4 WATER VAPOR

Contents

Water vapor is one of the gases in air. Unlike nitrogen and oxygen which are constant in the bottom 100 km of the atmosphere, water-vapor concentration can vary widely in time and space. Most people are familiar with relative humidity as a measure of water-vapor concentration because it affects our body's moisture and heat regulation. But other humidity variables are much more useful in other contexts.

Storms get much of their energy from water vapor — when water vapor condenses or freezes it releases latent heat. For this reason we carefully track water vapor as it rises in buoyant thermals or is carried by horizontal winds. The amount of moisture available to a storm also regulates the amount of rain or snow precipitating out.

What allows air to hold water as vapor in one case, but forces the vapor to condense in another? This depends on a concept called "saturation".

VAPOR PRESSURE AT SATURATION

Total atmospheric pressure is the sum of the **partial pressures** of all the constituents. For water vapor, the partial pressure is called the **vapor pressure**, e. Vapor pressure has units of kPa.

Gases in the air can mix with any relative proportions. However, for water vapor, there is a critical water-vapor concentration, known as the **saturation humidity**. Above this critical value, water vapor condenses faster than it evaporates, thereby reducing the water-vapor concentration back to the critical value. At this critical value, the air is said to be **saturated**, and the vapor is in equilibrium with liquid water. Let e_s represent the saturation vapor pressure at equilibrium over pure water having a flat surface.

It frequently happens that air holds less than the critical value, and is said to be **unsaturated**. For this case, $e < e_s$.

Although any portion of water vapor can be held in air, it is rare for the vapor pressure to be more than 1% greater than the saturation value. Air having $e > e_s$ is said to be **supersaturated**. Supersaturated air can occur as a transient condition while excess water vapor is condensing onto available surfaces such as on dust particles called cloud condensation

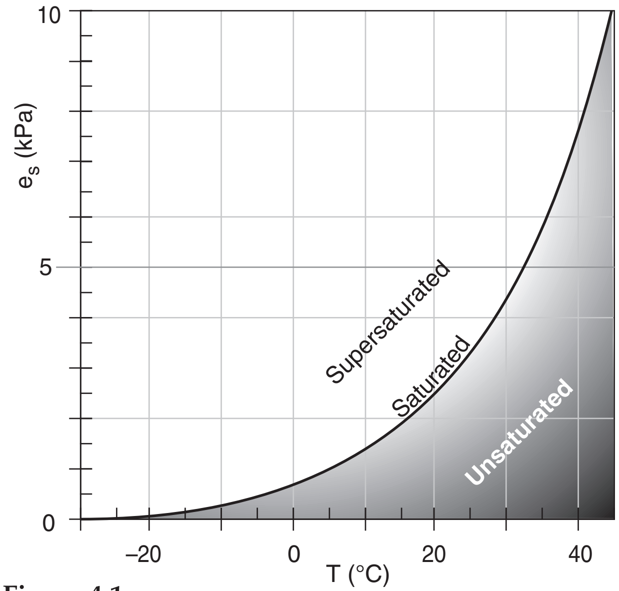


Figure 4.1
Pure-water saturation vapor pressure over a flat water surface.

Sample Application
Air of temperature 30°C has what value of e_s?

Find the Answer
Given: $T = 30°C = 303.15$ K
Find: e_s = ? kPa for over liq. water. Apply eq. (4.1a).

$$e_s = (0.6113\text{kPa})\cdot\exp\left[(5423\text{K})\cdot\left(\frac{1}{273.15\text{K}} - \frac{1}{303.15\text{K}}\right)\right]$$

$$= (0.6113\text{ kPa})\cdot\exp(1.96473) = \underline{\mathbf{4.36\ kPa}}$$

Check: Physics reasonable, & agrees with Table 4-1.
Exposition: At saturation for this *T*, water-vapor partial pressure is only 4.3% of total pressure $P \approx 101.3$ kPa.

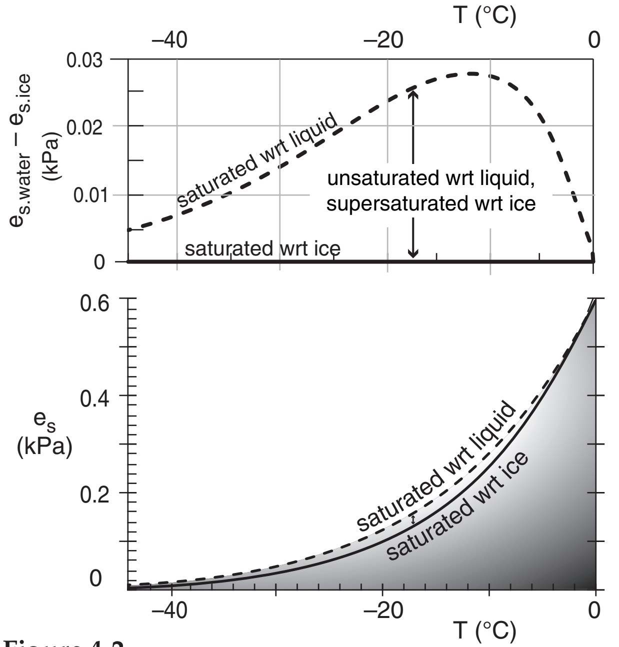


Figure 4.2
Ice & liquid saturation vapor pressures. (wrt = with respect to).

nuclei. However, photographs of air flow over aircraft wings for both subsonic and supersonic flight through humid air indicate that condensation to form cloud droplets occurs almost instantly.

During the equilibrium state (i.e., at saturation) there is a balance between the rate of evaporation from the liquid and the rate of condensation from vapor. Liquid-water temperature controls the rate of evaporation, and **humidity** (water-vapor concentration in air) controls the rate of condensation.

Warmer liquid temperatures cause greater evaporation rates, which allow the humidity in the air to increase until a new balance is attained. The opposite is true for colder temperatures. For a situation where the air and liquid water temperatures are equal, we conclude that colder air has a smaller capacity for holding water vapor than warmer air.

This relationship between saturation vapor pressure and temperature is approximated by the **Clausius-Clapeyron** equation:

$$e_s \approx e_o \cdot \exp\left[\frac{L}{\Re_v}\cdot\left(\frac{1}{T_o} - \frac{1}{T}\right)\right] \qquad \bullet(4.1a)$$

where the water-vapor gas constant is $\Re_v$ = 461 J·K^{-1}·kg^{-1}, T_o = 273.15 K, e_o = 0.6113 kPa, and *L* is a latent-heat parameter. Temperatures in this equation must have units of Kelvin.

This equation works for saturation over both liquid water and solid water (ice) surface if these surfaces are flat. For liquid water the latent heat of vaporization $L = L_v = 2.5\text{x}10^6$ J·kg^{-1}, giving $L_v/\Re_v$ = 5423 K. For ice the latent heat of deposition $L = L_d = 2.83\text{x}10^6$ J·kg^{-1} and $L_d/\Re_v$ = 6139 K.

The exponentially-shaped curve described by eq. (4.1a) is plotted in Fig. 4.1, with corresponding data values listed Table 4-1. One interpretation of this curve is that as unsaturated humid air is cooled, a temperature is reached at which point the air is saturated. Further cooling forces some water vapor to condense into liquid, creating clouds and rain and releasing latent heat. Hence, the Clausius-Clapeyron equation is important for understanding storms.

In the atmosphere it is possible for liquid water to remain unfrozen at temperatures down to –40°C. Such unfrozen cold water is said to be **supercooled**. The difference between saturation values of water vapor over supercooled liquid water and ice is plotted in Fig. 4.2.

The Clausius-Clapeyron equation also describes the relationship between actual (unsaturated) water-vapor pressure *e* and dew-point temperature (T_d, to be defined later):

$$e = e_o \cdot \exp\left[\frac{L}{\Re_v} \cdot \left(\frac{1}{T_o} - \frac{1}{T_d}\right)\right] \tag{4.1b}$$

where T_o = 273.15 K and e_o = 0.6113 kPa as before. Use, $L/\Re_v = L_v/\Re_v$ = 5423 K for liquid water, and use $L_d/\Re_v$ = 6139 K for ice.

The latent-heat parameter L varies slightly with temperature. Taking that into account, a different approximation known as **Tetens' formula** has been suggested for saturation vapor pressure e_s as a function of temperature (T, in Kelvins):

$$e_s = e_o \cdot \exp\left[\frac{b \cdot (T - T_1)}{T - T_2}\right] \tag{4.2}$$

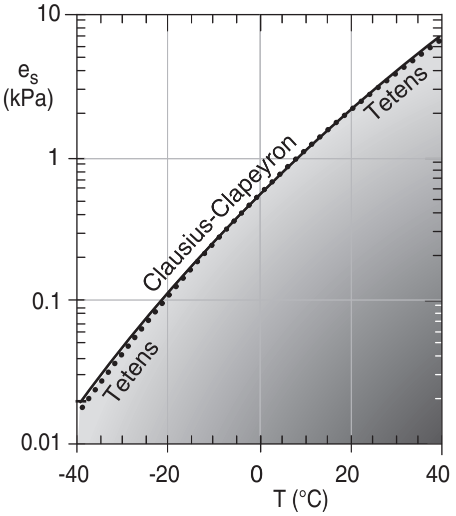
where b = 17.2694, e_o = 0.6113 kPa, T_1 = 273.15 K, and T_2 = 35.86 K.

Sample Application

How do the vapor pressure values from Tetens' formula differ from those of the Clausius-Clapeyron equation, as a function of temperature?

Find the Answer

Use a spreadsheet to solve eqs. (4.1a) and (4.2) .



Check: Physics and units are reasonable.

Exposition: The difference between both formulas are very small — often smaller than other uncertainties in measurements of temperature or humidity. So you can be confident using either equation.

HIGHER MATH • Clausius-Clapeyron Eq.

Rudolf Clausius and Benoît Paul Émilie Clapeyron were engineers trying to improve steam engines during the 1800's. They independently made laboratory measurements of water vapor pressure at saturation e_s, and found the following empirical relationship:

$$\frac{de_s}{dT} = \frac{L_v}{T}\left[\frac{1}{\rho_v} - \frac{1}{\rho_L}\right]^{-1}$$

where ρ_L is liquid-water density, and ρ_v is water-vapor density (i.e., **absolute humidity**). The liquid-water density is so much greater than the water-vapor density that the above equation can be approximated by:

$$\frac{de_s}{dT} \cong \frac{L_v}{T}\rho_v$$

The relationship between saturation vapor pressure and the absolute humidity is given by the ideal gas law for water vapor:

$$e_s = \rho_v \cdot \Re_v \cdot T$$

where the water-vapor gas constant is $\Re_v = 4.615\text{x}10^{-4}$ kPa K^{-1} (g $\text{m}^{-3})^{-1}$. You can solve the ideal gas law for ρ_v , which you can then use in the previous eq:

$$\frac{de_s}{dT} \cong \frac{L_v \cdot e_s}{\Re_v \cdot T^2}$$

This equation admits a solution if you separate the variables to put all terms involving T on the right, and all those involving e_s on the left:

$$\frac{de_s}{e_s} \cong \frac{L_v}{\Re_v}\frac{dT}{T^2}$$

Next, integrate between some known condition (e_o at T_o) to any arbitrary values of saturation vapor pressure (e_s) and temperature (T):

$$\int_{e_o}^{e_s} \frac{de_s}{e_s} \cong \frac{L_v}{\Re_v}\int_{T_o}^{T}\frac{dT}{T^2}$$

After integrating you get:

$$\ln\left(\frac{e_s}{e_o}\right) \cong -\frac{L_v}{\Re_v}\left[\frac{1}{T} - \frac{1}{T_o}\right]$$

To get eq. (4.1a) take the exponential (i.e., antilog) of both sides and then algebraically rearrange to give:

$$\boxed{e_s = e_o \cdot \exp\left[\frac{L}{\Re_v} \cdot \left(\frac{1}{T_o} - \frac{1}{T}\right)\right]} \tag{4.1}$$

Details are given in the *Atmospheric Thermodynamics* book by C. Bohren and B. Albrecht (1998, Oxford Univ. Press, 402 pp).

INFO • Boiling Point

Liquids boil when the saturation vapor pressure e_s equals the ambient pressure P of the atmosphere:

$$P = e_s$$

Variation of boiling temperature with Altitude

We know e_s as a function of temperature T from the Clausius-Clapeyon equation, so plug that into the right side of the equation above. We also know that ambient atmospheric pressure P decreases exponentially with increasing height z, as was given in Chapter 1, so we can plug that into the left side. This gives

$$P_o \cdot \exp\left[-\frac{z}{H_p}\right] = e_o \cdot \exp\left[\frac{L_v}{\Re_v}\cdot\left(\frac{1}{T_o}-\frac{1}{T}\right)\right]$$

where P_o = 101.325 kPa, $H_p \approx 7.29$ km, e_o = 0.6113 kPa, $L_v/\Re_v$ = 5423 K, and T_o = 273.15 K.

Next, divide the left and right sides of the equation by P_o. On the right side, note that $e_o/P_o = \exp[\ln(e_o/P_o)]$. Thus:

$$\exp\left[-\frac{z}{H_p}\right] = \exp\left[\ln\left(\frac{e_o}{P_o}\right) + \frac{L_v}{\Re_v}\left(\frac{1}{T_o}\right) - \frac{L_v}{\Re_v}\frac{1}{T}\right]$$

Create a new constant called a (dimensionless) that is the sum of the first two terms inside the square brackets, because those two terms are constant. Next, take the *ln* of the right and left sides to give:

$$\frac{z}{H_p} = \frac{L_v}{\Re_v}\frac{1}{T} - a$$

Solve this equation for T, which we can re-define as the boiling point $T_{boiling}$:

$$\boxed{T_{boiling} = \frac{L_v/\Re_v}{a + z/H_p}}$$

Knowing that at sea level (z = 0) the boiling temperature is 100°C (i.e., $T_{boiling}$ = 373.15 K) you can solve for the dimensionless constant, giving : a = 14.53.

Exposition:

If you solve this equation for various altitudes, you find that the boiling point decreases by 3.4°C km^{-1}. Thus, $T_{boiling}$ = 366.35 K = 93.2°C at 2 km altitude

To soften vegetables to the desired tenderness or to prepare meats to the desired doneness, foods must be cooked at a certain temperature over a certain time duration. Slightly cooler cooking temperatures must be compensated with slightly longer cooking times. Thus, you need to cook boiled foods for longer times at higher altitudes, because boiling happens at a lower temperature.

Table 4-1. Values of humidity variables at saturation (subscript s) over a liquid-water flat surface, for different air temperatures T. The same values also relate the dew-point temperature T_d to the actual humidity. Notation: e = vapor pressure, r = mixing ratio, q = specific humidity, ρ_v = absolute humidity. Note that r and q depend on pressure — this table shows their values for standard sea-level pressure.

		For P = 101.325 kPa		
T	e_s	q_s	r_s	ρ_{vs}
		or		
T_d	e	q	r	ρ_v
(°C)	(kPa)	(g kg^{-1})	(g kg^{-1})	(g m^{-3})
–40	0.0203	0.1245	0.1245	0.1886
–35	0.0330	0.2029	0.2029	0.301
–30	0.0528	0.324	0.3241	0.4708
–25	0.0827	0.5079	0.5082	0.7231
–20	0.1274	0.7822	0.7828	1.0914
–15	0.1929	1.1848	1.1862	1.6206
–10	0.2875	1.7666	1.7697	2.3697
–5	0.4222	2.5956	2.6024	3.4151
0	0.6113	3.7611	3.7753	4.8546
5	0.8735	5.3795	5.4086	6.8119
10	1.232	7.6005	7.6587	9.4417
15	1.718	10.62	10.73	12.94
20	2.369	14.67	14.89	17.53
25	3.230	20.07	20.48	23.5
30	4.360	27.21	27.97	31.2
35	5.829	36.58	37.97	41.03
40	7.720	48.8	51.3	53.48
45	10.13	64.66	69.13	69.1
50	13.19	85.18	93.11	88.56
55	17.04	111.7	125.7	112.6
60	21.83	145.9	170.8	142.2

Sample Application

For T = 10°C and **P = 70 kPa**, calculate q_s, r_s, & ρ_{vs}.

Find the Answer

Given: T = 10°C = 283.15 K, P = 70 kPa
Find: q_s = ? g kg^{-1}, r_s = ? g kg^{-1}, ρ_{vs} = ? g m^{-3}

Get e_s = 1.232 kPa from Table 4-1 (independent of P).

Apply eq. (4.8) to get q_s:
q_s = 0.622·(1.232 kPa)/(70 kPa) = 0.0109 g g^{-1} ≈ **11** g kg^{-1}

Apply eq. (4.5) to get r_s:
r_s = [0.622·(1.232 kPa)] / [70 kPa - 1.232 kPa]
= 0.0111 g g^{-1} = **11.1** g kg^{-1}

Apply eq. (4.12) to get ρ_{vs}:
ρ_{vs} = (1232. Pa)/[(461 J·K^{-1}·kg^{-1})·(283.15 K)]
= **0.00944** kg·m^{-3} = **9.44** g·m^{-3} .

Check: Physics and units are reasonable.
Exposition: Table 4-1 could have been used for ρ_{vs} .

MOISTURE VARIABLES

Table 4-2 (continued across several pages) shows most of the moisture variables used in meteorology. Other variables used in the table below include: m = mass, e = vapor pressure, P = total atmospheric pressure, $\Re_d$ = 2.871x10^{-4} kPa·K^{-1}·m^3·g^{-1} is the gas constant for dry air, $\Re_v$ = 4.61x10^{-4} kPa·K^{-1}·m^3·g^{-1} is the gas constant for pure water vapor, P_d is the partial pressure of dry air, ρ_d is the density of dry air (which is a function of pressure, altitude and temperature as given by the ideal gas law), and subscript s denotes saturation.

