Forward Modeling of Magnetic Fields

From last time we had:

$$① \text{---} \quad \Phi_m(P) = \frac{\mu_0}{4\pi} \int_R \bar{M} \cdot \bar{\nabla}_Q \left(\frac{1}{r}\right) dv \quad \leftarrow \text{scalar pot}^{l} \text{ due to magnetis}^{n} \ \bar{M}$$

and $\bar{B}$ can be obtained from $\bar{B} = -\nabla \Phi_m$.

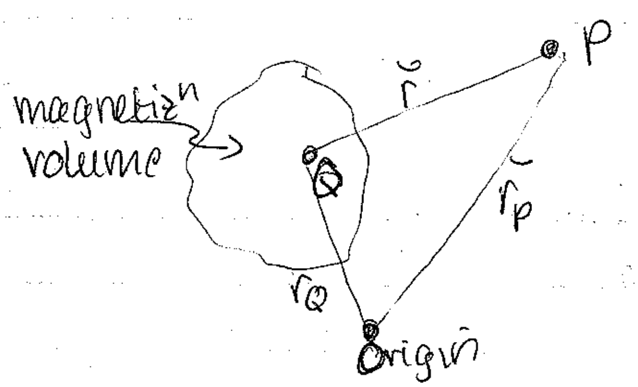
$\Rightarrow$ Thus for a generalized magnetis^{n} distribution we can compute Φ_m and $\bar{B}$

Special Case: Forward Modeling for regions of constant ρ and $\bar{M}$

If $\bar{M}$ is constant, then ① can be written as

$$\Phi_m(P) = -\frac{\mu_0}{4\pi} \bar{M} \cdot \bar{\nabla}_P \int_R \frac{1}{r} \, dv \quad \text{---} ②$$

since $\bar{\nabla}_Q\left(\frac{1}{r}\right) = -\bar{\nabla}_P\left(\frac{1}{r}\right)$ ← since



$$\bar{\nabla}_P\left(\frac{1}{r}\right) = \frac{\bar{r}_{QP}}{r^3} = \frac{\bar{r}_P - \bar{r}_Q}{r^3}$$

$$\bar{\nabla}_Q\left(\frac{1}{r}\right) = \frac{\bar{r}_{PQ}}{r^3} = \frac{\bar{r}_Q - \bar{r}_P}{r^3}$$

We showed previously that the gravitational potential for a volume/body of constant density is given by

$$\Phi_g(P) = G \int_R \frac{\rho}{r} \, dv = G\rho \int_R \frac{1}{r} \, dv$$

$$\Rightarrow \text{So} \quad \boxed{\int_R \frac{1}{r}\, dv = \frac{\Phi_g}{G\rho}} \quad \text{-----} (3)$$

Substituting (3) into (2) gives

$$\Phi_m = \frac{-\mu_0}{4\pi} \bar{M} \cdot \bar{\nabla}_p \left(\frac{\Phi_g}{\rho G} \right) = \frac{-\mu_0 M}{4\pi \rho G} \left(\hat{m} \cdot \bar{\nabla}_p \Phi_g \right)$$

where $\bar{M} = M \hat{M}$

$\hat{M}$ → unit vector in $\bar{M}$ direction

But since $\bar{\nabla} \Phi_g = \bar{g}$

$$\Rightarrow \boxed{\Phi_m = \frac{-\mu_0 M}{4\pi \rho G} \hat{m} \cdot \bar{g}} \quad \text{------} (4)$$

This known as Poisson's relation.

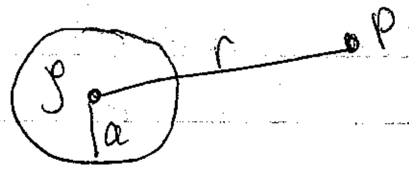
⟹ Thus if $\frac{\Delta \rho}{M}$ is constant, Φ_m can be calculated from the gravity field.

typically it is the density anomaly in a region that is of interest

<u>ALSO</u>: this allows us to compute <u>pseudo-gravity</u> maps from magnetic maps. This can be useful in interpreting magnetic data for geologic structure.

