Review of Some Finite Difference approximations for Advection
R. Stull, Aug 2018

=== Let:
Δt = time step (increment)
n = time step counter (0, 1, 2, 3, …, n, …, N)
t = time = 0, $1\Delta t$, $2\Delta t$, $3\Delta t$, …, $n\Delta t$, …$N\Delta t$
t = time =(0, t_1, t_2, t_3, …, t_n ,…, t_N)

Δx = horizontal grid size
j = spatial step counter (0, 1, 2, 3, …, j, …, J)
x = horizontal location = 0, $1\Delta x$, $2\Delta x$, $3\Delta x$, …, $j\Delta x$, …$J\Delta x$
x = spatial location = (0, x_1, x_2, x_3, …, x_j ,…, x_J)

U = horizontal wind speed
$U_{j,n}$ = speed at location = x_j and time = t_n

c = pollutant concentration
$c_{j,n}$ = concentration at location = x_j and time = t_n

t = 0 is the initial condition (must be provided)

=== Consider 1D Advection Eq.:

dc/dt = -U dc/dx

=== Desirable traits of finite-difference approximations
1) conserve pollutant mass
2) minimize numerical (non-physical) dispersion
3) have small errors in advection (phase) speed
4) be positive definite (negative concentrations are not allowed)
5) be monotonic (should not introduce additional extrema).

=== Finite difference approximations. Several examples:
(Assume constant wind speed U = U_0)

A. Forward in time, Centered in space:
(in the following, let T = c = concentration)

$$\frac{T_{j,n+1} - T_{j,n}}{\Delta t} = -U_0 \frac{T_{j+1,n} - T_{j-1,n}}{2\Delta x}.$$

Can rearrange to solve for concentration T at future time:

T(j, n+1) = T(j,n) - (Uo·Δt / Δx)·[T(j+1,n) - T(j-1,n)] / 2

Define (U_0·Δt / Δx) = Cr = Courant number.
For numerical stability (to prevent model from blowing up), need Cr ≤ 1.

(see Pedro Odon's movies of 1-D advection)

B. Forward in time, Backward in space

$$\frac{T_{j,n+1} - T_{j,n}}{\Delta t} = -U_0 \frac{T_{j,n} - T_{j-1,n}}{\Delta x}.$$

(see Pedro Odon's movies of 1-D advection)

C. Centered in time, Centered in space

$$\frac{T_{j,n+1} - T_{j,n-1}}{2\Delta t} = -U_0 \frac{T_{j+1,n} - T_{j-1,n}}{2\Delta x}.$$

(see Pedro Odon's movies of 1-D advection)

D. Third-order Runge-Kutta in time, Centered in space (used in WRF model)

$$\begin{aligned} T_{j,n+1} \quad = \quad & T_{j,n} - \tfrac{C_R}{2}\left(T_{j+1,n} - T_{j-1,n}\right) + \tfrac{C_R^2}{8}\left(T_{j+2,n} - 2T_{j,n} + T_{j-2,n}\right) \\ & - \tfrac{C_R^3}{48}\left(T_{j+3,n} - 3T_{j+1,n} + 3T_{j-1,n} - T_{j-3,n}\right). \end{aligned}$$

(see Pedro Odon's movies of 1-D advection)