a quick review / overview of

Governing Equations, Fluxes, Stress, Scaling Variables, and Similarity Theory

ATSC 595D

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2024

Governing Eqs. - Momentum

3.2.3 Conservation of Momentum (Newton's Second Law)

As presented at the end of section 2.8.2, one form for the momentum equation is

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\delta_{i3}\, g - 2\,\varepsilon_{ijk}\,\Omega_j\, U_k - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j} \qquad (3.2.3a)$$

tendency	advection	gravity	Coriolis	pressure	viscous
I	II	III	IV	V	VI
				gradient	(molecular) stress

Term I represents storage of momentum (inertia).
Term II describes advection.
Term III allows gravity to act vertically.
Term IV describes the influence of the earth's rotation (Coriolis effects).
Term V describes pressure-gradient forces.
Term VI represents the influence of viscous stress.

Governing Eqs. - Heat

$$\frac{\partial \theta}{\partial t} + U_j \frac{\partial \theta}{\partial x_j} = \nu_\theta \frac{\partial^2 \theta}{\partial x_j^2} - \frac{1}{\rho C_p}\left(\frac{\partial Q_j^*}{\partial x_j}\right) - \frac{L_p E}{\rho C_p} \qquad (3.2.5)$$

tendency	advection	molecular	radiation	latent
I	II	VI	VII	VIII
		conduction		heating

where ν_θ is the thermal diffusivity, and L_p is the latent heat associated with the phase change of E. The values for latent heat at 0°C are $L_v = 2.50 \times 10^6$ J/kg (gas:liquid), $L_f = 3.34 \times 10^5$ J/kg (liquid:solid), and $L_s = 2.83 \times 10^6$ J/kg of water (gas:solid).

Q^*_j is the component of net radiation in the j^{th} direction. The specific heat for ***moist*** air at constant pressure, C_p, is approximately related to the specific heat for dry air, $C_{pd} = 1004.67$ J kg^{-1} K^{-1}, by $C_p = C_{pd}(1 + 0.84\, q)$. Given typical magnitudes of q in the boundary layer, it is important not to neglect the moisture contribution to C_p.

Terms I, II, and VI are the storage, advection, and molecular diffusion terms, as before. Term VII is the "body source" term associated with radiation divergence. Term VIII is also a "body source" term associated with latent heat released during phase changes. These body source terms affect the whole volume, not just the boundaries.

Governing Eqs. - Pollutants

3.2.6 Conservation of a Scalar Quantity

Let C be the concentration (mass per volume) of a scalar such as a tracer in the atmosphere. The conservation of tracer mass requires that

$$\underset{\substack{\text{tendency}\\ \text{I}}}{\frac{\partial C}{\partial t}} + \underset{\substack{\text{advection}\\ \text{II}}}{U_j \frac{\partial C}{\partial x_j}} = \underset{\substack{\text{molecular}\\ \text{VI}\\ \text{diffusion}}}{\nu_C \frac{\partial^2 C}{\partial x_j^2}} + \underset{\substack{\text{source}\\ \text{VII}\\ \text{or sink}}}{S_C} \tag{3.2.6}$$

where ν_C is the molecular diffusivity of constituent C. S_C is the body source term for the remaining processes not already in the equation, such as chemical reactions. The physical interpretation of each term is analogous to that of (3.2.4c).

For Turbulent Flow, split each variable into mean and turbulent parts

$$\rho = \overline{\rho} + \rho', \quad T_v = \overline{T_v} + T_v', \quad p = \overline{P} + p'$$

etc. These can be rearranged to solve for the turbulent (gust) part:

$$u'(t) = U(t) - \overline{U}$$

$$v'(t) = V(t) - \overline{V}$$

$$w'(t) = W(t) - \overline{W}$$

$$T'(t) = T(t) - \overline{T}$$

$$r'(t) = r(t) - \overline{r}$$

turbulent component	actual instan-taneous value	mean component

Governing Eqs.