Table 4-2a. Moisture variables. *(continues on next page)*

Variable name:	Mixing Ratio	Specific Humidity	Absolute Humidity
Symbol:	r	q	ρ_v
Units:	$kg_{water\ vapor}\ kg_{dry\ air}^{-1}$	$kg_{water\ vapor}\ kg_{total\ air}^{-1}$	$kg_{water\ vapor}\ m^{-3}$
Alternative Units:	g kg^{-1} , g g^{-1} , kg kg^{-1}	g kg^{-1} , g g^{-1} , kg kg^{-1}	kg m^{-3}
Defining Equation: (& equation number)	$r = \frac{m_{water\ vapor}}{m_{dry\ air}}$ (4.3)	$q = \frac{m_{water\ vapor}}{m_{total\ air}}$ (4.6) $q = \frac{m_{water\ vapor}}{m_{dry\ air} + m_{water\ vapor}}$	$\rho_v = \frac{m_{water\ vapor}}{Volume}$ (4.9)
Relationship to Vapor Pressure:	$r = \frac{\varepsilon \cdot e}{P - e}$ (4.4)	$q = \frac{\varepsilon \cdot e}{P_d + \varepsilon \cdot e} = \frac{\varepsilon \cdot e}{P - e \cdot (1-\varepsilon)}$ $q \approx \frac{\varepsilon \cdot e}{P}$ (4.7)	$\rho_v = \frac{e}{\Re_v \cdot T}$ (4.10) $\rho_v = \frac{e \cdot \varepsilon \cdot \rho_d}{P - e} \approx \frac{e}{P} \cdot \varepsilon \cdot \rho_d$ (4.11)
If Saturated:	$r_s = \frac{\varepsilon \cdot e_s}{P - e_s}$ (4.5)	$q_s = \frac{\varepsilon \cdot e_s}{P_d + \varepsilon \cdot e_s}$ (4.8) $q_s = \frac{\varepsilon \cdot e_s}{P - e_s \cdot (1-\varepsilon)} \approx \frac{\varepsilon \cdot e_s}{P}$	$\rho_{vs} = \frac{e_s}{\Re_v \cdot T}$ (4.12) $\rho_{vs} = \frac{e_s \cdot \varepsilon \cdot \rho_d}{P - e_s} \approx \frac{e_s}{P} \cdot \varepsilon \cdot \rho_d$ (4.13)
Key Constants:	$\varepsilon = \Re_d/\Re_v$ $= 0.622\ g_{vapor}\ g_{dry\ air}^{-1}$ $= 622$ g kg^{-1}	$\varepsilon = \Re_d/\Re_v$ $= 0.622\ g_{vapor}\ g_{dry\ air}^{-1}$ $= 622$ g kg^{-1}	$\Re_v$ = 4.61x10^{-4} kPa·K^{-1}·m^3·g^{-1} ε = 622 g kg^{-1}
Typical Values:	See Table 4-1.	See Table 4-1.	See Table 4-1.
Relevance:	• r is conserved in unsaturated air parcels that move without mixing with their environment. • not affected by heating, cooling, pressure changes. • used in thermo diagrams.	• q is conserved in unsaturated air parcels that move without mixing with their environment. • not affected by heating, cooling, pressure changes.	• easy to measure using absorption of infrared, ultraviolet, or microwave radiation as a function of path length through the air. • is the concentration of water vapor in air.
Notes:	Derivation of eq. (4.4): Given eq. (4.3), divide numerator and denominator by volume. But *m/Volume* is density. Use ideal gas laws for water vapor and for dry air, assuming a common *T*.	Derivation of eq. (4.7) is similar to that for eq. (4.4).	• Eq. (4.10) is the ideal gas law for water vapor. • Eq. (4.11) uses the ideal gas law for dry air to replace temperature *T*. • *T* must be in Kelvin.

Table 4-2b. Moisture variables. *(continuation)*

Variable name:	Relative Humidity	Dewpoint
Symbol:	RH or $RH\%$	T_d
Units:	(dimensionless)	(K)
Alternative Units:	(%)	(°C)
Defining Equation: (& equation number)	$RH = \frac{e}{e_s}$ or $\frac{RH\%}{100\%} = \frac{e}{e_s}$ (4.14a)	"Temperature to which a given air parcel must be cooled at constant pressure and constant water-vapor content in order for saturation to occur."*
Alternative Definitions:	$RH = \frac{q}{q_s} = \frac{\rho_v}{\rho_{vs}} \approx \frac{r}{r_s}$ (4.14b) $\frac{RH\%}{100\%} = \frac{q}{q_s} = \frac{\rho_v}{\rho_{vs}} \approx \frac{r}{r_s}$ (4.14c)	$T_d = \left[\frac{1}{T_o} - \frac{\Re_v}{L} \cdot \ln\left(\frac{e}{e_o}\right)\right]^{-1}$ (4.15a) $T_d = \left[\frac{1}{T_o} - \frac{\Re_v}{L} \ln\left(\frac{r \cdot P}{e_o \cdot (r+\varepsilon)}\right)\right]^{-1}$ (4.15b)
If Saturated:	$RH = 1.0$ or $RH\% = 100\%$ (4.14d)	$T_d = T$
Key Constants:		$e_o = 0.6113$ kPa, $T_o = 273.15$ K $\Re_v/L_v = 1.844\text{x}10^{-4}$ K^{-1}. $\varepsilon = \Re_d/\Re_v = 0.622$ $g_{vapor}\ g_{dry\ air}{}^{-1}$
Typical Values:	$RH = 0.0$ to 1.0 or $RH\% = 0\%$ to 100%	See Table 4-1. $T_d \le T$
Relevance:	• regulates the max possible evaporation into the air. • easy to measure via: (1) capacitance changes across a plastic dielectric; (2) electrical resistance of an emulsion made of carbon powder; or (3) organic-fiber contraction/expansion. • most used by the general public.	• easy to measure via cooling a mirror to the point where condensation (dew) forms on it. The temperature at which this first happens is the dewpoint, as detected by measuring how well a light beam can reflect off the mirror. • a very accurate method for humidity measurement.
Notes:	• it is possible to have relative humidities as high as about 100.5%. This is called **supersaturation** (see the Precipitation Processes chapter).	• T_d also called **dewpoint temperature**. • $(T - T_d)$ = **dewpoint depression** = **temperature dewpoint spread** * Glickman, T. S., 2000: *Glossary of Meteorology*. American Meteorological Society.

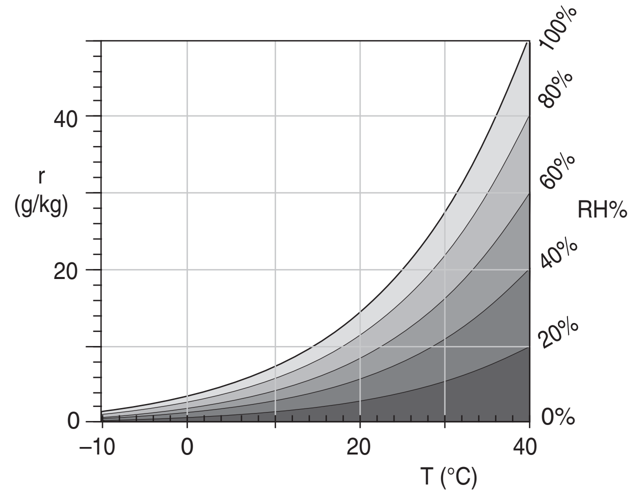


Figure 4.3
Relationship between relative humidity and mixing ratio at P = 101.325 kPa. Grey region is unsaturated. White region is unphysical (except sometimes for 0.5% of supersaturation).

Sample Application
Find T_d and $RH\%$ for r= 10 g kg^{-1}, P= 80 kPa, T=20°C.

Find the Answer
Given: $r = 0.01\ g_{vapor}\ g_{air}{}^{-1}$, $P = 80$ kPa, $T = 20°C$
Find: T_d = ? °C, and $RH\%$ = ? %

Use eq. (4.15b): $T_d = [(1/(273.15K) - (1.844\text{x}10^{-4}\ K^{-1}) \cdot$
$\ln\{ [(80kPa)\cdot(0.01 g_{vapor}\ g_{air}{}^{-1})] /$
$[(0.6113kPa)\cdot((0.01g\ g^{-1}) + (0.622g\ g^{-1})] \}]^{-1}$
$= [(1/(273.15K) - (1.844\text{x}10^{-4}K^{-1})\cdot\ln(2.07)]^{-1} = 283.78K.$
Thus, $T_d = 283.78 - 273.15 =$ or **10.6°C**

At $T_d = 10.6°C$, eq. 4.1b gives $e \approx 1.286$ kPa
At $T = 20°C$, Table 4-1 gives $e_s = 2.369$ kPa
Use eq. (4.14a): $RH\% = 100\%\cdot(1.286\ kPa/2.369\ kPa)$
$RH\%$ = **54.3%**

Check: Magnitude and units are reasonable.

Table 4-2c. Moisture variables. *(continuation)*

Variable name:	Lifting Condensation Level (LCL)	Wet-bulb Temperature
Symbol:	z_{LCL}	T_w
Units:	(km)	(K)
Alternative Units:	(m)	(°C)
Defining Equation: (& equation number)	$z_{LCL} = a\cdot(T - T_d)$ (4.16a)	$C_p\cdot(T - T_w) = -L_v\cdot(r - r_w)$ (4.17)
Applications:	$P_{LCL} = P\cdot\left[1 - b\cdot\left(\frac{T - T_d}{T}\right)\right]^{C_p/\Re}$ (4.16b)	$r = r_w - \beta\cdot(T - T_w)$ (4.18a) where $r_w = \frac{\varepsilon}{b\cdot P\cdot\exp\left(\frac{-c\cdot T_w(^\circ C)}{T_w(^\circ C)+\alpha}\right) - 1}$ (4.18b)
If Saturated:	$z_{LCL} = 0$, $P_{LCL} = P$	$T_w = T$, for T = **dry-bulb temperature**
Key Constants:	a = 0.125 km °C^{-1}. Γ_d = 9.8 °C km^{-1} $b = a\cdot\Gamma_d$ = 1.225 (dimensionless) $C_p/\Re$ = 3.5 (dimensionless)	ε = 622 g kg^{-1}, b = 1.631 kPa^{-1}, c = 17.67, α = 243.5°C, β = 0.40224 (g kg^{-1})/°C.
Typical Values:	0 to 5 km	See Figs. 4.4 & 4.5. $T_w \le T$
Relevance:	• for unsaturated air, it is the height (or pressure) to which air must be lifted to become just saturated (i.e., cloudy). • is **cloud-base** altitude for cumulus and other convective clouds.	• easy to measure, by placing a wet wick or sleeve around a thermometer bulb, and then blowing air past the wet bulb (an **aspirated psychrometer**) or moving the wet bulb through the air (a **sling psychrometer**).
Notes:	• does NOT give cloud base for stratiform clouds, because these clouds are caused by advection (nearly horizontal winds). • the LCL is also known as the **saturation level**. For saturated air, it is the height to which air must be lowered to become just unsaturated (eqs 4.16 do NOT apply).	• as water evaporates from the wet wick, the adjacent air temperature drops from T to T_w while the humidity in this air increases from r to r_w, until equilibrium heat balance is reached as described by eq. (4.17). • $(T - T_w)$ is called **wet-bulb depression**. • see **Normand's rule** on later pages.

Sample Application

For an air parcel at P = 90 kPa with T = 25°C and T_d = 8°C, what is the height and pressure of the LCL?

Find the Answer

Given: P = 90 kPa, T = 25°C = 298.15 K, T_d = 8°C .
Find: z_{LCL} = ? km , P_{LCL} = ? kPa

Use eq. (4.16a)

z_{LCL} = (0.125 km °C^{-1})·(25 – 8°C)
= **2.13 km higher than the initial height**

Use eq. (4.16b)

P_{LCL} = (90 kPa)·[1 – 1.225·(25–8°C)/298.15K]$^{3.5}$
= **69.85 kPa**

Check: Physics and units are reasonable.

Exposition: The pressure decreases about 10 kPa for each increase of 1 km of altitude near the surface, the pressure answer is also reasonable. Indeed, an air parcel moving from 90 to 70 kPa rises about 2 km.

Sample Application

You observe a dry-bulb (i.e., normal air) temperature of T = 25°C, and a wet-bulb temperature of 18°C. Use the equations to calculate the mixing ratio for P = 90 kPa. Don't use look-up tables or graphs.

Find the Answer

Given: P = 90 kPa, T = 25°C , T_w = 18°C
Find: r = ? g kg^{-1}

First, solve eq. (4.18b):

$$r_w = \frac{622\text{g/kg}}{(1.631\text{kPa}^{-1})\cdot(90\text{kPa})\cdot\exp\left(\frac{-17.67\cdot 18^\circ C}{18^\circ C + 243.5^\circ C}\right) - 1}$$

r_w = 14.6 g kg^{-1}.

Then solve eq. (4.18a):

r = (14.6g kg^{-1}) – [0.40224 (g kg^{-1})/°C]·(25–18°C)
= **11.78** g kg^{-1}.

Check: Physics and units are reasonable.

Exposition: Knowing r, use other eqs. to find any other humidity variable.

Sample Application

Find the mixing ratio and relative humidity for air temperature of 12°C and wet-bulb temperature of 10°C. Yes, you may use the graphs this time.

Find the Answer

Given: $T = 12°C$, $T_w = 10°C$
Find: $r = ?$ g kg^{-1} , and $RH = ?$ %

Assume $P = 101.3$ kPa, so we can use Figs. 4.4 & 4.5.
The wet-bulb depression is 12 – 10 = 2°C
Use Fig. 4.4. $r = \underline{7}$ g kg^{-1}.
Use Fig. 4.5. $RH = \underline{78\%}$.

Check: Physics and units are reasonable.
Exposition: Much easier than the Sample Applications on the previous page. Notice that in Fig. 4.4, r depends mostly on T_w, because the mixing-ratio lines are mostly horizontal. However, $RH\%$ depends mostly on $T - T_w$, because the lines in Fig. 4.5 are mostly vertical.

More Wet-bulb Temperature Info

The easiest way for you to find humidity from dry and wet-bulb temperature is to look-up the humidity in tables called **psychrometric tables**, which are often published in meteorology books. Figures 4.4 and 4.5 present the look-up information as **psychrometric graphs**, which were computed from the equations in this Chapter.

To create your own psychrometric tables or graphs, first generate a table of mixing ratios in a spreadsheet program, using eqs. (4.18) and (4.19). I assumed a standard sea-level pressure of $P = 101.325$ kPa for the figures here. Then contour the resulting numbers to give Fig. 4.4. Starting with the table of mixing ratios, use eqs. (4.2), (4.5), and (4.14) to create a new table of relative humidities, and contour it to give Fig. 4.5. All of these psychrometric tables and graphs are based on Tetens' formula (see eq. 4.2).

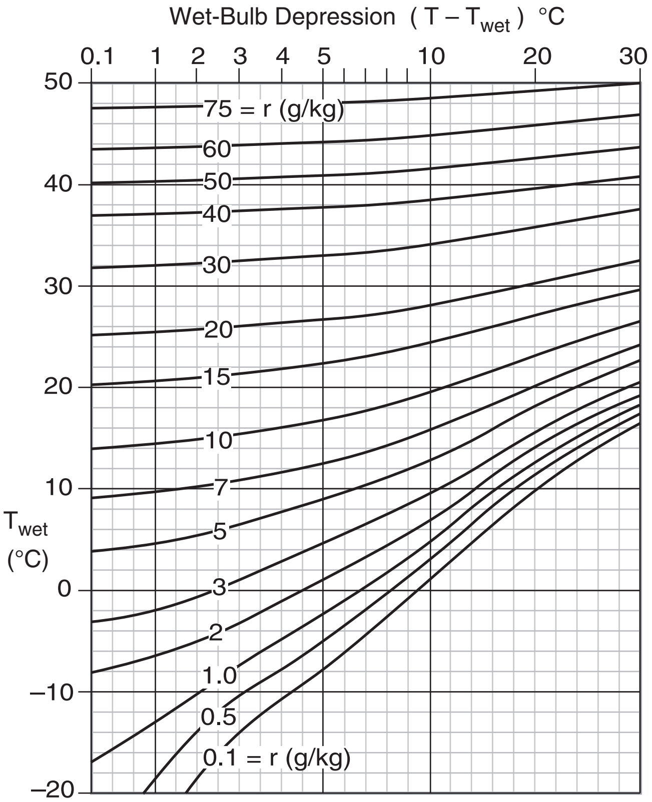


Figure 4.4
Psychrometric graph, to find mixing ratio r from wet and dry-bulb temperatures. Based on P = 101.325 kPa. Caution, the darker vertical lines mark scale changes along the abscissa.

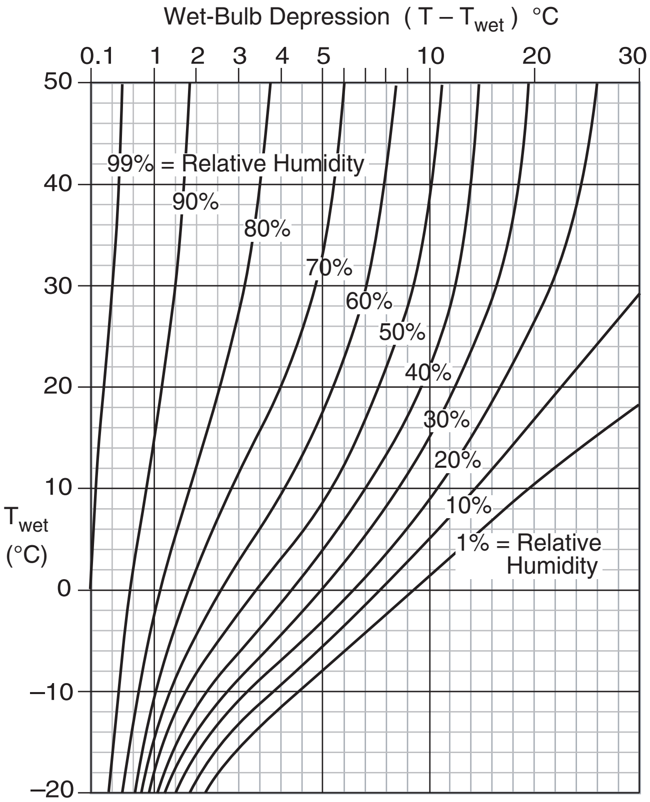


Figure 4.5
Psychrometric graph, to find relative humidity from wet and dry-bulb temperatures. Based on P = 101.325 kPa. Caution, the darker vertical lines mark scale changes along the abscissa.

It is easy to calculate other humidity variables such as mixing ratio or relative humidity from known values of T and T_w. Eqs. (4.18) and Figs 4.4 and 4.5 are examples of this.

However, going the opposite way is more difficult. If you are given other humidity variables it is hard to find the wet-bulb temperature. Namely, to use the equations or figures mentioned above, you would need to iterate to try to converge on the correct answer.

Instead, there are two methods to estimate T_w. One is an empirical approximation (given below), and the other is a graphical method called Normand's Rule (given on the next page).

The empirical approximation for T_w (in °C) at sea level is a function of air temperature T (in °C) and relative humidity $RH\%$ (e.g., using 65.8 to represent 65.8%):

$$\begin{aligned} T_w \approx\ & T\cdot\text{atan}[0.151977(RH\% + 8.313659)^{1/2}] - 4.686035 \\ & + \text{atan}(T + RH\%) - \text{atan}(RH\% - 1.676331) \\ & + 0.00391838\cdot(RH\%)^{3/2}\cdot\text{atan}(0.023101\cdot RH\%) \end{aligned} \tag{4.19}$$

where the arctangent (atan) function returns values in radians. [CAUTION: If your software returns arctan values in degrees, be sure to convert to radians before you use them in the equation above.]

Sample Application

Given an air temperature of 20°C and a relative humidity of 50%, use the empirical method to estimate the wet-bulb temperature at sea level.

