Problem Set 3 : Magnetic Field Modeling – Dipoles.

Due Monday Oct. 8. Submit a single .m script for the MATLAB parts with each problem set up in a new cell. Make sure your code produces ONLY the figures requested and doesn't output unecessary stuff to the screen. Other working can be submitted as a hand-written hard copy or a single pdf, whatever your preference. Derivations should be done by hand – if you use Mathematica or something similar this should only be to check your results.

GOALS: Build some tools to do simple forward modeling of magnetic field data in cartesian coordinates using dipoles. (1) Derive analytical expressions for the magnetic induction due to a dipole in cartesian coordinates. (2) Look at the solutions for some simple dipole orientations, and write MATLAB code to plot these solutions (profiles only). (3) Build MATLAB code to plot the solution for the full vector **B** measured in the x-y plane (e.g., a gridded data set of field observations) due to a single dipole.

1 Cartesian coordinates: general eqns for B_x, B_y, B_z

In class we showed that the magnetic induction due to a dipole, moment **m** can be written as

$$\mathbf{B} = \frac{\mu_0 m}{4\pi r^3}\left[3(\hat{m}\cdot\hat{r})\hat{r} - \hat{m}\right] \tag{1}$$

where $\hat{m} = \mathbf{m}/m$, $\hat{r} = \mathbf{r}/r$, and μ_0 is the permeability of free space. (This was equation (59) of the notes). Use cartesian coordinates i.e., $\mathbf{r} = x\hat{x} + y\hat{y} + z\hat{z}$ and $\mathbf{m} = m_x\hat{x} + m_y\hat{y} + m_z\hat{z}$ to show that

$$B_x = \frac{\mu_0}{4\pi\left(x^2+y^2+z^2\right)^{5/2}}\left[3\left(xm_x + ym_y + zm_z\right)x - \left(x^2+y^2+z^2\right)m_x\right] \tag{2}$$

and write down the corresponding expressions for B_y and B_z.

2 Vertically Oriented Dipole

(A): Assume the coordinate system is oriented such that $\hat{x}$ points North, $\hat{y}$ points East and $\hat{z}$ points down. Remember that the vector **r** in the above equation is the vector from the dipole location to the observer. So if our observer is at a location $\mathbf{r_p} = [x_p, y_p, z_p]$ and our dipole is at a location $\mathbf{r_q} = [x_q, y_q, z_q]$, then $x = x_p - x_q$ etc. Assume that the measurements are made on the Earth's surface (i.e. at positions $\mathbf{r_p} = [x_p, y_p, 0]$, and that the dipole is oriented vertically downward at the position $\mathbf{r_q} = [0, 0, d]$. Show that

$$B_x = \frac{\mu_0 m_z}{4\pi \left(x_p^2 + y_p^2 + d^2\right)^{3/2}} \left[\frac{-3dx_p}{\left(x_p^2 + y_p^2 + d^2\right)} \right] \tag{3}$$

and write down the equations for B_y, B_z.

(B): Now assume you are taking a profile in the x-direction along y=0. Show that the distance between the zero crossings of B_z for this vertical dipole is equal to $2d\sqrt{2}$, and that the distance between maximum and minimum values of B_x is equal to d.

(C): Assume the dipole depth, d = 25m, and that you are taking your $\mathbf{B}$ profile by making observations every 0.5m from x = -100m to x = 100m. Let the dipole have a moment of $10^5 Am^2$. Write MATLAB code to calculate and plot B_x and B_z as a function of distance, x. You'll notice that B_x and B_z are on the order of 10^{-6} Teslas, so make your final figure in nT (nanoTesla) for easier viewing. Plot B_x and B_z on the same figure so that you can see their relative magnitudes.

(D): Check that the distance between the max(B_x) and min(B_x) is indeed d. Do this numerically, not just by "eye-balling" the plots, and report your results. See what happens when you change your observation spacing to 1m and explain the result.