Reynolds-averaged equations for mean flow of turbulent air

$$\frac{\overline{P}}{\Re} = \overline{\rho}\,\overline{T_v} \tag{3.5.3a}$$

$$\frac{\partial \overline{U_j}}{\partial x_j} = 0 \tag{3.5.3b}$$

$$\frac{\partial \overline{U}}{\partial t} + \overline{U_j}\frac{\partial \overline{U}}{\partial x_j} = -f_c(\overline{V_g} - \overline{V}) - \frac{\partial(\overline{u_j'u'})}{\partial x_j} \tag{3.5.3c}$$

$$\frac{\partial \overline{V}}{\partial t} + \overline{U_j}\frac{\partial \overline{V}}{\partial x_j} = +f_c(\overline{U_g} - \overline{U}) - \frac{\partial(\overline{u_j'v'})}{\partial x_j} \tag{3.5.3d}$$

$$\frac{\partial \overline{q_T}}{\partial t} + \overline{U_j}\frac{\partial \overline{q_T}}{\partial x_j} = +S_{qT}/\overline{\rho_{air}} - \frac{\partial(\overline{u_j'q_T'})}{\partial x_j} \tag{3.5.3e}$$

$$\frac{\partial \overline{\theta}}{\partial t} + \overline{U_j}\frac{\partial \overline{\theta}}{\partial x_j} = -\frac{1}{\overline{\rho}\,C_p}\left[L_v E + \frac{\partial \overline{Q_j^*}}{\partial x_j}\right] - \frac{\partial(\overline{u_j'\theta'})}{\partial x_j} \tag{3.5.3f}$$

$$\frac{\partial \overline{C}}{\partial t} + \overline{U_j}\frac{\partial \overline{C}}{\partial x_j} = +S_c - \frac{\partial(\overline{u_j'c'})}{\partial x_j} \tag{3.5.3g}$$

I II VII X

The similarity between the last five equations reflects that the same forcings are present in each conservation equation:

Term I represents storage.
Term II represents advection.
Term VII represent sundry body forcings.
Term X describes the turbulent flux divergence.

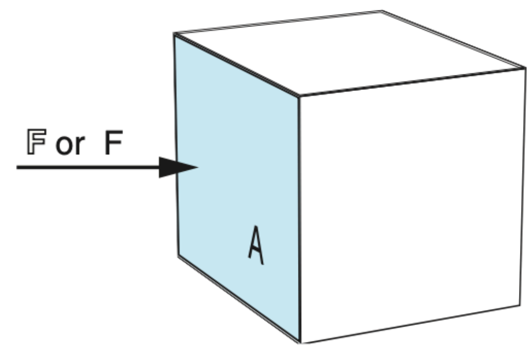
Term X in these eqs. represent a flux.

The next slides show why.

Fluxes

Flux density =
transport of a quantity per unit area per unit time.

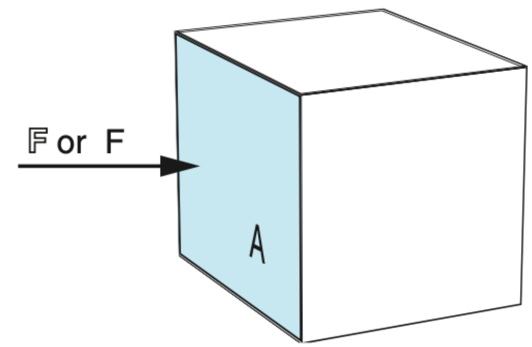
Flux	Symbol	Units
mass	$\tilde{M}$	$\left[\frac{kg_{air}}{m^2 \cdot s}\right]$
heat	$\tilde{Q}_H$	$\left[\frac{J}{m^2 \cdot s}\right]$
moisture	$\tilde{R}$	$\left[\frac{kg_{water}}{m^2 \cdot s}\right]$
momentum	$\tilde{F}$	$\left[\frac{kg \cdot (m \cdot s^{-1})}{m^2 \cdot s}\right]$
pollutant	$\tilde{\chi}$	$\left[\frac{kg_{pollutant}}{m^2 \cdot s}\right]$ or $\left[\frac{kg_{pollutant}}{m^3} \cdot \frac{m}{s}\right]$



Sometimes the moisture flux is rewritten as a latent heat flux, $\tilde{Q}_E$, where $\tilde{Q}_E = L_v \tilde{R}$ and L_v is the latent heat of vaporization of water ($L_v \cong 2.45 \times 10^6$ J/kg at a summertime BL temperature of 20 °C).

As a reminder, momentum is mass times velocity (kg·m/s); thus, a momentum flux is (kg·m/s)/(m^2·s). These units are identical to N/m^2, which are the units for stress. The nature of stress is reviewed in section 2.9.