Find the Answer

Given: T = 20°C , $RH\%$ = 50
Find: T_w = ? °C

Apply eq. (4.19:

$$\begin{aligned} T_w = & 20\cdot\text{arctan}[0.151977\cdot(50 + 8.313659)^{1/2} \\ & + \text{arctan}(20 + 50) \\ & - \text{arctan}(50 - 1.676331) \\ & + 0.00391838\cdot(50)^{3/2}\cdot\text{arctan}(0.023101\cdot 50) \\ & - 4.686035 \end{aligned}$$

T_w = **13.7 °C**

Check: Units reasonable. Agrees with Fig. 4.5.
Exposition: Although this equation had many terms, it needed to be solved only once. Contrast this with iterative methods, which require repeated solutions of equations in order to converge to an answer.

Sample Application

The air temperature is T = 20°C and the mixing ratio is 7.72 g/kg (which you might have found using a psychrometer with a wet-bulb temperature of 14°C). Use the equations to calculate the relative humidity for P = 100 kPa. Don't use look-up tables or graphs.

Find the Answer

Given: T = 20°C , r = 7.72 g kg^{-1}, P = 100 kPa
Find: RH = ? %

First, use Tetens' formula (4.2), with the trick that $\Delta T = T(\text{K}) - T_1(\text{K}) = [T(°\text{C})+273.15] - 273.15 = T(°\text{C})$, and remembering that for temperature differences: 1°C = 1 K .

$$e_s = 0.611(\text{kPa})\cdot\exp\left[\frac{17.2694\cdot(20\text{K})}{(20 + 273.15)\text{K} - 35.86\text{K}}\right]$$

e_s = 2.34 kPa.

Next, use this in eq. (4.5):

$$r_s = \frac{(622\text{g/kg})\cdot 2.34\text{kPa}}{[101.325 - 2.34]\text{kPa}} = 14.7\text{g/kg}$$

Finally, use eq. (4.14): $RH = 100\%\cdot(r/r_s)$
RH = 100% · (7.72/14.7) = **52.5%**

Check: Physics and units are reasonable.
Exposition: What a lot of work. If we instead had used psychrometric graph (Fig. 4.5) with T_w = 14°C and $T - T_w$ = 6°C , we would have found almost the same relative humidity much more easily.

Sample Application

For air of $T = 25°C$ and $T_d = 18°C$ at $P = 100$ kPa, find T_w. Use $\Gamma_s = 4.42$ °C km^{-1} in Normand's Rule.

Find the Answer

Given: $P = 100$ kPa, $T = 25°C$, $T_d = 18°C$,
$\Gamma_s = 4.42$ °C km^{-1}
Find: $T_w = ?°C$

According to Normand's Rule:
First, use eq. (4.16):
$z_{LCL} = (0.125 \text{ km °C}^{-1})\cdot(25 - 18°C) = 0.875$ km.
Next, use eq. (4.20):
$T_{LCL} = 25 - (9.8 \text{ K km}^{-1})\cdot(0.875 \text{ km}) = 16.43$ °C.
Finally, use eq. (4.21):
$T_w = 16.43 + (4.42 \text{ °C km}^{-1})\cdot(0.875 \text{ km}) = \underline{\mathbf{20.3°C}}$.

Check: Physics and units are reasonable.
Exposition: The resulting wet-bulb depression is $(T - T_w) = 4.7°C$.

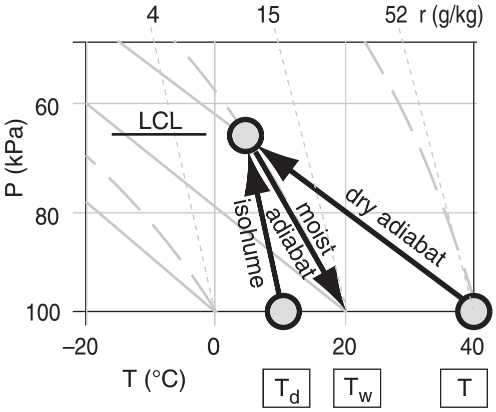


Figure 4.6
Demonstration of Normand's rule on a thermo diagram. It shows how to find wet-bulb temperature T_w, given T and T_d.

INFO • Summary of Humidity Variables

e = vapor pressure (kPa)
r = mixing ratio (g kg^{-1})
q = specific humidity (g kg^{-1})
ρ_v = absolute humidity (g m^{-3})
RH = relative humidity (%)
z_{LCL} = lifting condensation level (km)
T_d = dewpoint (temperature) (°C)
T_w = wet-bulb temperature (°C)

Notes:
- subscript s denotes saturation.
- most thermodynamic equations require temperatures to be converted into Kelvin.
- most thermo eqs. require mixing ratio in g g^{-1}.

You can obtain T_w from T_d via **Normand's Rule**:

- First: Find z_{LCL} using eq. (4.16).
- Next: $$T_{LCL} = T - \Gamma_d \cdot z_{LCL} \tag{4.20}$$
- Finally: $$T_w = T_{LCL} + \Gamma_s \cdot z_{LCL} \tag{4.21}$$

where the dry lapse rate is $\Gamma_d = 9.8$ K km^{-1} and the moist (saturated) lapse rate is Γ_s (magnitude varies, as explained later in this chapter). If you hypothetically lift an air parcel to its lifting condensation level, then its new temperature is T_{LCL}. The wet-bulb temperature is always constrained between the dry-bulb temperature and the dewpoint: $T_d \le T_w \le T$.

Normand's rule is easy to implement on a thermo diagram. Although isohumes and moist adiabats on thermo diagrams are not introduced until later in this chapter, I demonstrate Normand's rule here for future reference. Follow a dry adiabat up from the given dry-bulb temperature T, and follow an isohume up from the given dew point T_d (Fig. 4.6). At the LCL (where these two isopleths cross), follow a moist adiabat back down to the starting pressure to give the wet-bulb temperature T_w.

Converting Between Humidity Variables

Be sure to use g g^{-1} for q and r in the following equations:

$$q = \frac{r}{1+r} \tag{4.22}$$

$$q = \frac{\rho_v}{\rho_d + \rho_v} \tag{4.23}$$

$$q = \left(\frac{\varepsilon \cdot e_o}{P}\right)\cdot \exp\left[\frac{L}{\Re_v}\cdot\left(\frac{1}{T_o} - \frac{1}{T_d}\right)\right] \tag{4.24}$$

$$e = \frac{r}{\varepsilon + r}\cdot P \tag{4.25}$$

$$r = \frac{\rho_v}{\rho_d} \tag{4.26}$$

Also, when humidities are not high:

$$r \approx q \qquad \bullet(4.27a)$$

$$r_s \approx q_s \qquad \bullet(4.27b)$$

TOTAL WATER

Liquid and Solid Water

In clouds, fog, or air containing falling precipitation, one measure of the amount of liquid water in the air is the **liquid water content** (**LWC**). It is defined as

$$\rho_{LWC} = \frac{m_{liq.water}}{Vol} \qquad (4.28)$$

where $m_{liq.water}$ is mass of liquid water suspended or falling through the air, and *Vol* is the air volume. Typical values in cumulus clouds are $0 \le \rho_{LWC} \le 5$ g m^{-3}. It can also be expressed in units of kg$_{\text{liq.water}}$ m^{-3}. LWC is the liquid-water analogy to the absolute humidity for water vapor.

Another measure is the **liquid-water mixing ratio**:

$$r_L = \frac{m_{liq.water}}{m_{dry\ air}} \qquad (4.29)$$

where $m_{liq.water}$ is the mass of liquid water that is imbedded as droplets within an air parcel that contains $m_{dry\ air}$ mass of dry air. A similar **ice mixing ratio** can be defined:

$$r_i = \frac{m_{ice}}{m_{dry\ air}} \qquad (4.30)$$

Both mixing ratios have units of kg$_{\text{water}}$ kg$_{\text{air}}^{-1}$, or g$_{\text{water}}$ kg$_{\text{air}}^{-1}$.

Liquid water content is related to liquid-water mixing ratio by

$$r_L = \frac{\rho_{LWC}}{\rho_{air}} \qquad (4.31)$$

where ρ_{air} is air density.

Mixing Ratio of Total Water

The **total-water mixing ratio** r_T is defined as the sum of masses of all phases of water (vapor, liquid, solid) per dry-air mass:

$$r_T = r + r_L + r_i \qquad \bullet(4.32a)$$

where r is **mixing ratio for water-vapor**, r_L is **mixing ratio for liquid-water**, and r_i is **mixing ratio for ice**. Be sure to use common units for all terms in this equation; namely kg$_{\text{water}}$ kg$_{\text{air}}^{-1}$, or g$_{\text{water}}$ kg$_{\text{air}}^{-1}$. Total-water absolute humidity and total-water specific humidity are similarly defined.

Sample Application

Find the liquid water mixing ratio in air at sea level, given a liquid water content of 3 g m^{-3}.

Find the Answer

Given: ρ_{LWC} = 3 g$_{\text{water}}$ m^{-3}. Sea level.
Find: r_L = ? g$_{\text{water}}$ kg$_{\text{dry air}}^{-1}$.
Assume standard atmosphere, and use Table 1-5 from Chapter 1 to get: ρ_{air} = 1.225 kg$_{\text{air}}$ m^{-3} at sea level.

Use eq. (4.31):
r_L = (3 g$_{\text{water}}$ m^{-3})/(1.225 kg$_{\text{air}}$ m^{-3})
= 2.45 g$_{\text{water}}$ kg$_{\text{dry air}}^{-1}$. = **<u>2.45</u>** g kg^{-1}

Check: Physics, units & magnitude are reasonable.
Exposition: Liquid, solid and water vapor might exist together in a cloud.

Sample Application

At a pressure altitude of 50 kPa inside a thunderstorm, suppose that the air temperature is –5°C, and that each kg of air contains 4 g of water droplets and 2 g of ice crystals suspended in the air. What is the total-water mixing ratio value?

Find the Answer

Given: $P = 50$ kPa, $T = –5°C$, $r_L = 4$ g kg^{-1}, $r_i = 2$ g kg^{-1}
Find: $r_T = ?$ g kg^{-1}.
Assume cloudy (saturated) air inside the thunderstorm.

We need to solve eq. (4.32c), for which we have everything we need except r_s. To use eq. (4.5) to get r_s, we can read $e_s = 0.4222$ kPa at $T = –5°C$ from Table 4-1.

Now apply eq. (4.5): $r_s = \varepsilon \cdot e_s/(P–e_s)$ = (622 g·kg^{-1})·(0.4222 kPa)/(50– 0.4222 kPa) = 5.30 g kg^{-1}.

Finally, apply eq. (4.32c):
$r_T = r_s + r_L + r_i$ = 5.3 + 4 + 2 = **<u>11.3</u>** g·kg^{-1}

Check: Physics and units are reasonable.
Exposition: For altitudes where –40°C < T < 0°C, it is often the case in thunderstorms that both liquid water and ice crystals can be present in the air, along with water vapor.

Sample Application

Find the precipitable water in the troposphere if the bottom portion (below $P = 50$ kPa) has an average total water mixing ratio of 10 g kg^{-1} while the portion of troposphere between 50 and 20 kPa has an average total water mixing ratio of 2 g kg^{-1}.

Find the Answer

Assume: Bottom of troposphere is at 100 kPa.
Given: $r_T = 0.010$ kg kg^{-1}, $P_B = 100$ kPa, $P_T = 50$ kPa,
$r_T = 0.002$ kg kg^{-1}, $P_B = 50$ kPa, $P_T = 20$ kPa
Find: $d_W = ?$ m

Apply eq. (4.33) for the bottom and top segments of the column of tropospheric air, and sum the result.

$$d_W = [(9.8\ \text{m·s}^{-2})\cdot(1000\ \text{kg}_{water}\cdot\text{m}^{-3})]^{-1} \cdot$$
$$\{ (0.010\ \text{kg}_{water}\ \text{kg}_{air}^{-1})\cdot[100 - 50\ \text{kPa}] + (0.002\ \text{kg}_{water}\ \text{kg}_{air}^{-1})\cdot[50 - 20\ \text{kPa}]\ \}$$
$$= [1.02\text{x}10^{-4}\ \text{m}^2\ \text{s}^2\ \text{kg}_{water}^{-1}] \cdot \{(0.5 + 0.06)\ \text{kPa}\cdot\text{kg}_{water}\cdot\text{kg}_{air}^{-1}\}$$
$$= 5.71\text{x}10^{-5}\ \text{m}^2\ \text{s}^2\ \text{kPa}\ \text{kg}_{air}^{-1}\ .$$

To convert the units, use Appendix A info:
1 kPa = 1000 kg$_{air}$·m·s^{-2}
Thus, d_w = [(1000 kg$_{air}$·m·s^{-2})/(1 kPa)] · (5.71x10^{-5} m^2 s^2 kPa kg$_{air}^{-1}$) = **<u>0.057 m</u>**

Check: Physics and units are reasonable.
Exposition: The puddle depth of 5.7 cm is large because the total water mixing ratios were large.

Liquid-water cloud droplets can exist unfrozen in air of temperature less than 0°C. Thus, it is possible for ice and liquid water to co-exist in the same air parcel at the same time, along with water vapor.

Eq. (4.32a) can be simplified if there is no precipitation:

$$r_T = r \qquad \text{if the air is not cloudy} \qquad \bullet(4.32b)$$

$$r_T = r_s + r_L + r_i \qquad \text{if the air is cloudy} \qquad \bullet(4.32c)$$

By "not cloudy" we mean air that is **unsaturated** (i.e., $r < r_s$). By "cloudy" we mean air that is **saturated** (i.e., $r = r_s$) and has either or both liquid water drops and/or ice crystals suspended in it.

Suppose an air parcel has some total number of water molecules in it. Consider an idealized situation where an air parcel does not mix with its environment. For this case, we anticipate that all the water molecules in the parcel must move with the parcel. It makes no difference if some of these molecules are in the form of vapor, or liquid droplets, or solid ice crystals — all the water molecules must still be accounted for.

Hence, for this idealized parcel with no precipitation falling into or out of it, the amount of total water r_T must be constant. Any changes in r_T must be directly associated with precipitation falling into or out of the air parcel.

Suppose that the air parcel is initially unsaturated, for which case we can solve for the total water using eq. (4.32b). If this air parcel rises and cools and can hold less vapor at saturation (see eq. 4.5), it might reach an altitude where $r_s < r_T$. For this situation, eq. (4.32c) tells us that $r_L + r_i = r_T - r_s$. Namely, we can anticipate that liquid water droplets and/or ice crystals suspended in the air parcel must have formed to maintain the constant total number of water molecules.

Precipitable Water

Consider an air column between (top, bottom) altitudes as given by their respective air pressures (P_T, P_B). Suppose all the water molecules within that column were to fall to the bottom of the column and form a puddle. The depth of this puddle (namely, the **precipitable water**) is

$$d_W = \frac{r_T}{|g|\cdot\rho_{liq}}\cdot(P_B - P_T) \qquad \bullet(4.33)$$

where the magnitude of gravitation acceleration is $|g|$= 9.8 m·s^{-2} , the liquid-water density is ρ_{liq} = 1000 kg·m^{-3}, and a column-average of the total-water mixing ratio is r_T . For a column where r_T varies with altitude, split it into column segments each having unique r_T average, and sum over all segments.

Precipitable water is sometimes used as a humidity variable. The bottom of the atmosphere is warmer than the mid and upper troposphere, and can hold the most water vapor. In a pre-storm cloudless environment, contributions to the total-column precipitable water thus come mostly from the boundary layer. Hence, precipitable water can serve as one measure of boundary-layer total water that could serve as the fuel for thunderstorms later in the day. See the Thunderstorm chapters for a sample map of precipitable water.

Note that the American Meteorological Society *Glossary of Meteorology* considers only the water vapor in an air column for calculation of precipitable-water depth. However, some satellites can detect total water over a range of altitudes, for which eq. (4.33) would be applicable.

It is possible to have more precipitation reach the ground during a storm than the value of precipitable water. This occurs where moisture advection by the winds can replenish water vapor in a region.

LAGRANGIAN BUDGETS

Moist air parcels have two additional properties that were unimportant for dry air. One is the amount of water in the parcel, which is important for determining cloud formation and precipitation amounts. The second is the latent heat released or absorbed when water changes phase, which is critical for determining the buoyancy of air parcels and the energy of thunderstorms.

Water Budget

Lagrangian Water Conservation

Suppose that the amount of precipitation falling out of the bottom of an air parcel differs from the amount falling into the air parcel from above. This difference gives a net source or sink S^{**}, causing the total water to change inside the air parcel:

$$\frac{\Delta r_T}{\Delta t} = S^{**} \tag{4.34}$$

For situations where $S^{**} = 0$, then total water conservation requires that :

$$(r + r_i + r_L)_{initial} = (r + r_i + r_L)_{final} \qquad \bullet(4.35a)$$

Sample Application

Suppose an air parcel is stationary at sea level. While there, external processes cause it to cool from 30°C to 5°C. If the air is initially unsaturated with humidity of 15 g kg^{-1}, then what is the final disposition of water molecules in the air parcel? There is no precipitation into or out of the air parcel.

Find the Answer

Given: $(T_{initial}, T_{final})$ = (30, 5 °C), $r_{initial}$ = 15 g kg^{-1}
with $r_{L\ initial} = r_{i\ initial} = 0$ because unsaturated.
Find: $(r, r_L, r_i)_{final}$ = (?, ?, ?) g$_{liq}$ kg$_{air}^{-1}$

Because the final parcel temperature is warmer than freezing, we can assume **no ice**: $r_{i\ final}$ **= 0**. So this leaves r_{final} and $r_{L\ final}$ to be determined with eq. (4.35b):

$$(15 + 0) = (r_{final} + r_{L\ final}) \text{ g kg}^{-1}$$

Because this problem is set at sea level, we can skip some calculations by using the data in Table 4-1 for the final value of mixing ratio. At final temperature 5°C, Table 4-1 gives a saturation mixing ratio of $r_{s\ final}$ = 5.408 g$_{liq}$ kg$_{air}^{-1}$. Since this is less than the initial humidity, we know that the air is saturated with **water-vapor** content: $r_{final} = r_{s\ final}$ = **5.408** g$_{liq}$ kg$_{air}^{-1}$.

Finally, solving eq. (4.35b) for $r_{L\ final}$ gives:
$r_{L\ final}$ = 15 g kg^{-1} – 5.408 g kg^{-1} = **9.592** g kg^{-1}

Check: Physics and units are reasonable.
Exposition: Assuming that this final liquid water is suspended in the air as tiny droplets, the result is fog.

Sample Application

Use Fig. 4.7 to answer (**A**) these questions (**Q**).

Find the Answer

Q: What is the saturation mixing ratio for air at $P = 30$ kPa with $T = 20°C$?

A: Follow the $T = 20°C$ green isotherm vertically, and the $P = 30$ kPa green isobar horizontally, to find where they intersect. The saturation mixing ratio (blue diagonal) line that crosses through this intersection is the one labeled: $r_s \approx \underline{\mathbf{50}}$ g kg^{-1}.

