3 ODE's in r, θ, ϕ respectively:

$$\Psi(r,\theta,\phi) = R(r)\ \Theta(\theta)\ \Phi(\phi) \tag{5}$$

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} + uR = 0 \tag{6}$$

$$\frac{d^2\Phi}{d\phi^2} + v^2\Phi = 0 \tag{7}$$

$$\sin^2\theta\frac{d^2\Theta}{d\theta^2} + \sin\theta\cos\theta\frac{d\Theta}{d\theta} - \left(u\sin^2\theta + v^2\right)\Theta = 0 \tag{8}$$

where u and v are separation constants. The first two equations are not too bad to solve, the third involves several tricks and some algebra. We'll bypass all of this and just quote the solutions:

2.1 First: the radial solution

$$R_l(r) = a_l r^l + b_l r^{-(l+1)} \tag{9}$$

The radial part of the solution (analogous to the z component in cartesian geometry) has solutions that require that the separation constant in equation (6), $u = -l(l+1)$, where l is an integer. There are solutions that grow as r^l as r increases, and ones that decrease as $r^{-(l+1)}$. Recall that this is similar to our exponentially growing or decaying solutions in z ($e^{\pm 2\pi|k|z}$). l is analogous to the wavenumber ($|k|$): it is known as the *spherical harmonic degree.* We will see that unlike wavenumber it is dimensionless.

Note: (1) often n is used instead of l.

2.2 Second: the longitudinal solution

$$\Phi(\phi) = c_1 e^{im\phi} + c_2 e^{-im\phi} \tag{10}$$

where m>0. If we let $-\infty < m < \infty$ we can write this more compactly:

$$\Phi(\phi) = c_3 e^{im\phi} \tag{11}$$

Our separation constant, v, from equation (7), has to equal an integer m. The solutions in longitude, ϕ, are sines and cosines. m is a bit like the cartesian wavenumber in the x or y direction – k_x or k_y – except it is dimensionless. It is usually referred to as the *spherical harmonic order*.

2.3 Third: the latitudinal (colatitude) solution

$$\Theta_{l,m}(\theta) = c_4 P_l^m(\cos\theta) \tag{12}$$

$P_l^m(\cos\theta)$ are known as "Associated Legendre Functions" of (spherical harmonic) degree, l and order, m. WHAT are these? See Section 4

3 Solution to Laplace's Eqn in Spherical Coordinates

The general solution to Laplace's equation in spherical coordinates is

$$\Psi = \sum_{l=0}^{\infty} \sum_{m=0}^{l} \left[r^{-(l+1)} \left(C_l^m \cos(m\phi) + S_l^m \sin(m\phi) \right) + r^l \left(G_l^m \cos(m\phi) + H_l^m \sin(m\phi) \right) \right] P_l^m(\cos\theta) \tag{13}$$

where we have written the solution in longitude as sines and cosines explicitly. You'll see it written either as $e^{\pm im\phi}$ or as sines and cosines.

While this equation looks pretty nasty, if you compare it with the eqn that we came up with for a cartesian geometry, you'll notice that terms like $\sin(m\phi)P_l^m(\cos\theta)$ are basically "*shape functions*" in latitude and longitude, just like the term $e^{i(ux+vy)}$ (or $e^{2\pi i\vec{k}\cdot\vec{x}}$) is a shape function in x and y.

Just like in the cartesian case, the solution is much simpler if sources are above, or below a spherical shell of radius, R_a (usually the radius of the planet). Internal sources can only have solutions with radial terms that go as $r^{-(l+1)}$, since terms in $r^l \to \infty$ as $r \to \infty$. *i.e.*, the coefficients $a_l = 0$ in equation (9).

So the more general solution to Laplace's equation for the gravitational potential on or above the surface of a planet, radius R_a, is $(\Psi = \Psi(r, \theta, \phi))$:

$$\Psi = R_a \sum_{l=0}^{\infty} \sum_{m=0}^{l} \left(\frac{R_a}{r}\right)^{l+1} \left[C_{lm} cos(m\phi) + S_{lm} sin(m\phi)\right] P_l^m(cos\theta) \tag{14}$$

The use of the factor "R_a" outside the summation is convenient and a typical convention in gravity and magnetic spherical harmonic expansions. It sets up the dimensions of the problem such that when we write the spherical harmonic expansions for gravity and magnetics (the gradient of the potential), the spherical coefficients have units of gravity (e.g., $mgal$) or magnetic field (e.g., μT) respectively. R_a is often written just as a.

4 Associated Legendre Functions, Legendre Polynomials

We can get some insight into P_l^m by considering solutions to Laplace's equation that are symmetric about a rotation axis: e.g., no variation in longitude. When this is the case, all the m terms except $m = 0$ in equation (14) vanish and so $P_l^m = 0, m \neq 0$. We write

$$\Psi = R_a \sum_{l=0}^{\infty} \left(\frac{R_a}{r}\right)^{l+1} A_l P_l(\cos\theta) \tag{15}$$

where $P_l = P_l^0$ and $A_l = C_l^0$. This is just like equation (2) but with the r dependence included.

P_l are the Legendre polynomials. They are polynomials in $\cos(\theta)$. The first few Legendre polynomials are given by:

$$P_0(\cos\theta) = 1 \tag{16}$$

$$P_1(\cos\theta) = \cos\theta \tag{17}$$

$$P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1) \tag{18}$$

$$P_3(\cos\theta) = \frac{\cos\theta}{2}(5\cos^2\theta - 3) \tag{19}$$

These functions are progressively wigglier (as l, the spherical harmonic degree, increases) functions of the colatitude θ. We can choose the coefficients, A_l, so that the sum can approximate very accurately any observed dependence of V on θ. *Even-l* terms are symmetric about the equator, *odd-l* terms are antisymmetric about the equator.