Kinematic Fluxes



Kinematic Flux	Symbol	Equation	Units
mass	M	$= \dfrac{\tilde{M}}{\rho_{air}}$	$\left[\dfrac{m}{s}\right]$
heat	Q_H	$= \dfrac{\tilde{Q}_H}{\rho_{air}\, C_{p_{air}}}$	$\left[K\,\dfrac{m}{s}\right]$
moisture	R	$= \dfrac{\tilde{R}}{\rho_{air}}$	$\left[\dfrac{kg_{water}}{kg_{air}} \cdot \dfrac{m}{s}\right]$
momentum	F	$= \dfrac{\tilde{F}}{\rho_{air}}$	$\left[\dfrac{m}{s} \cdot \dfrac{m}{s}\right]$
pollutant	χ	$= \dfrac{\tilde{\chi}}{\rho_{air}}$	$\left[\dfrac{kg_{pollutant}}{kg_{air}} \cdot \dfrac{m}{s}\right]$

For dry air (subscript "$_d$") at sea level:

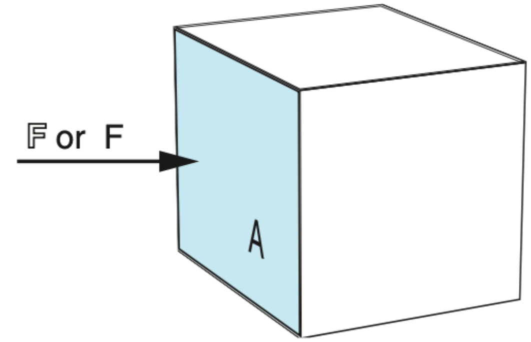
$$\rho_{air} \cdot C_{pd} = 1231\ (\text{W·m}^{-2}) \ /\ (\text{K·m·s}^{-1})$$

$$= 12.31\ \text{mb·K}^{-1}$$

$$= 1.231\ \text{kPa·K}^{-1} .$$

Kinematic flux is based on variables we can measure: wind (m/s), temperature (K), etc.

Fluxes due to Mean Winds



As you might guess, we can split the fluxes into mean and turbulent parts. For the flux associated with the mean wind (i.e., advection), it is easy to show, for example, that

$$\text{Vertical kinematic advective heat flux} = \overline{W}\cdot\overline{\theta} \tag{2.6.1f}$$

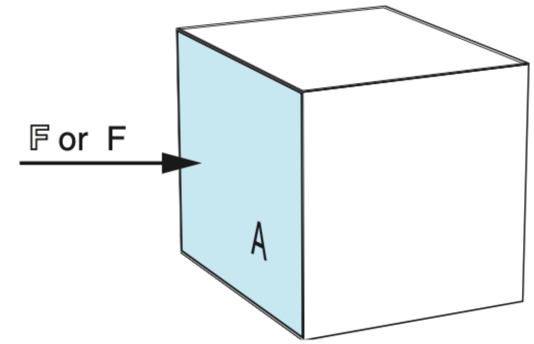
$$\text{Vertical kinematic advective moisture flux} = \overline{W}\cdot\overline{q} \tag{2.6.1g}$$

$$\text{x-direction kinematic advective heat flux} = \overline{U}\cdot\overline{\theta} \tag{2.6.1h}$$

$$\text{Vertical kinematic advective flux of U-momentum} = \overline{W}\cdot\overline{U} \tag{2.6.1i}$$

The last flux is also the kinematic flux of W-momentum in the x-direction.

Eddy Fluxes



Again we see the statistical nature of our description of turbulence. A kinematic flux such as $\overline{w'\theta'}$ is nothing more than a statistical covariance. We will usually leave out the word "kinematic" in future references to such fluxes.

As before, we can extend our arguments to write various kinds of eddy flux:

Vertical kinematic eddy heat flux $= \overline{w'\theta'}$ (2.7.1a)

Vertical kinematic eddy moisture flux $= \overline{w'q'}$ (2.7.1b)

x-direction kinematic eddy heat flux $= \overline{u'\theta'}$ (2.7.1c)

Vertical kinematic eddy flux of U-momentum $= \overline{u'w'}$ (2.7.1d)

The last flux is also the x-direction kinematic eddy flux of W-momentum.