Q: What is the actual mixing ratio for air at $P = 30$ kPa with $T_d = -20°C$?

A: Follow the green –20°C isotherm vertically to where it intersects the horizontal $P = 30$ kPa isobar. Interpolating between the blue diagonal lines that are adjacent to this intersection gives an actual mixing ratio of $r \approx \underline{\mathbf{3}}$ g kg^{-1}.

Q: What is the dew-point temperature for air at $P = 60$ kPa with $r = 0.2$ g kg^{-1}?

A: From the intersection of the blue diagonal isohume $r = 0.2$ g kg^{-1} and the green horizontal isobar for $P = 60$ kPa, go vertically straight down to find $T_d \approx \underline{\mathbf{-40°C}}$.

If the warm cloud contains no suspended ice crystals, then:

$$(r + r_L)_{initial} = (r + r_L)_{final} \qquad \bullet(4.35b)$$

Namely, an increase in the amount of water in one phase (ice, liquid, vapor) must be compensated by a decrease on other phases in order to satisfy total-water conservation if there are no sources or sinks. For an adiabatic process (i.e., no mixing of air or transfer of precipitation across the boundary of an air parcel), r_T must be conserved. For this reason, isohumes of total water are included on thermo diagrams.

Isohumes on a Thermo Diagram

Thermo diagrams were introduced as Fig. 3.3 in the Thermodynamics chapter. On that diagram, the **state** of the air was represented by two sets of thin solid green lines: isobars (horizontal lines) for pressure and isotherms (vertical lines) for temperature. To that background we will now add another state line: isohumes (thin dotted blue lines) for moisture state of the air (Fig. 4.7).

These isohumes are overloaded with information. As a "state" line, the isohume gives the saturation mixing ratio r_s at any given temperature and

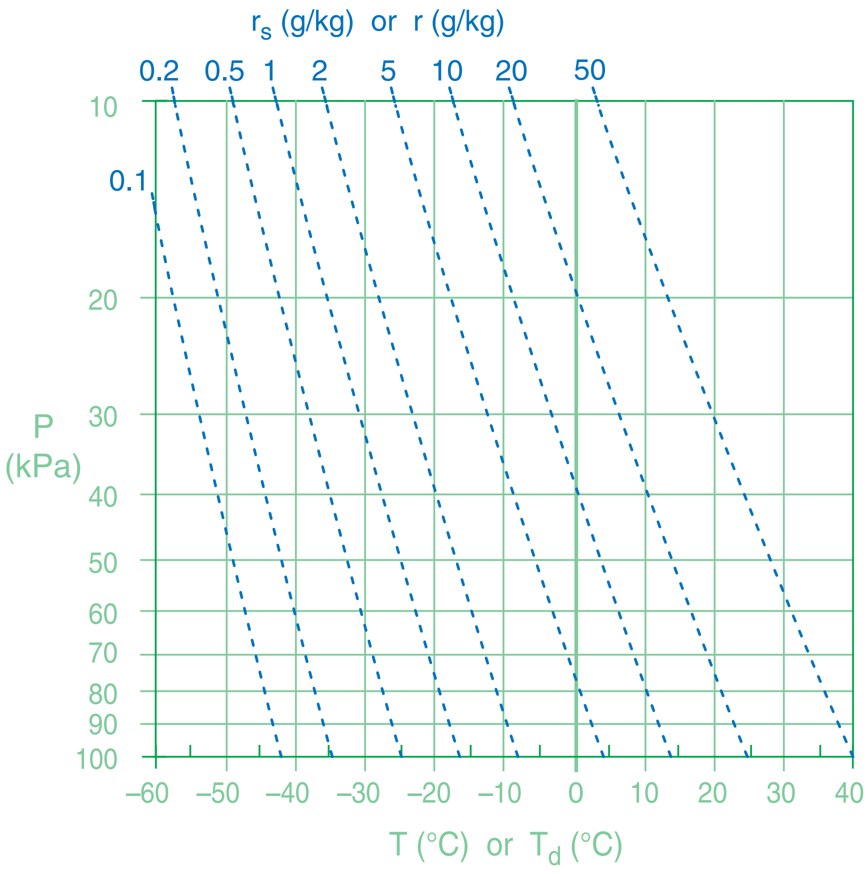


Figure 4.7
Isohumes are dotted blue diagonal lines, isobars are green horizontal lines, and isotherms are green vertical lines on this thermo diagram.

pressure. As a "**process**" line, the isohume shows how total water mixing ratio r_T is conserved for adiabatically rising or sinking air parcels.

To use the thermo-diagram background of P vs. T for isohumes, we need to describe r_s as a function of P and T [abbreviate as $r_s(P, T)$]. Eq. (4.5) gives $r_s(P, e_s)$ and the Clausius-Clapeyron eq. (4.1) gives $e_s(T)$. So combining these two equations gives $r_s(P, T)$.

But to draw any one isohume (i.e., for any one value of r_s), we need to rearrange the result to give $T(P, r_s)$:

$$T = \left[\frac{1}{T_o} - \frac{\Re_v}{L_v} \cdot \ln\left\{ \frac{r_s \cdot P}{e_o \cdot (r_s + \varepsilon)} \right\} \right]^{-1} \tag{4.36}$$

where T_o = 273.15 K, e_o = 0.6113 kPa, $\Re_v/L$ = 0.0001844 K^{-1}, and ε = 0.622 g g^{-1}. This eq. requires that r_s be in g g^{-1} (not g kg^{-1}). T has units of Kelvin.

Thus, pick any fixed r_s to plot. Then, for a range of P from the bottom to the top of the atmosphere, solve eq. (4.36) for the corresponding T values. Plot these T vs. P values as the isohume line on a thermo diagram. Use a spreadsheet to repeat this calculation for other values of r_s, to plot the other isohumes.

Eqs. (4.36) with (4.15b) are similar. Thus, you can use isohumes of $T(P, r_s)$ to also represent isohumes of $T_d(P, r)$. Namely, you can use isohumes to find the saturation state r_s of the air at any P and T, and you can also use the isohumes to describe the process of how T_d changes when an air parcel of constant r rises or descends to an altitude of different P.

Sample Application

Find the dew-point temperature of air having mixing ratio of 10 g kg^{-1} at an altitude where P = 40 kPa.

Find the Answer

Given: r = 10 g kg^{-1} , P = 40 kPa
Find: T_d = ?°C

Apply eq. (4.15b):

$$T_d = \left[\frac{1}{273.15\text{K}} - 0.000184\text{K}^{-1} \cdot \ln\left\{ \frac{(0.01\text{g/g})\cdot(40\text{kPa})}{(0.6113\text{kPa})\cdot(0.01+0.622\text{g/g})} \right\} \right]^{-1}$$

= [(0.003661 K^{-1}) – (0.0001844 K^{-1})·ln{1.035}]$^{-1}$
= [0.003655 K^{-1}]$^{-1}$ = 273.6 K ≈ **0.5°C**

Check: Physics and units are reasonable.
Exposition: You could have saved a lot of time and effort by just looking up the answer in Fig. 4.7. Namely, find the point where the 10 g kg^{-1} mixing-ratio diagonal line intersects the 40 kPa isobar, and then read vertically straight down to find T_d.

We would have found the same numerical answer if we asked for the temperature corresponding to air at 40 kPa that is saturated with r_s = 10 g kg^{-1} with r_L = 0. For this situation, we would have used eq. (4.36).

Heat Conservation for Saturated Air

Moist Adiabatic Lapse Rate

Saturated air is air that is foggy or cloudy, with an amount of water vapor equal to the maximum amount that air can hold given the parcel's temperature. The air parcel contains both vapor and small suspended liquid water droplets or ice crystals. Consider an adiabatic process for which: (1) the parcel does not mix with its surroundings; (2) there is no transfer of heat to/from the parcel from outside; and (3) no liquid (or solid) water falls out of or into the parcel.

An air parcel rising adiabatically has two competing processes that affect its temperature. As for the unsaturated parcel, the saturated parcel expands into regions of lower pressure, doing work on the atmosphere at the expense of thermal energy stored in the parcel. But the colder air parcel can hold less water vapor at saturation than it carried up from the altitude below. So more of the vapor condenses, for which: (1) the latent heating partially offsets the expansion cooling; and (2) the amount of condensed water droplets or ice increases.

The converse occurs for adiabatic descent, where cooling from evaporating liquid-water droplets partially offsets warming due to adiabatic compression.

For air rising across height increment Δz, a saturated parcel has less temperature decrease ΔT than does a dry (unsaturated) parcel. Conversely, for air descending across height increment Δz, a saturated parcel has less temperature increase ΔT than does a dry parcel.

While we previously saw that the dry adiabatic lapse rate was constant ($\Gamma_s = -\Delta T/\Delta z$ = 9.8 °C km^{-1}), we are not so lucky for the saturated lapse rate, which varies with altitude and air temperature.

A saturated air parcel that rises adiabatically has a temperature decrease with increasing height of $-\Delta T/\Delta z = \Gamma_s$, where Γ_s **saturated (or moist) adiabatic lapse rate**. This rate is given by:

$$\Gamma_s = \frac{|g|}{C_p} \cdot \frac{\left(1 + \frac{r_s \cdot L_v}{\Re_d \cdot T}\right)}{\left(1 + \frac{L_v^{\,2} \cdot r_s \cdot \varepsilon}{C_p \cdot \Re_d \cdot T^2}\right)} \tag{4.37a}$$

Sample Application

What is the value of moist-adiabatic lapse rate at $T = 10°C$ and $P = 70$ kPa? Do this calculation using both (a) $C_p = C_{p\ dry}$ and (b) the actual moist C_p. (c) Also find $\Delta T/\Delta P$ for a rising saturated air parcel.

Find the Answer

Given: $P = 70$ kPa, $T = 10°C = 283.15$ K ,

Find: Γ_s = ? °C km^{-1}

(a) Apply eq. (4.37b). But this eq. needs r_s, which we first can find using eq. (4.5). In turn, that eq. needs e_s, which we can get from Table 4-1: $e_s = 1.232$ kPa at $T = 10°C$. Plug this into eq. (4.5): $r_s = (0.622\ \text{g/g})\cdot(1.232\ \text{kPa}) / (70\text{kPa} - 1.232\ \text{kPa}) = 0.01114$ g/g.

Finally, use this in eq. (4.37b):

$$\Gamma_s = \frac{(9.8\frac{\text{K}}{\text{km}})\cdot\left[1+\frac{(0.01114\text{g/g})\cdot(8711\text{K})}{283.15\text{K}}\right]}{\left[1+\frac{(1.35\times10^7\,\text{K}^2)\cdot(0.01114\text{g/g})}{(283.15\text{K})^2}\right]}$$

Γ_s = (9.8 K km^{-1})·[1.3427] / [2.8758] = **<u>4.58 K km</u>$^{-1}$**.

(b) Apply eq. (3.3) for saturated air (for which $r = r_s$):

$C_p = C_{p\ dry}\cdot[1 + 1.84r]$ = (1004 J·kg^{-1}·K^{-1})·[1+1.84·(0.01114g/g] = 1024.6 J·kg^{-1}·K^{-1}.

Thus: $|g|/C_p$ = 9.565 K km^{-1}

L_v/C_p = 2440. K.

$L_v/\Re_d$ = 8711 K.

When these are applied in eq. (4.37a), the result is:

$$\Gamma_s = \frac{(9.565\frac{\text{K}}{\text{km}})\cdot\left[1+\frac{(0.01114\text{g/g})\cdot(8711\text{K})}{283.15\text{K}}\right]}{\left[1+\frac{(2440\text{K})\cdot(8711\text{K})\cdot(0.01114\text{g/g})\cdot0.622}{(283.15\text{K})^2}\right]}$$

Γ_s = (9.565 K km^{-1})·[1.3427]/[2.837] = **<u>4.53 K km</u>$^{-1}$**

(c) If we assume $C_p \approx C_{p\ dry}$, then apply eq. (4.38b):

$$\frac{\Delta T}{\Delta P} = \frac{[0.28571\cdot(283.15\text{K})+(2488.4\text{K}\cdot(0.01114\text{g/g}))]}{P\cdot\left[1+[1.35\times10^7\,K^2\cdot(0.01114\text{g/g})/(283.15\text{K})^2]\right]}$$

= [108.62 / (1 + 1.876)] / P = (37.77K) / P

At $P = 70$ kPa the result is = $\Delta T/\Delta P$ = **<u>0.54 K kPa</u>$^{-1}$**

Check: Physics and units are reasonable.

Exposition: Don't forget for answers (a) and (b) that lapse rates are the rate of temperature <u>decrease</u> with altitude. Parts (a) and (b) give nearly identical answers, implying that eq. (4.37b) is sufficiently accurate for most applications.

Normally, pressure <u>de</u>creases as altitude increases, thus answer (c) also gives cooling for negative ΔP.

Compare typical saturated adiabatic lapse-rate values of 4 to 7°C km^{-1} to the dry adiabatic lapse rate of 9.8°C km^{-1}.

Instead of a change of temperature with height, this saturated adiabatic lapse rate can be rewritten as a change of temperature ΔT with change of pressure ΔP:

$$\frac{\Delta T}{\Delta P} = \frac{\left[\left(\Re_d / C_p\right)\cdot T + \left(L_v / C_p\right)\cdot r_s\right]}{P\cdot\left(1+\frac{L_v{}^2\cdot r_s\cdot\varepsilon}{C_p\cdot\Re_d\cdot T^2}\right)} \qquad (4.38a)$$

For the equations above, don't forget that specific heat C_p varies with humidity (see eq. 3.2).

After plugging in the values for the thermodynamic constants and assuming $C_p \approx$ constant, eq. (4.37a) can be simplified as:

$$\Gamma_s = \Gamma_d \cdot \frac{[1+(a\cdot r_s / T)]}{[1+(b\cdot r_s / T^2)]} \qquad \bullet(4.37b)$$

where $a = 8711$ K, $b = 1.35\text{x}10^7$ K^2, and Γ_d = 9.8 K km^{-1}. Eq. (4.37b) differs from (4.37a) by roughly 1%, so it is often accurate enough for most applications. Use g g^{-1} for mixing ratio, and use Kelvin for temperature. Be aware that r_s is not constant, but is a function of temperature.

Similarly, eq. (4.38a) simplifies to:

$$\frac{\Delta T}{\Delta P} = \frac{[a\cdot T + c\cdot r_s]}{P\cdot[1+(b\cdot r_s / T^2)]} \qquad \bullet(4.38b)$$

with $a = 0.28571$, $b = 1.35\text{x}10^7$ K^2 , and $c = 2488.4$ K.

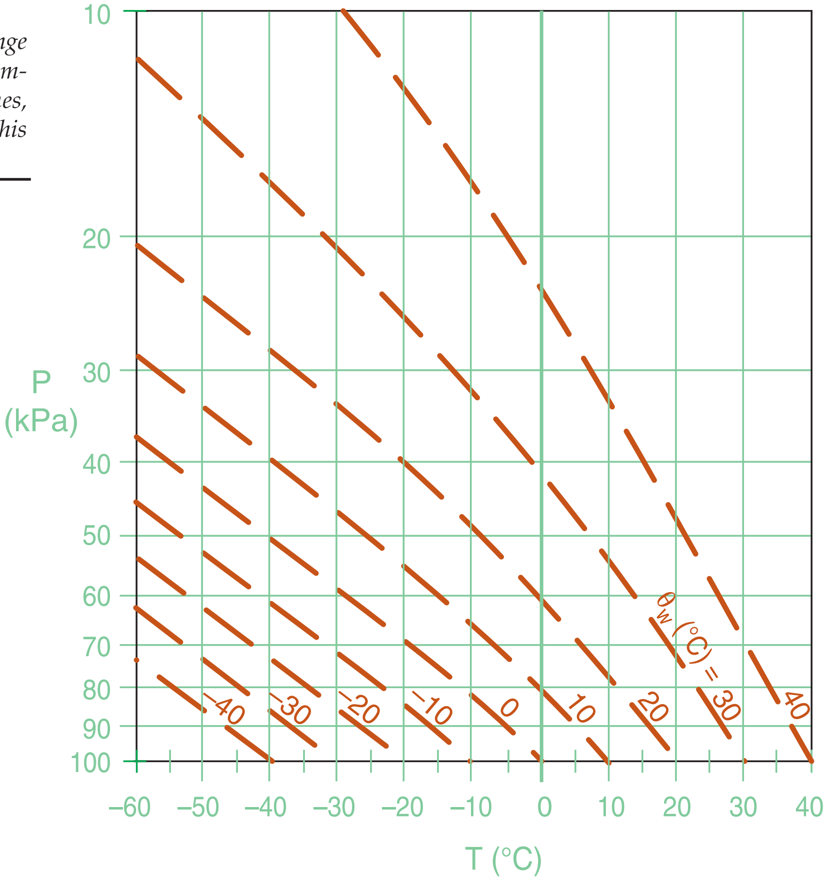
Moist Adiabats on a Thermo Diagram

In the Thermodynamics chapter we discussed the process of "dry" adiabatic vertical motion, where "dry" mean <u>un</u>saturated humid air. We had plotted those process lines as the dry adiabats in the thermo diagram of Fig. 3.4.

We can now use eq. (4.38) to calculate and plot the corresponding **moist adiabats** (also called **saturated adiabats**) that apply for saturated (cloudy or foggy) vertical motion. The saturated adiabats are rather complicated to calculate, because the equations above give the slope ($\Delta T/\Delta P$) for the moist adiabat rather than the desired value of T at each P. But don't despair — we can still find the moist-adiabat curves by iterating each curve upward.

Start at $P = 100$ kPa with some initial value for T. First use these with eq. (4.5) to get r_s. Use these in eq. (4.38) to get $\Delta T/\Delta P$. Then apply that over a small

Figure 4.8 (at right)
Moist (saturated) adiabats are thick dashed orange diagonal lines labeled by wet-bulb potential temperature θ_w*. Isobars are green horizontal lines, and isotherms are green vertical lines on this thermo diagram.*



increment of pressure (such as $\Delta P = P_2 - P_1 = -0.2$ kPa) to solve for the new T_2 at P_2 using:

$$T_2 = T_1 + \frac{\Delta T}{\Delta P}\cdot(P_2 - P_1) \qquad (4.39)$$

Repeat by using the new T_2 at P_2 to find the new r_{s2}, and use all these numbers to solve for a new $(\Delta T/\Delta P)_2$, and use eq. (4.39) again to take the next step. Repeat to iterate your way up the moist adiabat. The result is one of the curves in Fig. 4.8.

To get other curves, start over with a different initial value of T at P = 100 kPa, as shown in Fig. 4.8. Each of those curves is identified by its initial T at the reference pressure of 100 kPa. The next section shows that these labels are called wet-bulb potential temperature θ_w.

INFO • Create Your Own Thermo Diagram — Part 2: Moist Adiabats

As for the dry adiabats, it is useful to see how moist adiabats can be calculated using a computer spreadsheet. The following example is for the moist adiabat that starts at T = 30°C at P = 100 kPa.

