Problem Set 1b: Math Review. Due Thurs Sept 13, 2017

1 Vectors

(1) Let $\vec{a} = (0, 1, 3), \vec{b} = (2, -4, 0)$.

a) Find the vector $\vec{c} = \vec{a} \times \vec{b}$.

b) What is the unit vector in the direction of $\vec{c}$?

c) Write down an arbritrary vector in the direction of $\vec{c}$.

d) What physical measure does the cross product give? Provide a sketch.

e) What is the dot product of $\vec{a}$ and $\vec{b}$.

f) What physical measure does the dot product give? Provide a sketch.

(2) The expression $\vec{a} \times (\vec{b} \times \vec{c})$ is the "*vector triple product*" that we will come across often.

a) PROVE: $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$.

b) Is $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}$? Show your work.

(3) If $\vec{f}$ is a vector function with continuous second derivatives, show that $\nabla \times \left(\nabla \times \vec{f}\right) = \nabla \left(\nabla \cdot \vec{f}\right) - \nabla^2 \vec{f}$

(4) Do the gradient, divergence and curl of a vector returns a vector or a scalar. What do div, grad and curl mean physically? What are the mathematical definitions of each in cartesian and spherical coordinates? You can use sketches to help if you want.

(5) Assume a spherical coordinate system, and let $\vec{r}$ be a vector directed from the origin to a point P, with a magnitude equal to the distance from the origin to P. Prove the following relationships (they will be used in the class notes):

a) $\nabla \cdot \vec{r} = 3$ b) $\nabla r = \frac{\vec{r}}{r}$ c) $\nabla \cdot \left(\frac{\vec{r}}{r^3}\right) = 0, \quad r \neq 0$ d) $\nabla \frac{1}{r} = -\frac{\vec{r}}{r^3}, \quad r \neq 0$

e)$\nabla \times \vec{r} = 0$ f)$\vec{A} \cdot \nabla \frac{1}{r} = -\frac{\vec{A} \cdot \vec{r}}{r^3}, \quad r \neq 0$ g)$\left(\vec{A} \cdot \nabla\right) \vec{r} = A_r \frac{\vec{r}}{r}$.

2 Series

(6) In geophysics we can often use approximate answers. The solution can usually be cast in terms of a small quantity, ϵ, where ϵ is much less than 1 in value. A solution accurate to first order in ϵ retains terms with ϵ in but neglects terms in ϵ^2 and higher powers. For example, you might be asked to evaluate F to first order in ϵ where

$$F = \frac{(1+\epsilon)^4}{(1+5\epsilon)^{3/2}}$$

The trick here is to use the binomial expansion

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + ...$$

Your answer can always be simplified to look like $F \simeq 1 + C\epsilon$, where C is a number which you should be able to compute. Try $\epsilon = 10^{-3}$ and see how it compares with the actual answer the difference should be of order 10^{-3}, *i.e.,* ϵ^2.

(7) A Taylor series is something we often use and looks like

$$y(x_0 + h) = y(x_0) + h\frac{dy}{dx}(x_0) + \frac{h^2}{2!}\frac{d^2y}{dx^2}(x_0) + ...$$

where $\frac{dy}{dx}(x_0)$ is the first derivative of $y(x)$ evaluated at x_0. Similarly $\frac{d^2y}{dx^2}(x_0)$ is the second derivative of $y(x)$ evaluated at x_0 and so on. If h is small, we can truncate this series and get a good local approximation to $y(x)$ in the vicinity of x_0.

Try the equation $y(x) = 7x^4$ and set $x_0 = 2$ and $h = 0.1$. Evaluate y at $x = 2.1$ by using the Taylor series truncated at the second (linear) and then the third (quadratic) term and compare with the actual answer.

3 Fourier Series and Matlab

(8) Find the Fourier series of the function $f(x) = e^x$ on $-\pi < x < \pi$. Use MATLAB to plot the function and the Fourier series representation using the number of terms that give a "nice" fit. Start with one term. Show each choice. How do you decide on "nice"?

(9) Do the same thing for the function f(x) = $|x|$on the same interval.

(10) Do the terms in a Fourier series form an orthogonal basis? Why? (i.e., what is an orthogonal function)?

4 Heat flow, Laplace's equation, separation of variables and Fourier series

(11) A sheet of metal coincides with the square in the xy plane whose vertices are the points (0,0), (1,0), (1,1) and (0,1). The two (upper and lower) faces of the sheet are insulated. The sheet is sufficiently thin that heat flow within it may be regarded as two-dimensional. In other words, a 2D approximation is permitted because the thickness of the sheet is much less that its lateral extent, so $h \ll 1$. The edges parallel to the x-axis are insulated. The left-hand edge is maintained at the constant temperature 0. If the temperature distribution $T(1, y) = f(y)$ is maintained along the right-hand edge, find the steady-state temperature distribution throughout the sheet. If you can, plot your solution with MATLAB.

Hints:

1) Steady-state heat implies that there is no time-dependence in this problem

2) Use the technique of "separation of variables" to solve this linear PDE.

3) This question is quite long....

Main Steps: set up the PDE, find a general solution, write down the boundary conditions