Eddy Fluxes are a statistical covariances

2.4.5 Covariance and Correlation

In statistics, the covariance between two variables is defined as

$$\mathrm{covar}(A,B) \equiv \frac{1}{N} \sum_{i=0}^{N-1} (A_i - \overline{A}) \cdot (B_i - \overline{B})$$

Using our Reynolds averaging methods, we can show that:

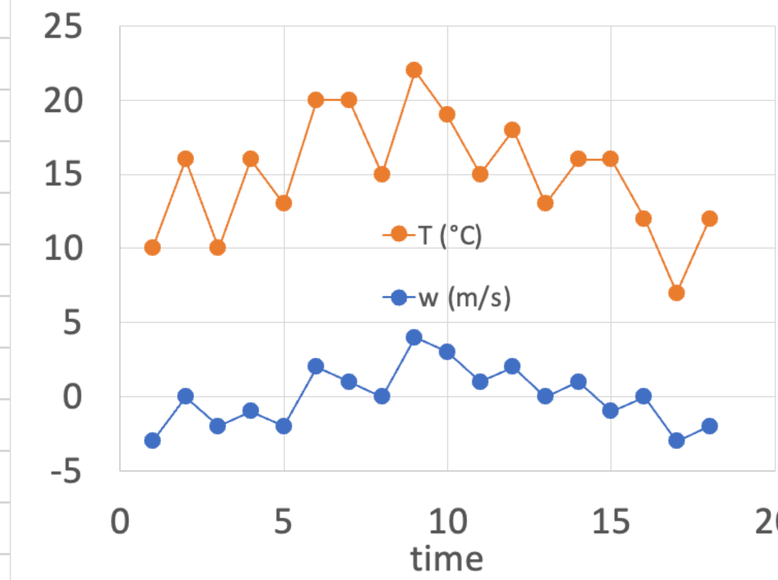
$$\mathrm{covar}(A,B) \equiv \frac{1}{N} \sum_{i=0}^{N-1} a_i' \, b_i'$$

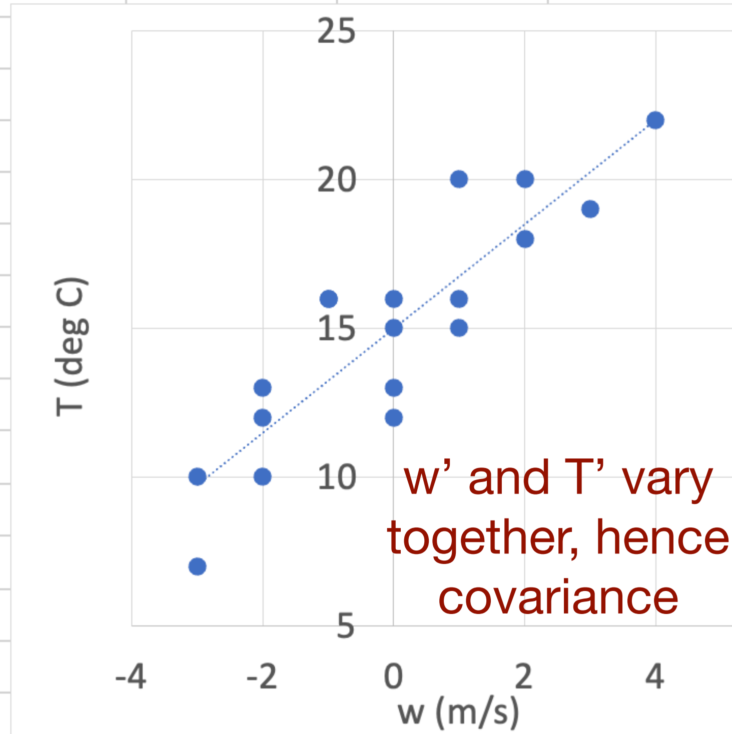
$$= \overline{a' \; b'}$$

Thus, kinematic heat flux $\overline{w'\theta'}$ is computed from data as a covariance.