	A	B	C	D	E
1	P (kPa)	T (°C)	e_s (kPa)	r_s (g/g)	ΔT/ΔP
2	eq:	(4.39)	(4.1a)	(4.5)	(4.38b)
3	**10.0**	–70.98	0.0006	0.0000	5.7176
4	10.2	–69.86	0.0007	0.0000	5.6297
5	10.4	–68.75	0.0008	0.0000	5.5440
6	10.6	–67.66	0.0009	0.0001	5.4603
...	...	...	...	...	...
450	99.4	29.82	4.3114	0.0282	0.3063
451	99.6	29.88	4.3270	0.0282	0.3056
452	99.8	29.94	4.3426	0.0283	0.3050
453	100.0	**30.00**	4.3582	0.0283	0.3043

In row 1, label the variables at the top of the 5 columns as I have done here. In row 2, identify which equation numbers you are using, as documentation for you or others who use your calculations later.

(continues in next column)

INFO • Moist Adiabats (continuation)

In row 3 column A, enter the pressure at the top of the atmospheric column of interest: 10.0 (kPa), shown in **red** in this example. In row 4 column A type the next pressure 10.2 (kPa), which is the starting pressure plus increment ΔP = 0.2 kPa. Then use automatic series generating methods in your spreadsheet to extend this series down to the point where the pressure is 100.0 (kPa), (on row 453 in my spreadsheet).

Next, in row 453 column B, type in the starting temperature of **30** (°C) for this moist adiabat. In (row, col) = (453, C), use the Clausius-Clapeyron eq. (4.1a) to calculate the saturation vapor pressure for the pressure in column A, and don't forget to convert temperature to Kelvin in your spreadsheet equation. Similarly, use eq. (4.5) to calculate the saturation mixing ratio in cell (453, D). Use eq. (4.38b) to find the moist-adiabat slope in cell (453, E). Again, use Kelvin in your eqs.

For the next row up, but only in cell (452,B), use eq. (4.39) to find the new temperature along the moist adiabat. Be sure to check that the sign is correct (i.e., that temperature is decreasing, not increasing). Then, use the "fill-up" spreadsheet command to fill the eqs. from cells (453, C through E) up one row. Finally, use the fill-up command to fill the eqs. from cells (452, B through E) up to the top pressure level in row 3.

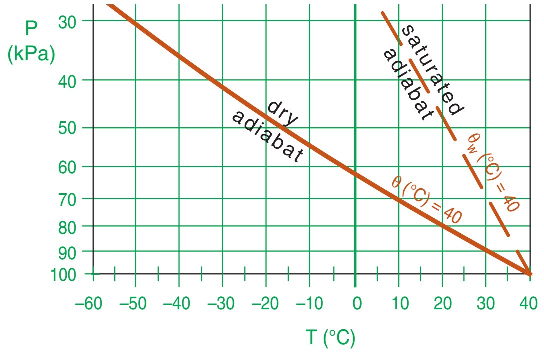
Check: Agrees with Fig. 4.8 for θ_w = 30°C.

Sample Application

What lines do dry & saturated air parcels follow on a thermo diagram if they start at P= 100 kPa, T= 40°C?

Find the Answer

Given: P = 100 kPa, T = 40°C initially.
Plot adiabatic process lines for θ = 40°C & θ_w = 40°C.
Copying the lines from Figs. 3.3 & 4.8.



Exposition: Even starting with the same temperature, a rising saturated air parcel becomes warmer than an unsaturated parcel due to latent-heat release.

Sample Application

Air at pressure 80 kPa and T = 0°C is saturated, & holds 2 g kg^{-1} of liquid water. Find θ_e and θ_L.

Find the Answer:

Given: P = 80 kPa, T = 0°C = 273.15K , r_L = 2 g kg^{-1}.
Find: θ_e = ? °C, θ_L = ? °C

First, do preliminary calculations shared by both eqs:
Rearrange eq. (3.12) to give:
$(\theta/T) = (P_o/P)^{0.28571} = (100\text{kPa}/80\text{kPa})^{0.28571} = 1.066$
Thus, θ = 291K ≈ 18°C
At 80 kPa and 0°C, solve eq. (4.5) for r_s = 4.7 g kg^{-1}
Then use eq. (3.2):
$C_p = C_{pd}\cdot(1 + 1.84\cdot r) = (1004.67\ \text{J}\cdot\text{kg}^{-1}\cdot\text{K}^{-1})$
$\cdot[1+1.84\cdot(0.0047\text{ g g}^{-1})] = 1013.4\ \text{J}\cdot\text{kg}^{-1}\cdot\text{K}^{-1}$.
Thus, $L_v/C_p = 2467\ \text{K}/(\text{g}_{\text{water}}\ \text{g}_{\text{air}}^{-1})$
and $(L_v/C_p)\cdot(\theta/T) \approx 2630\ \text{K}/(\text{g}_{\text{water}}\ \text{g}_{\text{air}}^{-1})$

Use eq. (4.40):
$\theta_e = (18°\text{C}) + (2630\ \text{K}/(\text{g}_{\text{water}}\ \text{g}_{\text{air}}^{-1}))\cdot(0.0047)$
= **<u>30.4 °C</u>**

Use eq. (4.41):
$\theta_L = (18°\text{C}) - (2630\ \text{K}/(\text{g}_{\text{water}}\ \text{g}_{\text{air}}^{-1}))\cdot(0.002)$
= **<u>12.7 °C</u>**

Check: Physics, units, & magnitude are reasonable.
Exposition: The answers are easier to find using a thermo diagram (after you've studied the Stability chapter). For θ_e , find the θ value for the dry adiabat that is tangent at the diagram top to the moist adiabat. For θ_L, follow a moist adiabat down to where it crosses the (2 + 4.7 = 6.7 g kg^{-1}) isohume, and from there follow a dry adiabat to P = 100 kPa.

Wet-Bulb, Equivalent, and Liquid-Water Potential Temperatures

Recall from the Thermodynamics chapter that potential temperature θ is conserved during unsaturated adiabatic ascent or descent. However, if an air parcel containing water vapor is lifted above its LCL, then condensation will add latent heat, causing θ to increase. Similarly, if the air contains liquid water such as cloud drops, when it descends some of the drops can evaporate, thereby cooling the air and reducing θ.

However, we can define new variables that are conserved for adiabatic ascent or descent, regardless of any evaporation or condensation that might occur. One is the **equivalent potential temperature** θ_e:

$$\theta_e \approx \theta + \left(\frac{L_v \cdot \theta}{C_p \cdot T}\right)\cdot r \quad \approx \quad \theta + \frac{L_v}{C_{pd}}\cdot r \qquad \bullet(4.40)$$

Another is **liquid water potential temperature**, θ_L:

$$\theta_L \approx \theta - \left(\frac{L_v \cdot \theta}{C_p \cdot T}\right)\cdot r_L \quad \approx \quad \theta - \frac{L_v}{C_{pd}}\cdot r_L \qquad \bullet(4.41)$$

where L_v = 2.5x10^6 J·kg^{-1} is the latent heat of vaporization, C_p is the specific heat at constant pressure for air (C_p is not constant, see the Thermodynamics chapter), T is the absolute temperature of the air, and mixing ratios (r and r_L) have units of ($\text{g}_{\text{water}}\ \text{g}_{\text{air}}^{-1}$). The last approximation in both equations is very rough, with L_v/C_{pd} = 2.5 K·($\text{g}_{\text{water}}/\text{kg}_{\text{air}})^{-1}$.

Both variables are conserved regardless of whether the air is saturated or unsaturated. Consider <u>unsaturated</u> air, for which θ is conserved. In eq. (4.40), water-vapor mixing ratio r is also conserved during ascent or descent, so the right side of eq. (4.40) is constant, and θ_e is conserved. Similarly, for unsaturated air, liquid water mixing ratio r_L = 0, hence θ_L is also conserved in eq. (4.41).

For <u>saturated</u> air, θ will increase in a rising air parcel due to latent heating, but r will decrease as some of the vapor condenses into liquid. The two terms in the right side of eq. (4.40) have equal but opposite changes that balance, leaving θ_e conserved. Similarly, the two terms on the right side of eq. (4.41) balance, due to the minus sign in front of the r_L term. Thus, θ_L is conserved.

By subtracting eq. (4.41) from (4.40), we can see how θ_e and θ_L are related:

$$\theta_e \approx \theta_L + \left(\frac{L_v\cdot\theta}{C_p\cdot T}\right)\cdot r_T \qquad (4.42)$$

for a total-water mixing ratio (in g g^{-1}) of $r_T = r + r_L$. Although θ_e and θ_L are both conserved, they are <u>not</u> equal to each other.

We can use θ_e or θ_L to identify and label moist adiabats. Consider an air parcel starting at P = 100 kPa that is saturated but contains no liquid water ($r = r_s = r_T$). For that situation θ_L is equal to its initial temperature T (which also equals its initial potential temperature θ at that pressure). A rising air parcel from this point will conserve θ_L, hence we could label the moist adiabat with this value (Fig. 4.9).

An alternative label starts from same saturated air parcel at P = 100 kPa, but conceptually lifts it to the top of the atmosphere (P = 0). All of the water vapor will have condensed out at that end point, heating the air to a new potential temperature. The potential temperature of the dry adiabat that is tangent to the top of the moist adiabat gives θ_e (Fig. 4.9). [*CAUTION: On some thermo diagrams, equivalent potential temperature is given in units of Kelvin.*]

In other words, θ_L is the potential temperature at the bottom of the moist adiabat (more precisely, at P = 100 kPa), while θ_e is the potential temperature at the top. Either labeling method is fine — you will probably encounter both methods in thermo diagrams that you get from around the world.

Wet-bulb potential temperature (θ_w) can also be used to label moist adiabats. For θ_w, use Normand's rule on a thermo diagram (Fig. 4.10). Knowing temperature T and dew-point T_d at initial pressure P, and plot these points on a thermo diagram. Next, from the T point, follow a dry adiabat up, and from the T_d point, follow an isohume up. Where they cross is the lifting condensation level LCL.

From that LCL point, follow a saturated adiabat back down to the starting altitude, which gives the **wet-bulb temperature** T_w. If you continue to follow the saturated adiabat down to a reference pressure (P = 100 kPa), the resulting temperature is the **wet-bulb potential temperature** θ_w (see Fig. 4.10). Namely, θ_w equals the θ_L label of the moist adiabat that passes through the LCL point. Labeling moist adiabats with values of wet-bulb potential temperature θ_w is analogous to the labeling of dry adiabats with θ, which is why I use θ_w here.

To find θ_e(K) for a moist adiabat if you know its θ_w(K), use:

$$\theta_e = \theta_w \cdot \exp(a_3 \cdot r_s / \theta_w)_o \tag{4.43a}$$

where a_3 = 2491 K·kg$_{air}$ kg$_{vapor}^{-1}$, and r_s is initial saturation mixing ratio (kg$_{vapor}$ kg$_{air}^{-1}$) at $T = \theta_w$ and P = 100 kPa (as denoted by subscript "o"). You can approximate (4.43a) by

$$\theta_e(\text{K}) \approx a_o + a_1 \cdot \theta_w(°\text{C}) + a_2 \cdot [\theta_w(°\text{C})]^2 \tag{4.43b}$$

where a_o = 282, a_1 = 1.35, and a_2 = 0.065, for θ_w in the range of 0 to 30°C (see Fig. 4.11). Also the θ_L label for the moist adiabat passing through the LCL equals this θ_w.

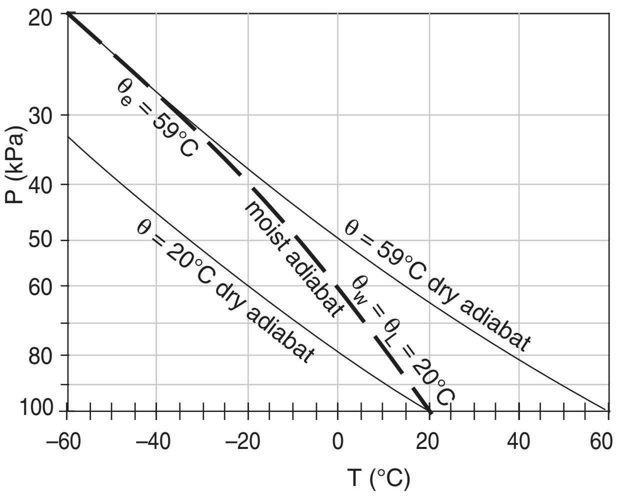


Figure 4.9
Comparison of θ_L , θ_w and θ_e values for the same moist adiabat.

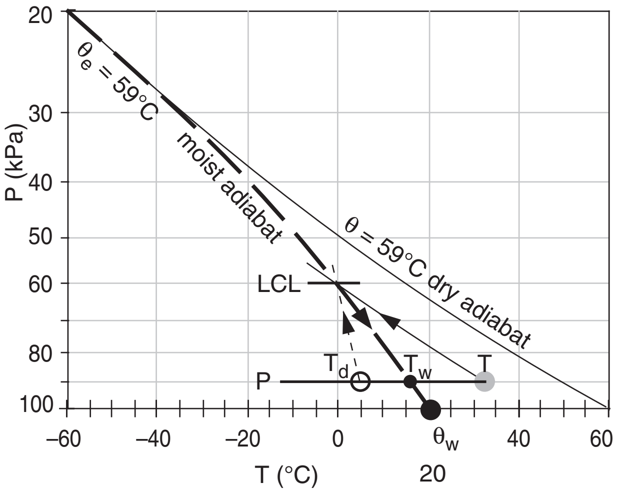


Figure 4.10
Thermo diagram showing how to use Normand's Rule to find wet-bulb potential temperature θ_w , and how it relates to θ_e.

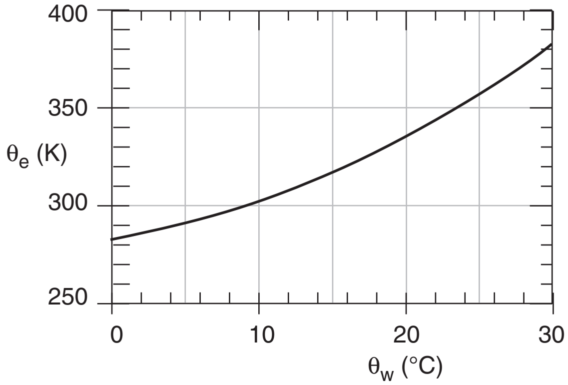


Figure 4.11
Approximate relationship between θ_w and θ_e.

Sample Application

Verify the labels on the moist adiabat that passes through the LCL in Fig. 4.10, given starting conditions $T = 33°C$ and $T_d = 5.4°C$ at $P = 90$ kPa.

Find the Answer

Given: $T = 33°C \approx 306K$, $T_d = 5.4°C \approx 278.4K$

Find: θ_e , θ_w and θ_L labels (°C) for the moist adiabat

First, find the initial θ, using eq. (3.12)

$\theta = T{\cdot}(P_o/P)^{0.28571} = (306K){\cdot}(100kPa/90kPa)^{0.28571}$

$= 315.4K = 42.4°C$

Next, find the mixing ratio using eq. (4.1b) & (4.4):

$e = 0.6114kPa{\cdot}\exp[5423{\cdot}(1/273.15 - 1/278.4)] = 0.889kPa$

$r \approx (622g\ kg^{-1}){\cdot}(0.889kPa)/[90{-}0.889kPa] \approx 6.2g\ kg^{-1}$

Next, use eq. (3.2) to find $C_p = C_{pd}{\cdot}(1 + 1.84{\cdot}r)$

$C_p = (1004.67\ J{\cdot}kg^{-1}{\cdot}K^{-1}) \cdot [1{+}1.84{\cdot}(0.0062\ g\ g^{-1})]$

$C_p = 1016.1\ \ J{\cdot}kg^{-1}{\cdot}K^{-1}$

Solve the more accurate version of eq. (4.40):

$\theta_e = (42.4°C) + \{(2500\ J\ g_{water}{}^{-1}){\cdot}(315.4K)/$

$[(1016.1\ \ J{\cdot}kg^{-1}{\cdot}K^{-1}){\cdot}(306K)]\} \cdot (6.2\ g_{water}\ kg_{air}{}^{-1})$

$= (42.4°C) + (2.536\ K{\cdot}kg_{air}\ g_{water}{}^{-1}){\cdot}(6.2\ g_{water}\ kg_{air}{}^{-1})$

$=$ **<u>58.1°C</u>** $= 331.1$ K

The approximate version of eq. (4.40) gives almost the same answer, and is much easier:

$\theta_e = (42.4°C) + (2.5\ K{\cdot}kg_{air}\ g_{water}{}^{-1}){\cdot}(6.2\ g_{water}\ kg_{air}{}^{-1})$

$=$ **<u>57.9°C</u>**

Eq. (4.43b) is a quadratic eq. that can be solved for θ_w. Doing this, and then plugging in $\theta_e = 331.1$ K gives:

$\theta_w \approx$ **<u>19°C</u>**. , which is the label on the moist adiabat.

Using $\theta_L \approx \theta_w$:

$\theta_L \approx$ **<u>19°C</u>**.

Check: Physics and units are reasonable.

Exposition: These values are within a couple degrees of the labels in Fig. 4.10. Disappointing that they aren't closer, but the θ_e results are very sensitive to the starting point.

Sample Application

For a moist adiabat of $\theta_w = 14°C$, find its θ_e.

Find the Answer

Given: $\theta_w = 14°C = 287$ K

Find: $\theta_e = ?$ K

First get r_s from Fig. 4.7 at $P = 100$ kPa and $T = 14°C$:

$r_s = 10\ g\ kg^{-1} = 0.010\ kg\ kg^{-1}$.

Next, use eq. (4.43a):

$\theta_e = (287K) \cdot \exp[2491(K{\cdot}kg_{air}\ kg_{vapor}{}^{-1}) \cdot$

$(0.010\ kg_{vapor}\ kg_{air}{}^{-1})\ /\ (287K)\ ] =$ **<u>313K</u>**

Check: Units are reasonable. Agrees with Fig. 4.11.

Exposition: This $\theta_e = 40°C$. Namely, if a saturated air parcel started with $\theta_w = T = 14°C$, and then if all the water vapor condensed, the latent heat released would warm the parcel to $T = 40°C$.

Sample Application

Suppose a psychrometer at 100 kPa measures dry-bulb and wet-bulb temperatures of 30°C and 15°C. Use a thermo diagram to find the values of T_d and r ?

Find the Answer

Given: $P = 100$ kPa, $T = 30°C$, $T_w = 15°C$.

Find: $r = ?\ g\ kg^{-1}$, $T_d = ?°C$

Hint: We can use the opposite of Normand's rule.

We will learn more about thermo diagrams in the next chapter. So this exercise gives us a preview.