WHERE do the P_l come from? What are they physically?

Think of the gravitational potential at a point P, that has position $\vec{r}$ due to a point mass at $\vec{r}_1$.

$$V = \frac{-GM}{|\vec{r} - \vec{r}_1|} \tag{20}$$

But the distance between the mass, m and the observer at point P is

$$|\vec{r} - \vec{r}_1| = \left(r^2 + r_1^2 - 2rr_1\cos\gamma\right)^{1/2} \tag{21}$$

where γ is the angle between the vectors $\vec{r}$ and $\vec{r}_1$. Using the cosine rule ($a^2 = b^2+c^2-2bc\cos\gamma$):

$$\frac{1}{|\vec{r} - \vec{r}_1|} = \frac{1}{r}\left(1 - 2\frac{r_1}{r}\cos\gamma + \left(\frac{r_1}{r}\right)^2\right)^{-1/2} \tag{22}$$

The last part of this is a series expansion in r, r_1 and $\cos\gamma$, and in fact

$$\left(1 - 2x\mu + x^2\right)^{-1/2} = \sum_{l=0}^{\infty} x^l P_l(\mu) \tag{23}$$

In our case $\mu = \cos\gamma$ and $x = r_1/r$. So the Legendre polynomials result from the series expansion of $\frac{1}{|\vec{r}-\vec{r}_1|}$, and are a natural way to express the gravitational potential due to a point mass. Usually of course they are not the simplest way to calculate V; however you can now see how this formulation is useful for the more general case of representing the gravitational potential due to an arbitrarily complicated density distribution.

5 Back to the more general solution to Laplace's equation for internal sources

$$\Psi = R_a \sum_{l=0}^{\infty} \sum_{m=0}^{l} \left(\frac{R_a}{r} \right)^{l+1} \left[C_{lm} cos(m\phi) + S_{lm} sin(m\phi) \right] P_l^m (cos\theta) \tag{24}$$

So we have a solution that depends on longitude as well as on colatitude.

- The P_l^m are calculated using a "recurrence relation". You don't need to know how to do this in this class, and you probably will never need to write this piece of code as there are plenty out there. The main thing to check should you need to use this is what the maximum spherical harmonic degree and order the code will work up to. There are often numerical problems for l, m beyond about 200. (This is not an issue for us!).
- The terms $P_l^m e^{\pm im\phi}$ are known as the *surface harmonics*. This is just like the cartesian case where the $e^{i\vec{k}\cdot\vec{x}}$ are surface harmonics.
- The terms $P_l^m e^{\pm im\phi} r^{-(l+1)}$ or $P_l^m e^{\pm im\phi} r^l$ are known as *solid harmonics*.
- The P_l^m are orthogonal – there are several different ways of normalizing them and unfortunately different ones are used in gravity and magnetics.
- Because the P_l^m are orthogonal then *we can calculate the spherical harmonic coefficients*, with a procedure similar to the one we would use in cartesian geometry to calcuate Fourier coefficients: Measure the function at some radius, r_{meas}, multiply this by $P_l^m cos(m\phi)$, and integrate over the surface of the sphere to get C_l^m. Similarly for S_l^m. Later in the class we will talk more about estimation procedures for determining spherical harmonic coefficients.
- **Once you have the spherical harmonic coefficents, the expression for your function is just a series. You can evaluate your function at any (r, θ, ϕ).** This is the beauty of a spherical harmonic representation. All you need are a set of coefficients and a (generic) spherical harmonic expansion code and you can evaluate any field anywhere. Forexample, you can make global map, or say predict the gravity of magnetic field along the track of a future satellite, and compare it with data when they are measured.

- In practice we sum over a finite number of spherical harmonic degrees, so $0 \leq l \leq l_{max}$. l_{max} is usually determined by the shortest wavelengths that we can, or expect to, resolve in our data. A good approximation is that a process with a wavelength of λ on a sphere of radius R (*e.g.*, a gravity field described on the surface of a planet) has an equivalent spherical harmonic degree given by

$$l = \frac{2\pi R}{\lambda} \tag{25}$$

Notice that l is dimensionless. (The actual relationship that this comes from is called "Jean's rule" and strictly speaking is $(l + \frac{1}{2}) = \frac{2\pi R}{\lambda}$, but for unless you are looking at very small l the approximation above in equation (25) is fine.)

Example: the Magellan spacecraft orbiting Venus had a typical altitude of 300 km in its circular orbit. We might expect to resolve features in the gravity field at the surface on the order of the spacecraft altitude, perhaps a little less – say 200km. So

$$l_{max} = \frac{2\pi R_{Venus}}{\lambda_{min}} = \frac{2\pi(6051.9)}{200} = 180 \tag{26}$$

Thus spherical harmonic models for the gravity field of Venus out to degree and order 180 are reasonable representations of the global gravity field.

- The term $r^{-(l+1)}$ is the upward /downward continuation term, the analog of $e^{-2\pi|k|z}$ from our cartesian solution. As in the cartesian solution, terms with higher spherical harmonic degree (*i.e.*, shorter wavelength as seen from equation (25) are attenuated more quickly with increasing radius (increasing height), and amplified more quickly in downward continuation.