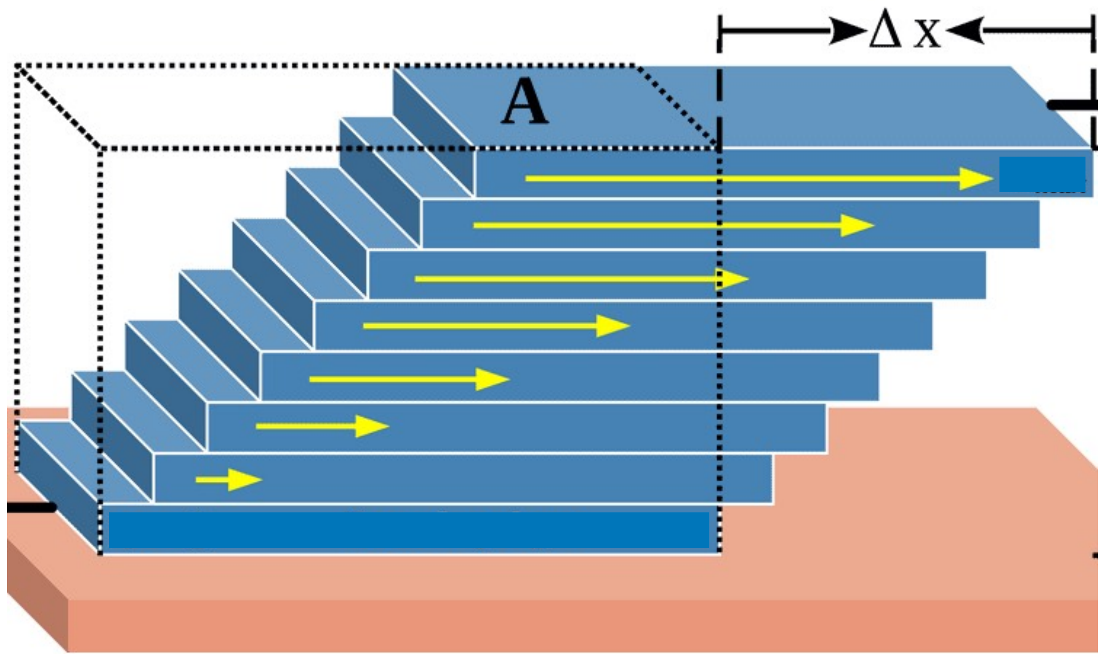
Demo of Covariance

time	w (m/s)	T (°C)	w' (m/s)	T' (°C)	w'T'
1	-3	10	-3	-5.00	15
2	0	16	0	1.00	0
3	-2	10	-2	-5.00	10
4	-1	16	-1	1.00	-1
5	-2	13	-2	-2.00	4
6	2	20	2	5.00	10
7	1	20	1	5.00	5
8	0	15	0	0.00	0
9	4	22	4	7.00	28
10	3	19	3	4.00	12
11	1	15	1	0.00	0
12	2	18	2	3.00	6
13	0	13	0	-2.00	0
14	1	16	1	1.00	1
15	-1	16	-1	1.00	-1
16	0	12	0	-3.00	0
17	-3	7	-3	-8.00	24
18	-2	12	-2	-3.00	6
mean =	0	15.00	0.00	0.00	6.61
					is (w'T')bar
					units °C·m/s
					is Vertical
					flux of heat
					= K m/s





Stress, tau (τ)



$$\tau = \frac{F}{A},$$

where:

F = the force applied;

A = the cross-sectional area.

The area involved corresponds to the material face parallel to the applied force vector, i.e., with surface normal vector perpendicular to the force.

Stress has same units as pressure: Pascals (Pa).

Shear-stress in a fluid is proportional to the shear: $\tau = \mu\,(\partial U/\partial z)$,

where μ = viscosity

Stress, tau (τ) = Momentum Flux

$$\tau = \frac{F}{A}$$

Stress has same units as momentum flux, which has the same units as pressure.

Units:

$$\mathrm{Pa} = \frac{\mathrm{N}}{\mathrm{m}^2} = \frac{\mathrm{kg \cdot m \cdot s^{-2}}}{\mathrm{m}^2} = \frac{\mathrm{kg \cdot m \cdot s^{-1}}}{\mathrm{m^2\, s}}$$

$$\frac{\text{mass} \cdot \text{velocity}}{\text{area} \cdot \text{time}}$$

$$= \frac{\text{momentem}}{\text{area} \cdot \text{time}}$$

$$= \text{momentum flux}$$

$\tau\,/\rho$ = kinematic stress. Units: $\dfrac{\mathrm{kg \cdot m \cdot s^{-2}}}{\mathrm{m}^2} \dfrac{\mathrm{m}^3}{\mathrm{kg}} = \dfrac{\mathrm{m}^2}{\mathrm{s}^2}$

Eddy momentum flux = Reynolds Stress

$$\left|\tau_{\text{Reynolds}}\right| / \bar{\rho} \equiv \left[\overline{u'w'}_s^{\,2} + \overline{v'w'}_s^{\,2} \right]^{1/2}$$

Statistical covariance of u and w is the kinematic form of Reynolds stress.