To use Normand's rule in reverse, follow a dry adiabat up from the starting dry-bulb temperature, and follow a saturated adiabat up from the starting wet-bulb temperature. Where they cross, follow the isohume down to the starting altitude to find the dew-point temperature, or follow the isohume up to read the isohume's mixing-ratio value.

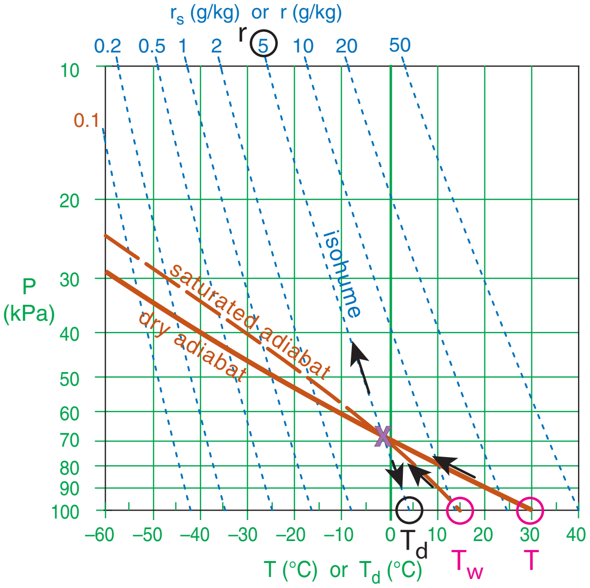
Fig. 3.4 shows many dry adiabats, but not the one we want. So I interpolated between the 20 and 40°C dry adiabats, and copied and pasted the result (as the solid dark-orange line for θ = 30°C) onto copy of Fig. 4.7, as shown below. A magenta circle indicates the starting temperature and pressure.

Fig. 4.8 shows many saturated adiabats, but not the one we want. So I interpolated between the 10 and 20°C saturated adiabats, and copied and pasted the result into the figure below as the dark-orange dashed line for $\theta_w = 15°C$. Another magenta circle indicates the starting wet-bulb temperature and pressure.

Those two lines cross (see purple "X" in the figure) almost exactly on the $r =$ **<u>5 g kg^{-1}</u>** isohume (see black circle at top of diagram. Following that same isohume down to $P = 100$ kPa gives $T_d \approx$ **<u>4°C</u>** (circled in black).

Check: Physics and unit are reasonable.

Exposition: The "X" is the location of the lifting condensation level (LCL), which is about 68 kPa in this diagram.



WATER BUDGET AT A FIXED LOCATION

Picture a cube of air at a fixed location relative to the ground (i.e., an **Eulerian** framework). Inflow and/or outflow of water (as vapor, liquid, or solid) can change the total water content r_T of air inside the hypothetical cube. As we did for heat, we can investigate each inflow and outflow process to determine which are significant. The insignificant ones include turbulent transport in the horizontal, molecular conduction (except close to the ground), and mean vertical advection (except in deep convective storms).

What remains is an approximate equation for the total water budget in an Eulerian framework:

•(4.44)

$$\frac{\Delta r_T}{\Delta t} = -\left[U \cdot \frac{\Delta r_T}{\Delta x} + V \cdot \frac{\Delta r_T}{\Delta y}\right] + \left(\frac{\rho_L}{\rho_d}\right)\frac{\Delta Pr}{\Delta z} - \frac{\Delta F_{z\ turb}(r_T)}{\Delta z}$$

storage *horiz. advection* *precipitation* *turbulence*

where precipitation is falling at rate Pr, horizontal advection is caused by wind components (U, V), the turbulent total-water flux (in kinematic units) is $F_{z\ turb}$ (r_T), and (Δx, Δy, Δz, Δt) are the Cartesian dimensions of the cube and the time interval, respectively.

Define **Standard Temperature and Pressure** (**STP**) as $T = 0°\text{C}$ and $P = 101.325$ kPa. Using air density at STP and liquid water density ($\rho_L = 1000\ \text{kg}_{\text{liq}}\ \text{m}^{-3}$) from Appendix B, you would find that $\rho_L/\rho_d = 836.7\ \text{kg}_{\text{liq}}\ \text{kg}_{\text{air}}^{-1}$, but this value would increase for hypothetical cubes at higher altitudes where air density is smaller .

In the subsections that follow, we will focus on the individual terms in eq. (4.44).

Horizontal Advection

If more total water (vapor + suspended cloud droplets + suspended ice crystals) is blown out of the cube than blows in, then the total water will decrease during time interval Δt.

Precipitation

Some solid and liquid water particles are large enough that they are not suspended in, and do not move with, the mean winds. Instead, they fall through the air as precipitation. If all the liquid and solid precipitation were collected in a rain gauge and then melted to be all liquid, the resulting water depth is called the **liquid water equivalent**. Thus, 2 mm of liquid precipitation plus 5 cm of snow (which might reduce to 5 mm of water when melted) would give a total liquid water depth of 7 mm.

A SCIENTIFIC PERSPECTIVE• Reasoning

"Reasoning" is the method by which we try to explain how things work. Namely, we try to infer an explanation or a theory that is reasonable. Two types of reasoning are inductive and deductive reasoning.

Characteristics of **inductive reasoning**:

- Going from specific examples to general theories.
- The general theory may be likely, but not certain.
- Often requires a "leap of faith".
- Could apply to situations not yet encountered.
- Can be proven wrong by new contrary evidence.
- Analogous to extrapolation.

Characteristics of **deductive reasoning**:

- Applying a general rule to a specific application.
- Confidence that the result is correct and will work well in new situations if used within the bounds of the general rule.
- Is a sound tool used in engineering.
- Analogous to interpolation.

As a contrived example, what if you observed that as adults, your grandparent was 1.4 m tall, your parent was 1.6 m tall, and you are 1.8 m tall. You could deduce that you are 0.2 m taller than your parent. In fact, you would consider this a "truth" based on the logic of mathematics.

However, you could induce that your unborn child will be 2.0 m tall as an adult. Namely, you saw a pattern from past observations: each offspring was 0.2 m taller than its parent. So extrapolating to your future unborn child, you might expect the child to become 0.2 m taller than you.

Obviously this last prediction carries some uncertainty. You could reduce the uncertainty by collecting more evidence from other families, or by going further back in your family tree. However, many factors over which you have little control could alter the growth of your child — your prediction is uncertain.

This same procedure is done by scientists proposing new theories. Namely, gain confidence that your theory is "universal" in the sense that it works at other places and times in the universe. Such a goal motivated Isaac Newton to design and build telescopes, so he could test his theories of motion by looking at other planets in the universe. The evidence convinced scientists to accept Newton's hypotheses as "laws of motion", yet they were later proven wrong by Einstein.

Let's try some inductive reasoning now. Eq. (4.44) says that change of total water depends on horizontal advection, change of turbulent fluxes in the vertical, and other source terms. Eq. (3.51) said that change of temperature depends on horizontal advection, change of turbulent fluxes in the vertical, and other source terms. By inductive reasoning, we might anticipate that the change of momentum (i.e., wind speed) depends on horizontal advection, change of turbulent fluxes in the vertical, and other source terms. We will check this later, in the Forces & Winds chapter.

Sample Application

Precipitation falls out of the bottom of a cloud at rate 0.5 cm h^{-1}. At 200 m below cloud base, the precipitation rate is 0.4 cm h^{-1}. In this 200 m thick layer of air below cloud base, what is the total-water change rate? Assume $\rho_{air} = 1$ kg m^{-3}.

Find the Answer

Given: $Pr_{bot} = 0.4$ cm $h^{-1} = 1.11\text{x}10^{-6}$ m s^{-1},
$Pr_{top} = 0.5$ cm $h^{-1} = 1.39\text{x}10^{-6}$ m s^{-1},
$\rho_{air} = 1$ kg m^{-3}, and
$\Delta z = 200$ m
Find: $\Delta r_T/\Delta t$ = ? (g kg^{-1})/s.

Because no other processes are specified, let's neglect them for simplicity. Thus, we can apply eq. (4.44):

$$\Delta r_T/\Delta t = (\rho_L/\rho_d)\cdot(Pr_{top}-Pr_{bot})/(z_{top}-z_{bot}) =$$

$$= \frac{(1.0\times10^6\,\text{g}_{\text{water}}/\text{m}^3)}{(1\ \text{kg}_{\text{air}}/\text{m}^3)}\cdot\frac{[(1.39-1.11)\times10^{-6}\,\text{m/s}]}{200\text{m}}$$

$$= \underline{0.0014\ (\text{g kg}^{-1})/\text{s}} = 5\ (\text{g kg}^{-1})/\text{h}$$

Check: Physics and units are reasonable.
Exposition: This is typical of **virga**, a weather element where some of the precipitation evaporates before reaching the ground. In virga, the evaporating rain makes the air more humid below cloud base.

Sample Application

Suppose the latent heat flux is 250 W·m^{-2}. What are the values of water-vapor flux, kinematic water-vapor flux and evaporation rate? Assume $\rho_{air} = 1\ \text{kg}_{\text{air}}\ \text{m}^{-3}$.

Find the Answer

Given: $\rho_{air} = 1\ \text{kg}_{\text{air}}\cdot \text{m}^{-3}$, $\mathbb{F}_E = 250\ \text{W}\cdot\text{m}^{-2}$
Find: $\mathbb{F}_{water}$ = ? $\text{kg}_{\text{water}}\cdot\text{m}^{-2}\cdot\text{s}^{-1}$, $Evap$ = ? mm day^{-1} ,
F_{water} = ? $(\text{kg}_{\text{water}}\cdot \text{kg}_{\text{air}}^{-1})\cdot(\text{m}^{-2}\cdot\text{s}^{-1})$

Apply eq. (4.45):
$\mathbb{F}_{water} = (250\ \text{J}\cdot\text{m}^{-2}\cdot\text{s}^{-1})/(2.5\text{x}10^6\ \text{J kg}^{-1})$
$= \underline{0.0001\ \text{kg}}_{\text{water}}\cdot\text{m}^{-2}\cdot\text{s}^{-1}$

Apply eq. (4.48a):
$F_{water} = (0.0001\ \text{kg}_{\text{water}}\cdot\text{m}^{-2}\cdot\text{s}^{-1})/\ (1\ \text{kg}_{\text{air}}\ \text{m}^{-3})$
$= \underline{0.0001}\ (\text{kg}_{\text{water}}\cdot \text{kg}_{\text{air}}^{-1})\cdot(\text{m}\cdot\text{s}^{-1})$

Apply eq. (4.49): $Evap = a\cdot\mathbb{F}_E$
$= [\ 0.0346\ (\text{m}^2\ \text{W}^{-1})\cdot(\text{mm day}^{-1})]\cdot(\ 250\ \text{W m}^{-2}\ )$
$= \underline{8.65\ \text{mm day}}^{-1}$

Check: Physics and units are reasonable. Namely, the first answer has units of water mass per area per time, the second is like a mixing ratio times velocity, and the third is rate of decreased of water depth in a puddle.
Exposition: If this evaporation rate continues day after day, crops could become desiccated.

If more precipitation falls out of the bottom of the air cube than is falling into the top, then this change of precipitation with altitude would tend to reduce the total water in the cube. Thus, the change of **precipitation rate** (Pr in mm h^{-1} or m s^{-1}) between the top and bottom of the cube of air is important.

Not all the precipitation will reach the ground, because some might evaporate on the way down. The precipitation rate (Pr) at the ground is given a special name: the **rainfall rate** (RR).

Moisture Flux at the Earth's Surface

In the previous chapter we discussed ways to estimate sensible and latent heat fluxes at the Earth's surface. But latent heat flux is tied to the movement of water molecules. Here we show how you can use knowledge of the latent heat flux to estimate the moisture flux at the Earth's surface.

Recall that vertical flux is the movement of something across a horizontal unit area per unit time. Let $\mathbb{F}_E$ be the **latent heat flux** in units of ($\text{J}\cdot\text{m}^{-2}\cdot\text{s}^{-1}$ or $\text{W}\cdot\text{m}^{-2}$) and $\mathbb{F}_{water}$ be the **vertical water-vapor flux** in units of ($\text{kg}_{\text{water}}\cdot\ \text{m}^{-2}\cdot\text{s}^{-1}$). The relationship between these fluxes is:

$$\mathbb{F}_{water} = \mathbb{F}_E\ /\ L_v \qquad \bullet(4.45)$$

or

$$\mathbb{F}_{water} = \rho_{air}\cdot(C_p/\ L_v)\cdot F_E \qquad (4.46)$$

or

$$\mathbb{F}_{water} = \rho_{air}\cdot\gamma\cdot F_E \qquad (4.47)$$

Constants and parameters in these equations are:
F_E = kinematic latent heat flux ($\text{K}\cdot\text{m}\cdot\text{s}^{-1}$);
$L_v = 2.5\text{x}10^6\ \text{J kg}_{\text{water}}^{-1}$ = latent heat of vaporization;
$\gamma = C_p/L_v = 0.4\ (\text{g}_{\text{water}}\ \text{kg}_{\text{air}}^{-1})\cdot\text{K}^{-1}$
= **psychrometric constant**.

If you divide these equations by air density (ρ_{air}), then you can get the water flux in kinematic form:

$$F_{water} = \mathbb{F}_{water}\ /\ \rho_{air} \qquad (4.48a)$$

For example, eq. (4.45) becomes:

$$F_{water} = \mathbb{F}_E\ /\ (\ \rho_{air}\cdot L_v\ ) = \gamma\cdot F_E \qquad (4.48b)$$

The advantage of **kinematic water flux** (F_{water}) is that its units are similar to (mixing ratio) x (wind speed). Thus, the units are $(\text{kg}_{\text{water}}\ \text{kg}_{\text{air}}^{-1})\cdot(\text{m}\cdot\text{s}^{-1})$.

When water from a puddle on the Earth's surface evaporates, the puddle depth decreases. The rate of this depth decrease (mm day^{-1}) is the **evaporation rate**, ($Evap$), which is related to the moisture flux:

$$Evap = \mathbb{F}_{water}/\rho_L = \mathbb{F}_E/(\rho_L\cdot L_v) = a\cdot\mathbb{F}_E \qquad (4.49)$$

$$Evap = (\rho_{air}/\rho_L)\cdot F_{water} = (\rho_{air}/\rho_L)\cdot\gamma\cdot F_E \qquad \bullet(4.50)$$

For these equations, ρ_L = 1000 kg_{liq} m^{-3}, a = 0.0346 (m^2 W^{-1})·(mm day^{-1}), and $a = 4.0\text{x}10^{-10}$ $m^3 \cdot W^{-1} \cdot s^{-1}$.

For windy conditions, another way to estimate the water vapor flux (in kinematic units) is with a **bulk-transfer relationship** similar to eq. (3.35), such as

$$F_{water} = C_H \cdot M \cdot \left(r_{sfc} - r_{air} \right) \tag{4.51}$$

where the wind speed at 10 m above the surface is M, and the water-vapor mixing ratio in the air at 2 m above the surface is r_{air}. The **bulk-transfer coefficient** for water vapor is roughly the same as the one for heat, C_H, which ranges between $2\text{x}10^{-3}$ (for smooth surfaces) to $2\text{x}10^{-2}$ (for rough surfaces).

In calmer conditions with sunny skies, convective thermals of warm rising air can form, which are effective at transporting moisture in the vertical. The resulting kinematic water-vapor flux is:

$$F_{water} = b_H \cdot w_B \cdot \left(r_{sfc} - r_{ML} \right) \tag{4.52}$$

for a mid-mixed layer mixing ratio of r_{ML} and a **convective heat-transport coefficient** of $b_H = 5\text{x}10^{-4}$ (dimensionless). Eq. (3.38) in the previous chapter gives the expression for buoyancy-velocity scale w_B.

While the two equations above seem physically reasonable, they have a problem in that the mixing ratio at the surface r_{sfc} (literally at the Earth's surface skin) is an abstract concept that is not measurable. For the special case of a river or lake surface or rain-saturated ground, scientists often approximate $r_{sfc} \approx r_s(T_{sfc})$; namely the surface mixing ratio equals the saturation mixing ratio r_s (eq. 4.5) for air at a temperature T_{sfc} equal to that of the surface skin.

Moisture Transport by Turbulence

Recall from the Thermodynamics chapter that turbulence is the quasi-random movement and mixing of air parcels by swirls and eddies — analogous to the effects of an egg beater. By mixing together dry and moist air parcels, turbulence causes a net moisture transport — namely, a **moisture flux**. If the moisture flux out of the top of a layer of air is different than the moisture flux into the bottom, then there is a **moisture flux divergence** $\Delta F_{z\ turb}(r_T)/\Delta z$ that changes the total water content in the layer.

For hot sunny days with light winds, rising thermals create a mixed layer of depth z_i, within which:

$$\frac{\Delta F_{z\ turb}(r_T)}{\Delta z} = \frac{F_{z\ turb\ at\ zi}(r_T) - F_{z\ turb\ at\ surface}(r_T)}{z_i - z_{surface}} \tag{4.53}$$

where eq. (4.53) says that the local flux divergence equals the average flux divergence across the whole turbulent mixed layer (ML).

Sample Application

A cold front recently passed over your farm field during which it rained hard, but now it is windy with M = 15 m s^{-1}. Air behind the cold front is relatively dry (r_{2m} = 5 g kg^{-1}), but the ground surface is still relatively warm (14°C). Assume your farm is near sea level. What kinematic water-vapor flux do you expect?

Find the Answer

Given: M = 15 m s^{-1}, r_{air} = 5 g kg^{-1}, T_{sfc} = 14°C
Find: F_{water} = ? (kg_{water} kg_{air}^{-1})·(m·s^{-1})

The farm field is probably not perfectly smooth, so let's assume $C_H \approx 5\text{x}10^{-3}$ (dimensionless). But assume the recent rain left the soil nearly saturated with water.

To apply eq. (4.51), we need to estimate $r_{sfc} \approx r_s(T_{sfc})$ = r_s (14°C) = 10.0 g kg^{-1} from eqs (4.1) & (4.5). Thus, eq. (4.51) is:

F_{water} = ($5\text{x}10^{-3}$)·(15 m s^{-1})·(10.0 – 5 g kg^{-1})
= 0.375 (g_{water}·kg_{air})·(m·s^{-1})
= **<u>0.000375</u>** (kg_{water} kg_{air}^{-1})·(m·s^{-1})

Check: Physics and units are reasonable.
Exposition: If you apply eq. (4.50) you find this is associated with an evaporation rate of 40 mm day^{-1}.

Sample Application

What is the value of F_{water} on a sunny day with no winds? Assume mixed-layer depth z_i = 3 km, θ_{ML} = 290 K, and θ_{sfc} = 320 K. The air and ground are somewhat dry, so r_{sfc} = 5 g kg^{-1}, and r_{ML} = 3 g kg^{-1}.

Find the Answer

Given: r_{sfc} = 5 g kg^{-1}, r_{ML} = 3 g kg^{-1}, z_i = 3 km,
θ_{sfc} = 320 K, θ_{ML} = 290 K
Find: F_{water} = ? (kg_{water} kg_{air}^{-1})·(m·s^{-1})

Sunny, calm conditions suggests a convective mixed layer, so we can apply eq. (4.52). The temperature info in this sample application is the same as for a sample application in the Surface Fluxes section of Chapter 3, where we found from eq. (3.38) that $w_B \approx 55.1$ m s^{-1}. Thus, eq. (4.52) is:

F_{water} = ($5\text{x}10^{-4}$)·(55.1 m s^{-1})·(5 – 3 g kg^{-1})
= 0.055 (g_{water}·kg_{air})·(m·s^{-1})
= **<u>$5.5\text{x}10^{-5}$</u>** (kg_{water} kg_{air}^{-1})·(m·s^{-1})

Check: Physics and units are reasonable.
Exposition: If you apply eq. (4.50) you find this is associated with an evaporation rate of 5.83 mm day^{-1} at sea level.

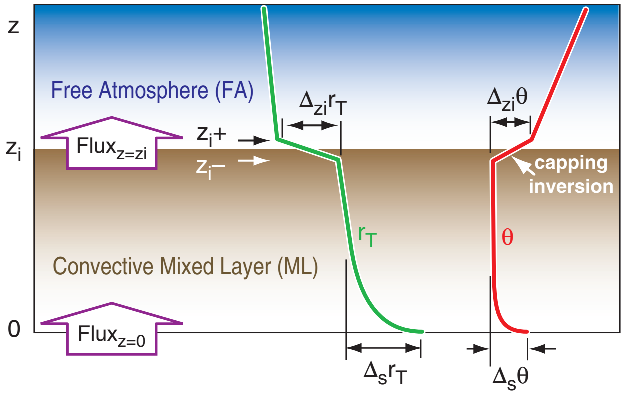


Figure 4.12
Idealized vertical profiles of potential temperature $\Delta\theta$ *and total water mixing ratio* Δr_T *across the turbulent boundary layer (known as a mixed layer; shaded brown in this figure). The differences (jumps: Δ) of these values at the top* (z_i) *and bottom (s) of the mixed layer can be used to estimate turbulent vertical fluxes in those regions.*

Sample Application

Winds are light and solar heating is strong, creating a convective mixed layer (ML) having depth 1.5 km and convective transport of $b_H \cdot w_B$ = 0.03 m s^{-1}. The potential temperature jump near the surface is 7°C while the corresponding jump near the top of the ML is 4°C. The jumps of total water mixing ratio are 6 and –3 g kg^{-1}, respectively. (a) Find the turbulent flux divergence of r_T. (b) If that was the only process active, then at what rate does total water increase in the ML?

Find the Answer

Given: z_i = 1.5 km = 1500 m, $b_H \cdot w_B$ = 0.03 m s^{-1}.
$|\Delta_s\theta|$ = 7°C, $|\Delta_{zi}\theta|$ = 4°C,
$|\Delta_s r_T|$ = 6 g kg^{-1}, $\Delta_{zi} r_T$ = –3 g kg^{-1},
No other processes acting.
Find: (a) $\Delta F_{z\ turb}(r_T)/\Delta z$ = ? (g kg^{-1})/s,
(b) $\Delta r_T/\Delta t$ = ? (g kg^{-1})/s,

(a) Apply eq. (4.56):

$$\frac{\Delta F_{z\ turb}(r_T)}{\Delta z} = -\frac{b_H \cdot w_B}{z_i}\left[0.2\cdot\left(\Delta_{z_i} r_T\right)\cdot\left|\frac{\Delta_s\theta}{\Delta_{z_i}\theta}\right| + \left|\Delta_s r_T\right|\right]$$

$$= -\frac{(0.03\text{m/s})}{(1500\text{m})}\left[0.2\cdot\left(-3\frac{\text{g}}{\text{kg}}\right)\cdot\left|\frac{7°\text{C}}{4°\text{C}}\right| + \left(6\frac{\text{g}}{\text{kg}}\right)\right]$$

= <u>**–9.9x10**</u>$^{-5}$ (g kg^{-1})/s

(b) Apply eq. (4.44):
$\Delta r_T/\Delta t$ = – $\Delta F_{z\ turb}(r_T)/\Delta z$ = <u>**9.9x10**</u>$^{-5}$ (g kg^{-1})/s
= 0.36 (g kg^{-1})/hour

Check: Physics and units are reasonable.
Exposition: Entrainment is bringing dry air down into the mixed layer [$F(r_T)$ = –], but that drying rate is offset by moisture flux entering the ML from the Earth's surface, leading to increase of r_T with time.

At the top of the ML, entrainment brings in warmer drier air from just above the ML top (at z_i+) and mixes it with air just below the ML top (at z_i–), see Fig. 4.12. That entrainment is driven by thermals associated with the effective surface heat flux F_H (in kinematic units of K m s^{-1}) and is modulated by the strength of the capping temperature inversion $\Delta_{zi}\theta$, where $\Delta_{zi} = (\ )_{zi+} - (\ )_{zi-}$. Thus,

$$\frac{\Delta F_{z\ turb}(r_T)}{\Delta z} = -\frac{\left[0.2\cdot(\Delta_{z_i} r_T / \Delta_{z_i}\theta)\cdot F_H\right] + F_{water}}{z_i} \tag{4.54}$$

Also in eq. (4.54) is F_{water}, the effective surface water-vapor flux [units kinematic (kg$_{water}$ kg$_{air}^{-1}$)·(m s^{-1})], which can be rewritten using the psychrometric constant [γ = 0.4 (g$_{water}$ kg$_{air}^{-1}$)·K^{-1}] in terms of the latent heat flux F_E (eq. 4.48), at the surface:

$$\frac{\Delta F_{z\ turb}(r_T)}{\Delta z} = -\frac{0.2\cdot(\Delta_{z_i} r_T / \Delta_{z_i}\theta)\cdot F_H + \gamma\cdot F_E}{z_i} \tag{4.55}$$

The effective surface sensible and latent heat fluxes in the eq. above can be parameterized using the convective transport coefficient (b_H, see the Thermodynamics chapter) and the buoyancy velocity scale w_B (see eq. 3.38):

$$\frac{\Delta F_{z\ turb}(r_T)}{\Delta z} = -\frac{b_H\cdot w_B}{z_i}\left\{\left[0.2\cdot\left(\Delta_{z_i} r_T\right)\cdot\left|\frac{\Delta_s\theta}{\Delta_{z_i}\theta}\right|\right] + \left|\Delta_s r_T\right|\right\} \tag{4.56}$$

where $\Delta_s(\ ) = (\ )_{surface} - (\ )_{mid\ mixed\text{-}layer}$.

Any one of the four eqs. above can be used to find moisture-flux divergence for $0 < z < z_i$. Above the convective ML (at $z > z_i$) where turbulence is weak or nonexistent:

$$\frac{\Delta F_{z\ turb}(r_T)}{\Delta z} \approx 0 \tag{4.57}$$

During night, turbulence is often weak at most heights (except very close to the surface), allowing you to use eq. (4.57) as a reasonable approximation at $z > 0$.

Next, consider a different type of mixed layer — one that is mixed mechanically by wind shear on windy days (not by thermals on calm sunny days). Further, suppose that there is such a strong capping temperature inversion at the top of this layer (at height z_i) that there is negligible entrainment (thus near zero moisture flux) into the top of this ML. For this case, the turbulent moisture flux divergence is driven solely by F_{water}, the flux at the Earth's surface: $\Delta F_{z\ turb}(r_T)/\Delta z \approx - F_{water}/z_i$. For example, in winter a humid ML might lose moisture due to condensation of water onto a cold snow-covered landscape. Or a cool dry ML might gain moisture by evaporation from a warm lake.

Thunderstorms are deep convective storms that turbulently mix air over the whole depth of the troposphere. The fuel for such storms is humidity in the prestorm mixed layer. When this ML air is drawn into the storm via the storm updrafts, water vapor condenses and falls out as rain. Meanwhile, drier air from the mid-troposphere is often brought down towards the surface to create a new, but drier, post-storm atmospheric boundary layer (ABL). Some of the falling rain can evaporate into this drier ML, increasing the humidity there.

HUMIDITY INSTRUMENTS

Instruments that measure humidity are called **hygrometers**. Don't confuse the word with "hydrometers", which are used to measure specific gravity of fluids such as battery acid.

Dew-point hygrometers, also known as **chilled-mirror hygrometers**, reflect a beam of light off a tiny metal mirror. When the mirror is cooled to the dew-point temperature, dew forms on the mirror and the light beam scatters instead of reflecting into a detector. Electronics in the instrument cool or heat the mirror to maintain the surface precisely at the dew point temperature, which is provided as an output. These are accurate instruments with relatively slow response. For cold temperatures and low humidities, **frost-point hygrometers** are used instead.

Hair hygrometers use organic fibers such as long hairs, anchored at one end and attached at the other end by amplifying levers to a dial that reads out relative humidity. As the RH increases, the hairs get longer, causing the dial to turn. These are inaccurate, but are inexpensive and are the most common hygrometers for home use. Other hygrometers use other materials that also change their dimensions when they absorb water molecules.

Psychrometers are instruments with two liquid-in-glass thermometers attached to a board or frame. The bulb of one thermometer is surrounded by a sleeve or wick of cloth that is saturated with distilled water, while the other bulb remains dry. After both thermometers are actively ventilated [by whirling the instrument through the air on a hand-held axel (**sling psychrometer**), or by using a spring or electrically driven fan to blow air past the thermometers (**aspirated psychrometer**)], the two thermometers are read to give the **wet and dry-bulb temperatures**. The wet-bulb is cooler than the dry, because of the latent heat absorbed when water evaporates. This thermodynamic information can be used with psychrometric tables or charts (Figs. 4.4 & 4.5) to determine the humidity. These instruments are extremely slow response, but relatively simple. Modern psychrometers replace the liquid-in-glass thermometers with electronic thermometers such as thermistors.

In old **radiosondes** (balloon-borne weather instruments), the electrical resistance across a carbon-coated glass slide was measured. In more humid air, this **carbon-film hygrometer** becomes more resistive. Modern radiosondes often measure the capacitance across a very thin dielectric plastic that is coated on both sides with a porous metallic grid. Both approaches are small and light enough to be carried aloft, but both sensors can be easily contaminated by chemical vapors that change their electrical properties.

Microwave refractometers draw air into a small chamber filled with microwaves. The refraction (bending) of these microwave beams depends on humidity (see the radar section of the Satellites & Radar chapter), and can be measured. These are very fast-response sensors.

Spectral absorption hygrometers, also known as **optical hygrometers**, transmit frequencies of electromagnetic radiation that are strongly absorbed by water vapor. By passing the beam of radiation across a short path of air to a detector, the amount of attenuation can be measured to allow calculation of the absolute humidity. One such instrument, the **Lyman-alpha hygrometer,** uses ultraviolet light of wavelength 0.121567 µm, corresponding to an absorption/emission line of hydrogen. Another, the **krypton hygrometer**, uses emissions at 0.12358 µm, generated by glow tube filled with the noble gas krypton. Other instruments use absorption of infrared light (**infrared hygrometers**). These are all fast-response instruments. See the Satellites & Radar chapter for absorption spectra across the atmosphere.

Some **lidars** (laser radars) have been developed to transmit two neighboring wavelengths of electromagnetic radiation, one of which is affected by water vapor and the other which is not. Such **differential absorption lidars** (**DIAL**) can remotely measure humidity along vertical or slant paths, and can scan the atmosphere to measure the humidity in a volume or in a plane.

Weather radars and other microwave profilers can be used to measure profiles of humidity in the atmosphere, because the speed and/or polarity of microwaves through air depends on humidity.

Some sensors measure path-averaged humidity. One example is the **water-vapor channel on weather satellites**, which measures infrared emissions from water vapor in the air. As discussed in the Satellites & Radar chapter, such emissions come from a layer of air several kilometers thick in the top third of the troposphere. These instruments have the advantage of remotely sampling the atmosphere at locations that are difficult to reach otherwise, such as over the oceans. A disadvantage is that they have difficulty seeing through clouds.

Transmissions from **Global Positioning System** (**GPS**) satellites are slightly delayed or refracted by humidity along the path of the beam through the atmosphere. However, data from many such crossing beams from the constellation of GPS satellites can be computationally inverted to yield vertical profiles of humidity, similar to the medical X-ray tomography methods used for brain scans.

REVIEW

Many different variables can be used as measures of water-vapor content in air, including: water-vapor partial pressure (known as vapor pressure in meteorology), mixing ratio, absolute humidity, specific humidity, relative humidity, dew-point temperature, lifting condensation level, and wet-bulb temperature. Some of these variables can be used to quantify total water (vapor + liquid + ice).

Formulas exist to convert between these different variables. Some of these humidity variables can be easily measured by instruments called hygrometers, others are useful in conservation equations, while others are commonly known by the general public.

The amount of water actually being held in the air might be less than the maximum amount that could be held at equilibrium, where this equilibrium value is known as the saturation humidity. Cooler air can hold less water vapor at saturation than warmer air — a fact that is critical in understanding why clouds and storms form in rising, cooling air.

By following such a rising or sinking air parcel we can write a Lagrangian moisture budget and calculate adiabatic temperature changes for air that is saturated (foggy or cloudy). A graphical description of this process can be represented as moist adiabats on a thermo diagram. Saturated rising air does not cool as rapidly with altitude as dry air. Thermo diagrams can also include isohume lines that relate humidity state to temperature and pressure.

An alternative frame of reference is Eulerian, which is fixed relative to a location on the Earth's surface. To forecast humidity in such a fixed frame, we can account for the advection of moisture by the winds, the fluxes of moisture due to turbulence, and effects due to precipitation. Clouds and precipitation are discussed in subsequent chapters.

HOMEWORK EXERCISES

Broaden Knowledge & Comprehension

B1. Use the internet to acquire the current humidity at a weather station near you. What type of humidity variable is it?

B2. Use the internet to acquire a current weather map of humidity contours (isohumes) for your region. Print this map and label regions of humid and dry air.

B3. At the time this chapter was written, a web page was available from the National Weather Service in El Paso, Texas, that could convert between different weather variables (search on "El Paso weather calculator"). How do the formulas for humidity on this web page compare to the ones in this chapter?

B4. Use your internet search engine to find additional "weather calculators" that can convert between different units, other than the calculator mentioned in the previous problem. Which one do you like the best? Why?

B5. Use the internet to acquire the company names and model numbers of at least two different instruments for each of 3 different methods for sensing humidity, as was discussed on the previous page.

B6. Two apparent-temperature indices (humidex and heat index) describing heat stress or summer discomfort were presented in the Thermodynamics chapter. Use the internet to acquire journal articles or other information about any two additional indices from the following list:

- **apparent temperature**,
- **discomfort index**,
- **effective temperature**,
- **humisery**,
- **humiture**,
- **index of thermal stress**.
- **livestock weather safety index**,
- **summer simmer index**,
- **temperature-humidity index** (THI),
- **wet-bulb globe temperature**,

B7. Use the internet to acquire a weather map for your region showing isohumes (either at the surface, or at 85 or 70 kPa). For one Eulerian location chosen by your instructor, use the winds and horizontal humidity gradient to calculate the horizontal moisture advection. State if this advection would cause the air to become drier or more humid. Also, what other factors in the Eulerian water balance equation could counteract the advection (by adding or removing moisture to the Eulerian volume)?

B8. A **meteogram** is a graph of a weather variable (such as humidity or temperature) along the vertical axis as a function of time along the horizontal axis. Use the internet to either acquire such a humidity meteogram for a weather station near your location, or create your own meteogram from a sequence of humidity observations reported at different times from a weather station.

B9. Use the internet to search on "**upper air sounding**", where a sounding is a plot of weather variables vs. height or pressure. Some of these web sites allow you to pick the upper-air sounding station of interest (such as one close to you), and to pick the type of sounding plot.

There are several different types of thermo diagram frameworks for plotting soundings, and so far we discussed only one (called an Emagram). We will learn about the other thermo diagrams in the next chapter -- for example, the Stuve diagram looks very similar to an Emagram. Find a web site that allows you to view and plot an emagram or Stuve for a location near you.

Apply

A1. Compare the saturation vapor pressures (with respect to liquid water) calculated with the Clausius-Clapeyron equation and with Tetens' formula, for T (°C):

a. 45 b. 40 c. 35 d. 30 e. 25 f. 20 g. 15
h. 10 i. 5 j. 0 k. –5 l. –10 m. –15 n. –20

A2. Calculate the saturation vapor pressures with respect to both liquid water and flat ice, for T (°C) =

a. –3 b. –6 c. –9 d. –12 e. –15 f. –18
g. –21 h. –24 i. –27 j. –30 k. –35 l. –40

A3. Find the boiling temperature (°C) of pure water at altitudes (km) of:

a. 0.2 b. 0.4 c. 0.6 d. 0.8 e. 1.2 f. 1.4
g. 1.6 h. 1.8 i. 2.2 j. 2.4 k. 2.6 l. 2.8

A4. Calculate the values of e_s (kPa), r (g kg^{-1}), q (g kg^{-1}), ρ_v (g m^{-3}), RH (%), T_d (°C), LCL (km), T_w (°C), r_s (g kg^{-1}), q_s (g kg^{-1}), and ρ_{vs} (g m^{-3}), given the following atmospheric state:

	P (kPa)	T (°C)	e (kPa)
a.	100	45	5
b.	90	35	2
c.	80	25	1
d.	70	15	0.8
e.	80	5	0.5
f.	90	5	0.2
g.	80	15	1
h.	100	25	2.5

A5(§). Some of the columns in Table 4-1 depend on ambient pressure, while others do not. Create a new version of Table 4-1 for an ambient pressure P (kPa) of

a. 95 b. 90 c. 85 d. 80 e. 75 f. 70 g. 65
h. 60 i. 55 j. 50 k. 45 l. 40 m. 35 n. 30

A6. Given the following initial state for air outside your home. If your ventilation/heating system brings this air into your home and heats it to 22°C, what is the relative humidity (%) in your home? Assume that all the air initially in your home is replaced by this heated outside air, and that your heating system does not add or remove water.

T(°C), RH(%)
a. 15, 80
b. 15, 70
c. 15, 60
d. 10, 80
e. 10, 70
f. 10, 60

T(°C), RH(%)
g. 5, 90
h. 5, 80
i. 5, 70
j. –5, 90
k. –5, 80
m. –5, 70

A7. Given the temperatures and relative humidities of the previous exercise, what is the mixing ratio value (g kg^{-1}) for this "outside" air?

A8. For air with temperature and dew-point values given below in °C, find the LCL value (km).

T, T_d
a. 15, 12
b. 15, 10
c. 15, 8
d. 10, 8
e. 10, 5
f. 10, 2

T, T_d
g. 5, 4
h. 5, 0
i. 5, –5
j. 20, 15
k. 20, 10
m. 20, 5

A9. Given the following dry and wet-bulb temperatures, use equations (not graphs or diagrams) to calculate the dew-point temperature, mixing ratio, and relative humidity. Assume P = 95 kPa.