Coding hint 1: Recall `find` *and /or logical indexing from MATLAB class to find the positions in the B_x array corresponding to the min value of B_x.*

Coding hint 2: Remember `fprintf` *for nice formatting to output stuff to the screen.*

3 Inclined dipole in the x-z plane

(A): Now assume your dipole moment $\mathbf{m}$ is oriented at an angle I to the horizontal (I is positive downwards), and that it lies in the x-z plane. Show that along the x-axis the magnetic induction $\mathbf{B}$ is given by

$$B_x = \beta \left[\left(2\alpha^2 - 1\right) m_x - 3\alpha \; m_z \right] \tag{4}$$

$$B_y = 0 \tag{5}$$

$$B_z = \beta \left[-3\alpha \; m_x + \left(2 - \alpha^2\right) m_z \right] \tag{6}$$

where $m_x = m\ cos(I)$, $m_z = m\ sin(I)$, $\alpha = x/d$ (dimensionless) and

$$\beta = \frac{\mu_0}{4\pi d^3\ (\alpha^2 + 1)^{5/2}} \tag{7}$$

(B): Write MATLAB code to calculate and plot B_x and B_z as a function of distance, x. Plot B_x and B_z on a single figure. Comment on how the profiles compare with those for a vertical dipole.

4 Arbitrarily oriented dipole

You can see that the analytical expressions quickly become cumbersome as the problem becomes more generalized (dipoles at varying positions and orientations relative to the observations) and that it is not easy to visualize the form of the resulting profiles. In this section we will develop the code to calculate the full vector **B** measured on the x-y plane at any height, z, due to a dipole moment **m**. The dipole moment **m** is specified by its magnitude and 2 directions: its azimuth, D in the x-y plane, and inclination, I, with respect to the horizontal, such that

$$m_x = mcos(I)cos(D) \tag{8}$$

$$m_y = mcos(I)sin(D) \tag{9}$$

$$m_x = msin(I) \tag{10}$$

(A) Download the skeleton script magdip.m and the function dipm2b.m from the website. The script lays out the plan for writing the code to compute the magnetic induction due to an arbitrary oriented dipole, via comment lines. Make sure you understand what the function dipm2b.m does (one line at a time). In the script magdip.m, write code below each of the comment lines to set up the variables and / or do the calculations indicated by the comment.

Assume that the dipole orientation and position are as in question 2 above, *i.e.,* (1) the dipole is vertically oriented (set D, I accordingly) with moment $m = 10^5$ Am as previously. (2) the dipole is at position vector $[x_q, y_q, z_q]$. Assume (3) that observations are take on the plane $z = 0, x = -100 : 1 : 100, y = -100 : 1 : 100$. Make a figure that shows four contour plots, one each for B_x, B_y, B_z, B_h in the plane of the observations. B_h is the horizontal component of the magnetic field, $B_h = (B_x^2 + B_y^2)^{1/2}$. Make sure your figure has labels, units, etc.

(B) Take a profile along the line y=0 (i.e. along the x axis). Plot B_x, B_y, B_z on a single figure. Check that your B_x, B_z profiles look exactly the same as those you obtained in question (2C).

B_y should of course be zero. This is the kind of check you should always do in coding up more complicated problems – check that your results match a simple case that you were able to code more easily.

(C) Now make the corresponding contour plots (1 figure, 4 subplots) and profiles (1 figure) for a dipole oriented in the x-z plane, inclined at 60^o. Make sure your profiles match those in question (3B).

(D) Make contour plots for a horizontal dipole oriented in the $+\hat{x}$ direction.

5 Rearranging the equations – towards estimation and inverse problems

Take your equations for B_x, B_y and B_z from question #1 and rearrange them so that you have a matrix equation in the following form

$$\vec{B} = G\vec{m} \tag{11}$$

where $\vec{B}$ is a 3N x 1 vector of observations: $[B_x^i, B_y^i, B_z^i]$, where i corresponds to the $i'th$ observation. $\vec{m}$ is a 3 x 1 vector $[m_x, m_y, m_z]$. G will be a matrix with what dimensions? Notice that G encompasses only the "geometry" of the problem - i.e. information about the position of the i'th observation (and the constant μ_0).