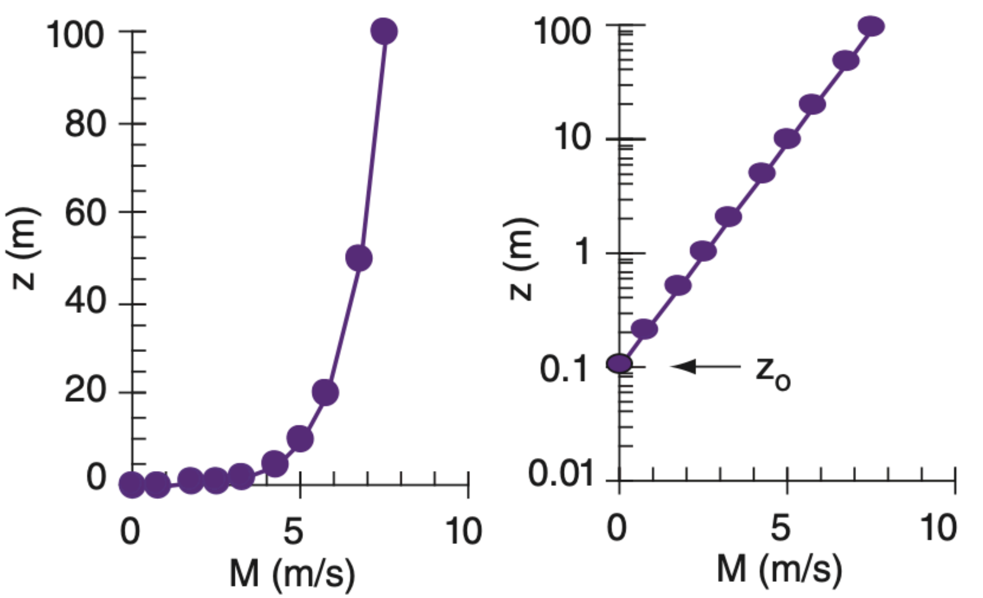
The "friction velocity" u_* is based on the vertical flux of horizontal momentum.

$$u_*^2 \equiv \left[\overline{u'w'}_s^{\,2} + \overline{v'w'}_s^{\,2} \right]^{1/2}$$

$$u_*^2 = \left|\overline{u'w'}_s\right| = \left|\tau_{\text{Reynolds}}\right| / \bar{\rho}$$

u_* is a velocity scale for the surface layer (not the whole BL.

Useful for a "similarity theory" relationship called the "law of the wall", which yields the log wind profile for statically neutral surface layers.



18.5.3. Log Profile in the Neutral Surface Layer

Wind speed M is zero at the ground (more precisely, at a height equal to the aerodynamic roughness length). Speed increases roughly logarithmically with height in the statically neutral surface layer (bottom 50 to 100 m of the ABL), but the shape of this profile depends on the surface roughness:

$$M(z) = \frac{u_*}{k} \ln\left(\frac{z}{z_o}\right) \quad \text{for } z \geq z_o \qquad \bullet(18.14a)$$

Alternately, if you know wind speed M_1 at height z_1, then you can calculate wind speed M_2 at any oth-

Wind-profile example

$$M_2 = M_1 \cdot \frac{\ln(z_2 / z_o)}{\ln(z_1 / z_o)}$$

Sample Application

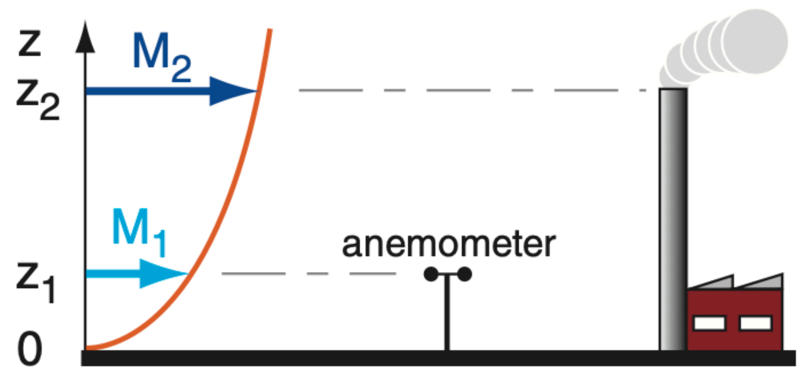
On an overcast day, a wind speed of 5 m s^{-1} is measured with an anemometer located 10 m above ground within an orchard. What is the wind speed at the top of a 25 m smoke stack?

Find the Answer

Given: $M_1 = 5$ m s^{-1} at $z_1 = 10$ m
Neutral stability (because overcast)
$z_o = 0.5$ m from Table 18-1 for an orchard

Find: $M_2 = ?$ m s^{-1} at $z_2 = 25$ m

Sketch:



Use eq. (18.14b):

$$M_2 = 5(\mathrm{m/s}) \cdot \frac{\ln(25\mathrm{m} / 0.5\mathrm{m})}{\ln(10\mathrm{m} / 0.5\mathrm{m})} = \underline{\mathbf{6.53\ m}}\ \mathrm{s}^{-1}$$

Check: Units OK. Physics OK. Sketch OK.