T, T_w
a. 15, 14
b. 15, 12
c. 15, 10
d. 10, 9
e. 10, 7
f. 10, 5

T, T_w
g. 5, 4
h. 5, 3
i. 5, 2
j. 20, 18
k. 20, 15
m. 20, 13

A10. Same as the previous exercise, but you may use the psychrometric graphs (Figs. 4.4 or 4.5).

A11. Given the following temperatures and dewpoint temperatures, use the equations of Normand's Rule to calculate the wet-bulb temperature.

T, T_d	T, T_d
a. 15, 12	g. 5, 4
b. 15, 10	h. 5, 0
c. 15, 8	i. 5, –5
d. 10, 8	j. 20, 15
e. 10, 5	k. 20, 10
f. 10, 2	m. 20, 5

A12. Same as the previous exercise, but you may use a thermo diagram to apply Normand's rule.

A13. For air at sea level, find the total-water mixing ratio for a situation where:

a. $T = 3°C$, $r_L = 3$ g kg^{-1}
b. $r = 6$ g kg^{-1}, $r_L = 2$ g kg^{-1}
c. $T = 0°C$, $r_L = 4$ g kg^{-1}
d. $T = 12°C$, $r_L = 3$ g kg^{-1}
e. $r = 8$ g kg^{-1}, $r_L = 2$ g kg^{-1}
f. $r = 4$ g kg^{-1}, $r_L = 1$ g kg^{-1}
g. $T = 7°C$, $r_L = 4$ g kg^{-1}
h. $T = 20°C$, $r_L = 5$ g kg^{-1}

A14. Given air with temperature (°C) and total water mixing ratio (g kg^{-1}) as given below. Find the amount of liquid water suspended in the air: r_L (g kg^{-1}). Assume sea-level.

T, r_T	T, r_T
a. 15, 16	g. 25, 30
b. 15, 14	h. 25, 26
c. 15, 12	i. 25, 24
d. 10, 12	j. 20, 20
e. 10, 11	k. 20, 18
f. 10, 10	m. 20, 16

A15. If the mixing ratio is given below in (g kg^{-1}), then use equations (not figures or graphs) find the dew point (°C), assuming air at $P = 80$ kPa.

a. 28 b. 25 c. 20 d. 15 e. 10 f. 5 g. 3
h. 2 i. 1 j. 0.5 k. 0.3 l. 0.2 m. 0.1 n. 0.05

A16. Same as the previous exercise, but you may use Fig. 4.7.

A17. Given an air parcel starting at 100 kPa with dewpoint (°C) given below, use Fig. 4.7 to find the parcel's final dewpoint (°C) if it rises to a height where $P = 60$ kPa.

a. 40 b. 35 c. 30 d. 25 e. 20 f. 15 g. 10
h. 5 i. 0 j. –5 k. –10 l. –15 m. –20 n. –25

A18. What is the value of relative humidity (%) for air with the following state:

T (°C), T_d (°C), P (kPa)	T (°C), T_d (°C), P (kPa)
a. 15, 10, 90	g. 25, 20, 100
b. 15, 10, 90	h. 25, 15, 100
c. 15, 5, 80	i. 25, 10, 90
d. 10, 0, 80	j. 20, 15, 90
e. 10, 0, 70	k. 20, 10, 80
f. 10, –5, 70	m. 20, 5, 70

A19. Calculate the saturated adiabatic lapse rate (°C km^{-1}) at the temperatures and pressures given below. Use the equations, not the thermo diagram.

T (°C), P (kPa)	T (°C), P (kPa)
a. 15, 90	g. 25, 100
b. 15, 90	h. 25, 100
c. 15, 80	i. 25, 90
d. 10, 80	j. 20, 90
e. 10, 70	k. 20, 80
f. 10, 70	m. 20, 70

A20. Same as the previous exercise, but use equations to find the saturated adiabatic lapse rate as a change of temperature with pressure (°C kPa^{-1}).

A21. For air parcels with initial state as given below, use the thermo diagram (Fig. 4.8) to find the final air-parcel temperature after it is lifted to an altitude where $P = 50$ kPa. Assume the air parcels are saturated at all times.

T (°C), P (kPa)	T (°C), P (kPa)
a. 15, 80	g. 25, 90
b. 15, 70	h. 25, 90
c. 15, 60	i. 25, 80
d. 10, 70	j. 20, 80
e. 10, 60	k. 20, 70
f. 10, 60	m. 20, 60

A22. For air parcels with initial state as given in the previous exercise, use the thermo diagram (Fig. 4.8) to find the final air-parcel temperature after it is lowered to an altitude where $P = 100$ kPa. Assume the air parcels are saturated at all times.

A23. Using Fig. 4.8 and other figures for dry adiabats in the Thermodynamics chapter, determine the values of the liquid-water potential temperature and equivalent potential temperature for the initial air parcel of exercise A21.

A24. Same as the previous exercise, except use equations instead of the thermo diagram.

A25. Given an air parcel that starts at a height where $P = 100$ kPa with $T = 25°C$ and $r = 12$ g kg^{-1} (i.e., it is initially unsaturated). After rising to its final height, it has an r_L (g kg^{-1}) value listed below. Assuming no precipitation falls out, find the final value for r (g kg^{-1}) for this now-saturated air parcel.

a. 0.5 b. 1 c. 1.5 d. 2 e. 2.5 f. 3 g. 3.5
h. 4 i. 4.5 j. 5 k. 5.5 l. 6 m. 6.5 n. 7

A26. Imagine a horizontally uniform wind given below, which is blowing in to the west side of a fixed cubic domain and blowing out of the east side. The cube is 200 km on each side. The total water mixing ratio for (incoming, outgoing) air is r_T = (12, 8) g kg^{-1}. What is the rate of change of r_T inside the volume due to this advection?

a. 2 b. 4 c. 5 d. 7 e. 10 f. 12 g. 15
h. 18 i. 20 j. 21 k. 23 l. 25 m. 27 n. 30

A27. Imagine a fixed cube of air 200 km on each side. Precipitation is falling at rate 4 mm h^{-1} into the top of this volume, and is falling out of the bottom of the volume at the rate (mm h^{-1}) given below. What is the rate of change of r_T inside the volume due to this precipitation gradient?

a. 6 b. 5.5 c. 5 d. 4.5 e. 4.3 f. 4.2 g. 4.1
h. 4 i. 3.9 j. 3.8 k. 3.7 l. 3.5 m. 3 n. 2.5

A28. Given below the value of latent heat flux (W·m^{-2}) at the surface. Find the kinematic value of latent flux (K·m s^{-1}), the vertical flux of water vapor (kg$_{water}$·m^{-2}·s^{-1}), the vertical flux of water vapor in kinematic form (kg$_{water}$ kg$_{air}$$^{-1}$)·(m s^{-1}), and the evaporation rate (mm d^{-1}).

a. 100 b. 150 c. 200 d. 250 e. 300 f. 350
g. 80 h. 75 i. 70 j. 60 k. 50 l. 40 m. 25

A29. For windy, overcast conditions, estimate the kinematic latent heat flux at the surface, assuming $C_H = 5\text{x}10^{-3}$.

	M (m s^{-1})	r_{sfc} (g kg^{-1})	r_{air} (g kg^{-1})
a.	2	25	10
b.	2	20	10
c.	2	15	10
d.	5	25	10
e.	5	20	10
f.	5	15	10
g.	12	25	10
h.	12	20	10
i.	12	15	10

A30. For sunny, free-convective conditions, estimate the kinematic latent heat flux at the surface.

	w_B (m s^{-1})	r_{sfc} (g kg^{-1})	r_{ML} (g kg^{-1})
a.	5	25	10
b.	5	20	10
c.	5	15	10
d.	12	25	10
e.	12	20	10
f.	12	15	10
g.	25	25	10
h.	25	20	10
i.	25	15	10

A31. Suppose the atmospheric mixed layer (ML) is as sketched in Fig. 4.12, having $|\Delta_s r_T|$ = 8 g kg^{-1}, $|\Delta_s\theta|$ = 5°C, $\Delta_{zi} r_T$ (g kg^{-1}) as given below, and $|\Delta_{zi}\theta|$ = 3°C. The mean winds are calm for this daytime convective boundary layer, for which $b_H{\cdot}w_B$ = 0.02 m s^{-1}. What is the rate of change of total water mixing ratio in the ML due to the turbulence?

a. –10 b. –8 c. –5 d. –3 e. –2 f. –1 g. 0
h. +1 i. 2 j. 4 k. 6 l. 7 m. 9 n. 12

Evaluate & Analyze

E1. When liquid water evaporates into a portion of the atmosphere containing dry air initially, water vapor molecules are added into the volume as quantified by the increased vapor pressure e. Does this mean that the total pressure increases because it has more total molecules? If not, then discuss and justify alternative outcomes.

E2(§). a. Make your own calculations on a spreadsheet to re-plot Fig. 4.2 (both the bottom and top graphs).

b. What factor(s) in the equations causes the vapor pressure curves to differ for the different phases of water?

E3. Suppose that rain drops are warmer than the air they are falling through. Which temperature should be used in the Clausius-Clapeyron equation (T_{air} or T_{water})? Why? (For this exercise, neglect the curvature of the rain drops.)

E4. Notice the column labels in Table 4-1. In that table, why are the same numbers valid for (T, r_s) and for (T_d, r)?

E5(§). a. Plot a curve of boiling temperature vs. height over the depth of the troposphere.

b. The purpose of pressure cookers is to cook boiled foods faster. Plot boiling temperature vs. pressure for pressures between 1 and 2 times $P_{sea\ level}$.

E6(§). Plot r_s vs. T and q_s vs. T on the same graph. Is it reasonable to state that they are nearly equal? What parts of their defining equations allow for this characteristic? For what situations are the differences between the mixing ratio and specific humidity curves significant?

E7(§). On a graph of T vs. T_d, plot curves for different values of RH.

E8. Why is it important to keep thermometers dry (i.e., placing them in a ventilated enclosure such as a **Stevenson screen**) when measuring outside air temperature?

E9(§). For any unsaturated air-parcel, assume you know its initial state (P, T, r). Recall that the LCL is the height (or pressure) where the following two lines cross: the dry adiabat (starting from the known P, T) and the isohume (starting from the known P, r). Given the complexity of the equations for the dry adiabat and isohume, it is surprising that there is such a simple equation (4.16a or b) for the LCL.

Confirm that eq. (4.16) is reasonable by starting with a variety of initial air-parcel states on a thermo diagram, lifting each one to its LCL, and then comparing this LCL with the value calculated from the equation for each initial parcel state. Comment on the quality of eq. (4.16).

E10. Some of the humidity variables [vapor pressure, mixing ratio, specific humidity, absolute humidity, relative humidity, dew-point temperature, LCL, and wet-bulb temperature] have maximum or minimum limits, based on their respective definitions. For each variable, list its limits (if any).

E11. A **swamp cooler** is a common name for an air-conditioning system that lowers the air temperature to the wet-bulb temperature by evaporating liquid water into the air. But this comes with the side-effect of increasing the humidity of the air. Consider the humidex as given in the Thermodynamics chapter, which states that humid air can feel as uncomfortable as hotter dry air. Suppose each cell in Table 3-5 represents a different initial air state. For which subset of cells in that table would a swamp cooler take that initial air state and change it to make the air feel cooler.

E12. We know that $T_w = T$ for saturated air. For the opposite extreme of totally dry air ($r = 0$), find an equation for T_w as a function of (P, T). For a few sample initial conditions, does your equation give the same results you would find using a thermo diagram?

E13(§). Create a table of T_d as a function of (T, T_w). Check that your results are consistent with Figs. 4.4 and 4.5.

E14(§). Do eqs. (4.20) and (4.21) give the same results as using **Normand's Rule** graphically on a thermo diagram (Fig. 4.6)? Confirm for a few different initial air-parcel states.

E15. Derive eqs. (4.23) through (4.27) from eqs. (4.4, 4.7, and 4.10).

E16. Rain falls out of the bottom of a moving cloudy air parcel. (a) If no rain falls into the top of that air parcel, then what does eq. (4.35) tell you? (b) If rain falls out of the bottom at the same rate that it falls into the top, then how does this affect eq. (4.35).

E17(§). Create a thermo diagram using a spreadsheet to calculate isohumes (for $r = 1, 3, 7, 10, 30$ g kg^{-1}) and dry adiabats (for $\theta = -30, -10, 10, 30$ °C), all plotted on the same graph vs. P on an inverted log scale similar to Figs. 3.3 and 4.7.

E18. Start with Tetens' formula to derive equation (4.36). Do the same, but starting with the Clausius-Clapeyron equation. Compare the results.

E19. It is valuable to test equations at extreme values, to help understand limitations. For example, for the saturated adiabatic lapse rate (eq. 4.37b), what is the form of that equation for $T = 0$ K, and for T approaching infinity?

E20(§). Create your own thermo diagram similar to Fig. 4.8 using a spreadsheet program, except calculate and plot the following saturated adiabats ($\theta_w = -30, -10, 10, 30$°C).

E21(§). Create a full thermo diagram spanning the domain ($-60 \le T \le 40$°C) and ($100 \le P \le 10$ kPa). This spreadsheet graph should be linear in T and logarithmic in P (with axes reversed so that the highest pressure is at the bottom of the diagram). Plot

- isobars (drawn as thin green solid lines) for P(kPa) = 100, 90, 80, 70, 60, 50, 40, 30, 20
- isotherms (drawn as thin green solid lines) for T(°C) = 40, 20, 0, –20, –40 °C.
- dry (θ, solid thick orange lines) and moist adiabats (θ_w, dashed thick orange lines) for the same starting temperatures as for T.
- isohumes (thin dotted orange lines) for r (g kg^{-1}) = 50, 20, 10, 5, 2, 1, 0.5, 0.2

E22. T and θ are both in Kelvins in eqs. (4.40 and 4.41). Does this mean that these two temperatures cancel each other? If not, then what is the significance of θ/T in those equations?

E23. Consider an air parcel rising adiabatically (i.e., no mixing and no heat transfer with its surroundings). Initially, the parcel is unsaturated and rises dry adiabatically. But after it reaches its LCL, it continues its rise moist adiabatically. Is θ_w or θ_L conserved (i.e., constant) below the LCL? Is it constant above the LCL? Are those two constants the same? Why or why not. Hint, consider the following: In order to conserve total water ($r_T = r + r_L$), r must decrease if r_L increases.

E24. For each of the saturated adiabats in Fig. 4.8, calculate the corresponding value of θ_e (the equivalent potential temperature). Why is it always true that $\theta_e \geq \theta_L$?

E25. a. By inspection of the horizontal advection terms in eq. (4.44), write the corresponding term for vertical advection. b. Which term of that equation could account for evaporation from a lake surface, if the Eulerian cube of air was touching the lake?

E26. Based on the full (un-simplified) Eulerian heat budget equation from the Thermodynamics chapter, create by inspection a full water-balance equation similar to eq. (4.44) but without the simplifications.

E27. Why does the condensation-caused latent heating term in the Eulerian heat balance equation (see the Thermodynamics chapter) have a different form (or purpose) than the precipitation term in the water balance eq. (4.44)?

E28(§). Plot curves kinematic latent flux vs. evaporation rate for different altitudes.

E29. The flux of heat and water due to entrainment at the top of the mixed layer can be written as $F_{H\ zi} = w_e \cdot \Delta_{zi}\theta$ and $F_{water\ zi} = w_e \cdot \Delta_{zi} r_T$, respectively. Also, for free convection (sunny, calm) conditions in the mixed layer, a good approximation is $F_{H\ zi} = 0.02 \cdot F_H$. If the entrainment velocity is w_e, then show how the info above can be used to create eq. (4.54).

E30. Derive eq. (4.56) from (4.55).

E31. Which humidity sensors would be best suited for measuring the rapid fluctuations of humidity in the turbulent boundary layer? Why?

Synthesize

S1. Describe how the formation and evolution of clouds would differ if colder air could hold more water at saturation than warmer air. During a typical daytime summer day, when and at what altitudes would you expect clouds to form?

S2. Describe how isohumes on a thermo diagram would look if saturation mixing ratio depended only on temperature.

S3. Consider the spectral-absorption hygrometers described earlier (e.g., optical hygrometers, Lyman-alpha hygrometer, krypton hygrometer, infrared hygrometers). What principle(s) or law(s) from the Solar and IR Radiation chapter describe the fundamental way that these instruments are able to measure humidity?

S4. Describe the shape (slope and/or curvature) of moist adiabats in a thermo diagram if water-vapor condensation released more latent heat than the cooling associated with adiabatic expansion of the air parcel. Describe any associated changes in climate and weather.

S5. What if the evaporation rate of water from the surface was constant, and did not depend on surface humidity, air humidity, wind speed, or solar heating (i.e., convection). Describe any associated changes in climate and weather.

S6. a. Describe the depth of liquid water in rain gauges at the ground if all the water vapor in the troposphere magically condensed and precipitation out. Assume a standard atmosphere temperature profile, but with a relative humidity of 100% initially (before rainout), and with no liquid or solid water initially suspended in the atmosphere.

b. Oceans currently cover 70.8% of the Earth's surface. If all the water from part (a) flowed into the oceans, describe the magnitude of ocean-depth increase.

c. Do a similar exercise to (a) and (b), but for a saturated standard-atmosphere stratosphere (ignoring the troposphere).

S7. The form of Clausius-Clapeyron equation presented near the beginning of this chapter included both T_o and T as arguments of the exponential function.

a. Use algebra to separate T_o and T into separate exponential functions. Once you have done that, your equation should look like: $e_s = C \cdot \exp[-(L/\Re_v) \cdot (1/T)]$, where C contains the other exponential. Write the expression for C.

b. The **Boltzmann constant** is $k_B = 1.3806\text{x}10^{-23}$ J·K^{-1}·molecule^{-1} . This can be used to rewrite the water-vapor gas constant as $\Re_v \approx k_B/m_v$, where m_v represents the mass of an individual water-molecule. Substitute this expression for $\Re_v$ into $e_s = C\cdot\exp[-(L/\Re_v)\cdot(1/T)]$, leaving C as is in this eq.

c. If you didn't make any mistakes in this alternative form for the Clausius-Clapeyron equation, and after you group all terms in the numerator and all terms in the denominator of the argument then your equation should contain a new form for the ratio in the argument of the exponential function. One can interpret the numerator of the argument as the potential energy gained when you pull apart the bond that holds a water molecule to neighboring molecules in a liquid, so as to allow that one molecule to move freely as water vapor. The denominator can be interpreted as the kinetic energy of a molecule as indicated by its temperature.

With that in mind, the denominator is energy available, and numerator is energy needed, for one molecule of water to evaporate. Describe why this ratio is appropriate for understanding saturation vapor pressure as an equilibrium.

S8. Devise an equation to estimate surface water flux that works for a sunny windy day, which reduces to eqs. (4.51 & 4.52) in the limits of zero convection and zero mean wind, respectively.