Example #1 :

Magnetic Field of a Uniformly Magnetized Sphere



- Sphere, radius a
- constant density ρ, and

- constant magnetiz^n, $\vec{M}$

$$\vec{g} = -\frac{G\,(\text{mass sphere})\,\hat{r}}{r^2}$$

$$= -\frac{4}{3}\pi a^3 \rho G \frac{\hat{r}}{r^2}$$

$$\Phi_M = -\frac{\mu_0 M}{4\pi \rho G}\, \hat{m} \cdot \vec{g}$$

gamma on next line should be G

$$= \frac{-\mu_0 M}{4\pi \rho G}\left(-\frac{4}{3}\pi a^3 \rho \gamma\right) \hat{m} \cdot \frac{\hat{r}}{r^2}$$

$$= \frac{\mu_0}{4\pi}\left(\frac{4}{3}\pi a^3 M\right) \hat{m} \cdot \frac{\hat{r}}{r^2}$$

and substitute $\vec{m} = \frac{4}{3}\pi a^3 \vec{M}$

$\vec{m}$ ↓ magnetic dipole centered @ center of sphere

$\vec{M}$ ↓ magnetis^n

so $\boxed{\Phi_m = \frac{\mu_0}{4\pi} \frac{\vec{m} \cdot \hat{r}}{r^2}}$ ------ (5)

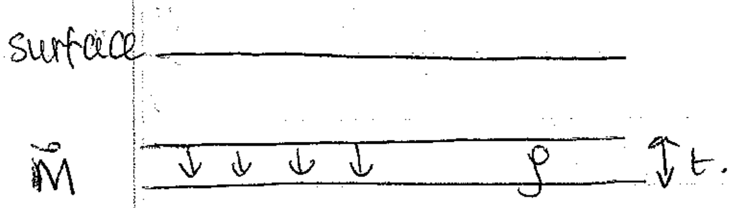
So a uniformly magnetized sphere has the same magnetic field outside the body as a point dipole @ the center of the sphere.

Analogy with gravity ($\vec{g}$ due to uniform density sphere $\equiv$ to that due to point mass @ center)

Remark: Note that measuring the field due to a uniformly magnetized sphere allows us to determine only the
i) centre of the sphere (dipole location)
ii) strength of the dipole, $\frac{4}{3}\pi a^3 M$

We can't uniquely determine a, or M without more information.
The same non-uniqueness problem arises w/ gravity data.

Example #2: Magnetic field due to a uniformly magnetized layer



see later in course

$$\vec{g} = 2\pi G \rho t \hat{z}$$

Suppose $\vec{M} = M\hat{z}$ (vertical).

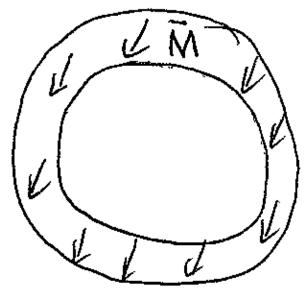
$$\Phi_m = \frac{-\mu_0}{4\pi \rho G} \vec{M} \cdot \vec{g} = \frac{-\mu_0}{4\pi \rho G} 2\pi G \rho t \hat{z} \cdot (M\hat{z})$$

$$\Rightarrow \boxed{\Phi_m = -\frac{\mu_0}{2} t M}$$ ← magnetic potential is constant.

Thus $\boxed{\vec{B} = -\nabla \Phi_m = 0}$ ——— (6)

Hence the magnetic field of a uniformly magnetized slab is zero. Hence we cannot detect such a structure from magnetic field measurements.

As we will see later, the same conclusion holds for a uniformly magnetized spherical shell.



$\longrightarrow$ $\vec{B}_{\text{outside the shell}} = 0.$

This is known as Runcorn's theorem.

Runcorn used this, together with early observations of the Moon's magnetic field (that indicated $\vec{B} \approx 0$) to argue that the following two hypotheses could not be distinguished:

1) The Moon does not currently have a global (core dynamo) field, and has never had one
2) The Moon does not currently have a global field <u>but</u> had one early in its history that resulted in a uniformly magnetized crustal shell.

The important ~~[illegible]~~ point is that non-zero magnetic field measurements at or above a planet's surface indicate magnetization <u>contrasts</u> laterally.