Exposition: Hopefully the anemometer is situated far enough from the smoke stack to measure the true undisturbed wind.

Wind profile for statically-stable surface layer

$$M(z) = \frac{u_*}{k}\left[\ln\left(\frac{z}{z_o}\right) + 6\frac{z}{L}\right] \qquad \bullet(18.15)$$

where M is wind speed at height z, $k = 0.4$ is the von Kármán constant, z_o is the aerodynamic roughness length, and u_* is friction velocity. As height increases, the linear term dominates over the logarithmic term, as sketched in Fig. 18.20.

An **Obukhov length** L is defined as:

$$L = \frac{-u_*^{\;3}}{k \cdot (|g| / T_v) \cdot F_{Hsfc}} \qquad \bullet(18.16)$$

where $|g| = 9.8$ m s^{-2} is gravitational acceleration magnitude, T_v is the absolute virtual temperature, and F_{Hsfc} is the kinematic surface heat flux. L has units of m, and is positive during statically stable conditions (because F_{Hsfc} is negative then). The Obukhov length can be interpreted as the height in the stable surface layer below which shear production of turbulence exceeds buoyant consumption.

Scaling Variables for the surface layer

Length: Obukhov Length L

Velocity: friction velocity u*

Temperature: $$\theta_*^{SL} = \frac{-\overline{w'\theta'}_s}{u_*}$$

Moisture: $$q_*^{SL} = \frac{-\overline{w'q'}_s}{u_*}$$

Scaling Variables for the convective mixed layer

Length: Capping Inversion height, z_i

Velocity: The Deardorff velocity (eq. 3.39)

$$w_* = \left[\frac{|g|}{T_v} \cdot z_i \cdot F_{Hsfc} \right]^{1/3}$$

where $F_{Hsfc} = \overline{w'\theta_v'}_s$

Temperature: $\theta_*^{ML} = \overline{w'\theta_v'}_s / w_*$

Moisture: $q_*^{ML} = \overline{w'q'}_s / w_*$

Value of Similarity Theory

illustrated with an example of a convective BL

Suppose you measured w’2bar vs z on different days, as plotted below. They look “similar” to each other.

By using appropriate scaling variables to create dimensionless groups, all the curves collapse into a single curve.

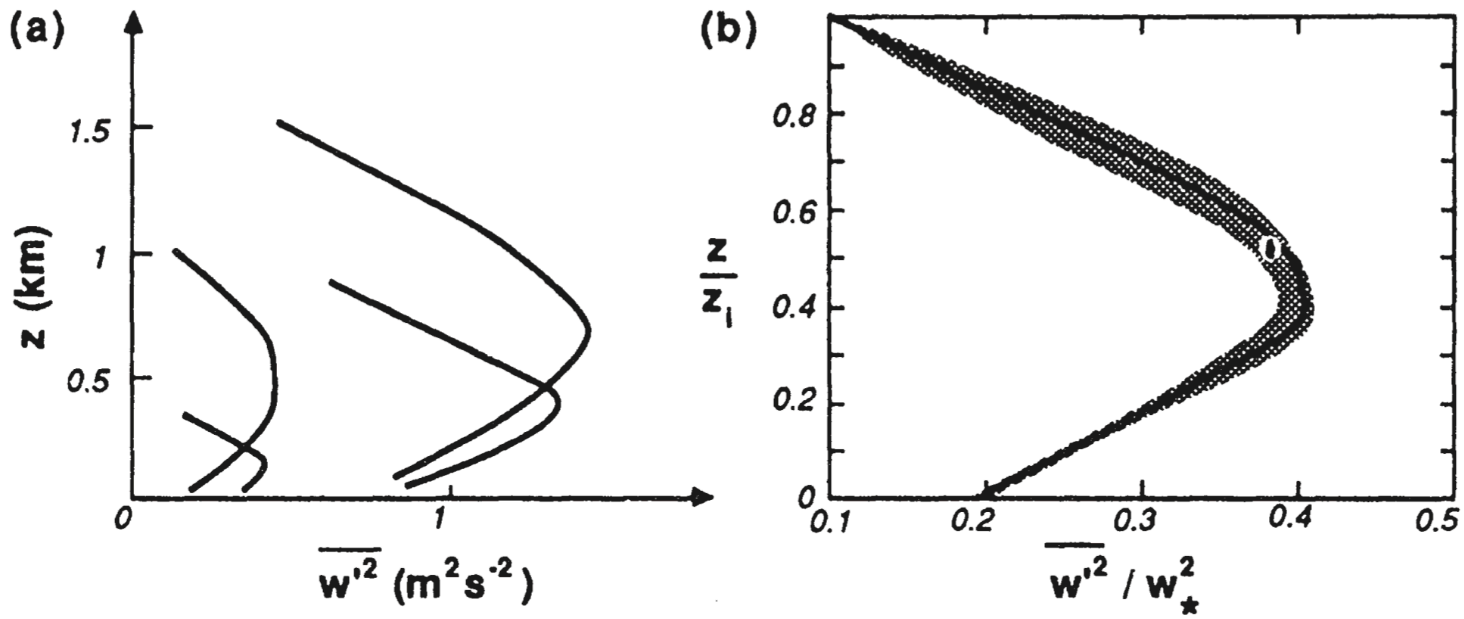


Fig. 9.3 (a) Hypothetical sample of vertical profiles of $\overline{w'^2}$ (b) Profiles of $\overline{w'^2}$ from (a) scaled by free convection similarity. The range of the curves is shaded.

Note: w’^{2}bar is the vertical velocity variance (sigma_w)^2. Sigma_w isan important parameter for smoke dispersion.

An empirical eq. can be found for that curve:

$$\frac{\overline{w'^2}}{w_*^2} = 1.7\left(\frac{z}{z_i}\right)^{2/3}\cdot\left(1 - 0.8\frac{z}{z_i}\right)^2$$

which should work for any convective BL.

Value of Similarity Theory

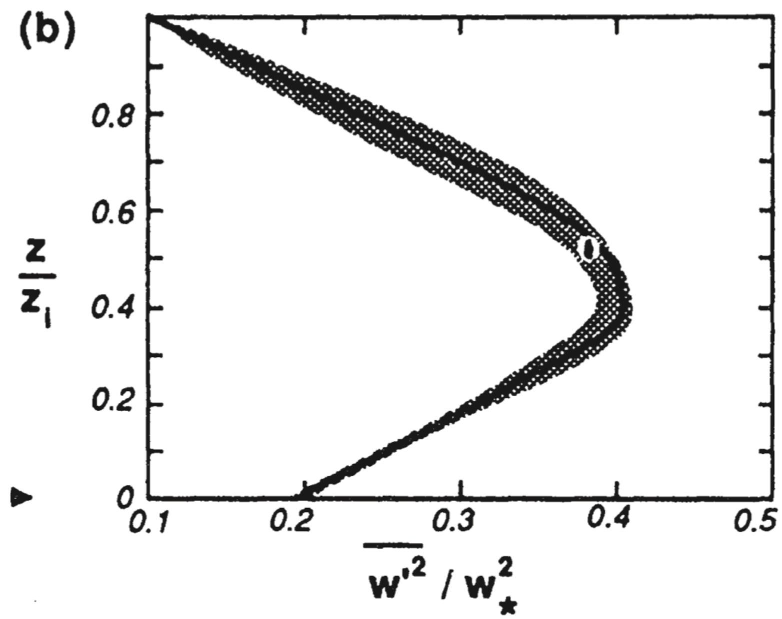
illustrated with an example of a convective BL

Hypothesis:
This same empirical equation (where BL scaling variables were used to create dimensionless groups) will work for all other boundary layers that have similar characteristics.

For this example, the hope is that this eq. works for ALL other purely convective boundary layers (i.e., having negligible shear generation of turbulence) over flat uniform terrain.

This is the value. It allows us to estimate vertical velocity variance (which is used to find pollutant spread sigma_z), knowing z, Zi and the vertical heat flux at the surface for **any** convective BL.

By using appropriate scaling variables to create dimensionless groups, all the curves collapse into a single curve.



An empirical eq. can be found for that curve:

$$\frac{\overline{w'^2}}{w_*^2} = 1.7 \left(\frac{z}{z_i}\right)^{2/3} \cdot \left(1 - 0.8\,\frac{z}{z_i}\right)^2$$

which should work for any convective BL.

Other similarity eqs. can be empirically found for other variables, such as v'^2 bar, Ubar, etc.

Governing Eqs., Fluxes, Stress, Scaling Variables, and Similarity Theory

For more details, take ATSC 500 Atmospheric Boundary Layers, or Read Stull (1988) An Intro. to Boundary Layer Meteorology

Any Questions?

References: