C.
~~B~~ ~~A.~~ c. G.I. Taylors Statistical Theory

1) Background : Autocorrelation

a) Definition
b) Special cases
c) Real examples
d) Lagrangian Integral timescale, t_L.

2) Development

3) Behavior
a) Intuitive
b) Short Time scales
c) Long time scales

4) Spectral Form

5) Correlation vs. Randomness

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A. ~~C.~~ G.I. Taylor's Statistical Theory

1921 Diffusion by Continuous Movements
Proceed. of Royal Society

2) Development

G.I. Taylor's Statistical Theory

(1921 Diffusion by Continuous Movements Proceed. of London Math Soc. (2), 20, 196-211.)

~~Notes:~~
Look at just 1-D for simplicity.

Neglect mean advection " " ,

Lagrangian framework (ie, we follow a particle)

Development:

If $u'(t)$ is an instantoneous gust velocity at time t then the total distance traveled by a particle $x(t)$ during time t is (in x-dir only)

$$x(t) = \int_0^t u'(\tau)\, d\tau$$

Multiply this distance by the instantaneous gust velocity of that same partical at time t : $(u'(t))$ LAGRANGIAN

$$u'(t)x(t) = u'(t)\int_{\tau=0}^{t} u'(\tau)\, d\tau$$

But $u'(t)$ is independent of τ, $\therefore$

$$u'(t)\, x(t) = \int_{\tau=0}^{t} u'(t)u'(\tau)\, d\tau$$

But $u'(t) = \frac{d\,x(t)}{dt}$

Thus $u'(t)x(t) = x(t)\,\frac{d\,x(t)}{dt} = \frac{1}{2}\frac{d\,x^2}{dt}$

$\therefore$

$$\frac{1}{2}\,\frac{d\,x^2(t)}{dt} = \int_{\tau=0}^{t} u'(t)u'(\tau)\, d\tau$$

~~But remember (eq.) $\sigma_x^2 \equiv$~~

Next, average over N particles:

$$\frac{d\,\overline{x^2(t)}}{dt} = 2\int_{\tau=0}^{t} \overline{u'(t)u'(\tau)}\, d\tau$$

But remember (eq.) $\overline{x^2} \equiv \sigma_x^2$ Assuming $\bar{x}=0$

so

$$\frac{d\,\sigma_x^2(t)}{dt} = 2\int_{\tau=0}^{t} \overline{u'(t)u'(\tau)}\, d\tau$$

Define $\xi \equiv t-\tau$ = time lag

$\therefore\ \tau = t-\xi$

$d\tau = -d\xi$

$\xi = t \quad @ \quad \tau = 0$

$\xi = 0 \quad @ \quad \tau = t$

$$\frac{d\,\sigma_x^2(t)}{dt} = -2\int_{\xi=t}^{0} \overline{u'(t)\,u'(t-\xi)}\, d\xi$$

or

$$\frac{d\,\sigma_x^2(t)}{dt} = +2\int_{0}^{t} \overline{u'(t)\,u'(t-\xi)}\, d\xi$$

But the variance at time T is

$$\sigma_x^2(T) = \int_0^T \left[\frac{d\,\sigma_x^2(t)}{dt}\right] dt$$

$$\sigma_x^2(T) = 2\int_0^T\int_0^t \overline{u'(t)\,u'(t-\xi)}\,d\xi\,dt$$

But for stationary turbulence (see Aside)

$$\overline{u'(t)\,u'(t-\xi)} = \overline{u'(t)\,u'(t+\xi)}$$

Thus

$$\sigma_x^2(T) = 2\int_0^T\int_0^t \overline{u'(t)\,u'(t+\xi)}\,d\xi\,dt$$

Next, mult. & divide RHS by $\overline{u'^2}$: (Note $\overline{u'^2}$ = const for stationary turb)

$$\sigma_x^2(T) = 2\,\overline{u'^2}\int_0^T\int_0^t \frac{\overline{u'(t)\,u'(t+\xi)}}{\overline{u'^2}}\,d\xi\,dt$$

But the Lagrangian autocorrelation coef. is defined as (for stationary turb.)

$$R_{uu}^L(\xi) \equiv \frac{\overline{u'(t)\,u'(t+\xi)}}{\overline{u'^2}}$$

Thus

$$\sigma_x^2(T) = 2\,\overline{u'^2}\int_0^T\int_0^t R_{uu}^L(\xi)\,d\xi\,dt$$

But

$$\overline{u'^2} \equiv \sigma_u^2$$

Thus

$$\boxed{\sigma_x^2(T) = 2\,\sigma_u^2\left[\int_0^T\int_0^t R_{uu}^L(\xi)\,d\xi\,dt\right]}$$

IV.B.5.5 The answer

Aside: Proof that $\overline{u'(t_1)u'(t_1-\xi)} = \overline{u'(t_1)u'(t_1+\xi)}$

$$\overline{u'(t_1)\,u'(t_1-\xi)} = \overline{u'(t_1-\xi)\,u'(t_1)}$$

Let $t_1 - \xi \equiv t_2$

Thus

$$\overline{u'(t_1)\,u'(t_1-\xi)} = \overline{u'(t_2)\,u'(t_2+\xi)}$$

But if the turbulence is stationary, then its statistical features do not change with time

Thus

$$\overline{u'(t_2)\,u'(t_2+\xi)} \equiv \overline{u'(t_1)\,u'(t_1+\xi)}$$

↑ for stationary turb.

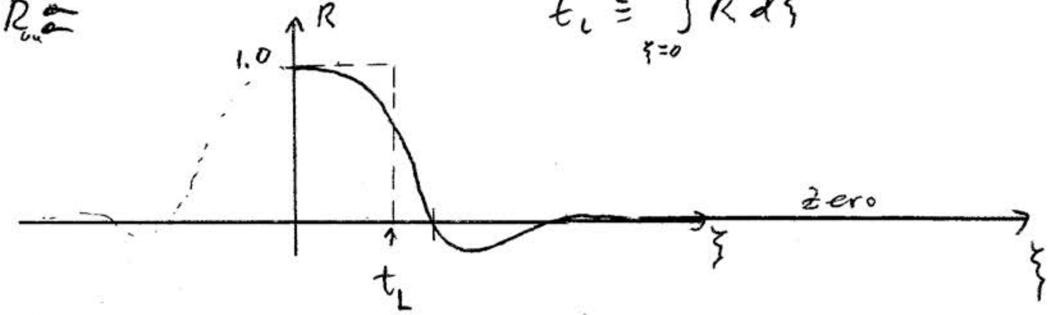
Plugging this into previous eq. gives

$$\boxed{\overline{u'(t_1)\,u'(t_1-\xi)} = \overline{u'(t_1)\,u'(t_1+\xi)}} \qquad \text{IV.B.5.4}$$

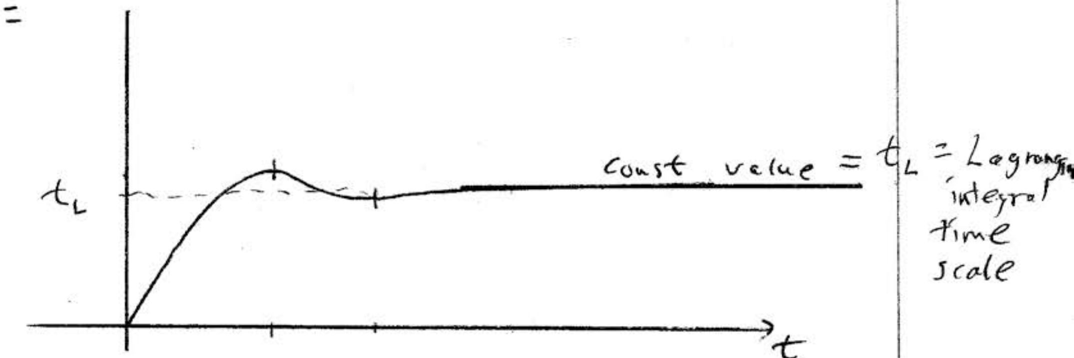
for stationary turb.

3) Behavior – a) Intuitive

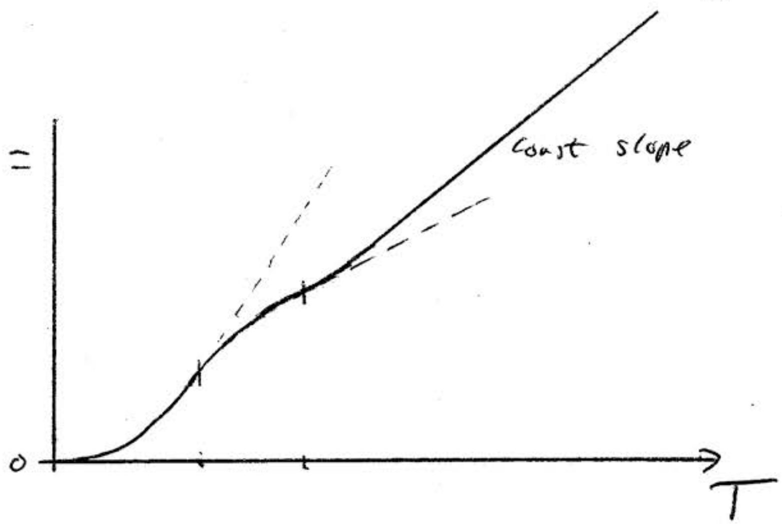
If $R^L_{uu} =$ $\qquad t_L \equiv \int_{\xi=0}^{\infty} R \, d\xi$ (lag: ξ)

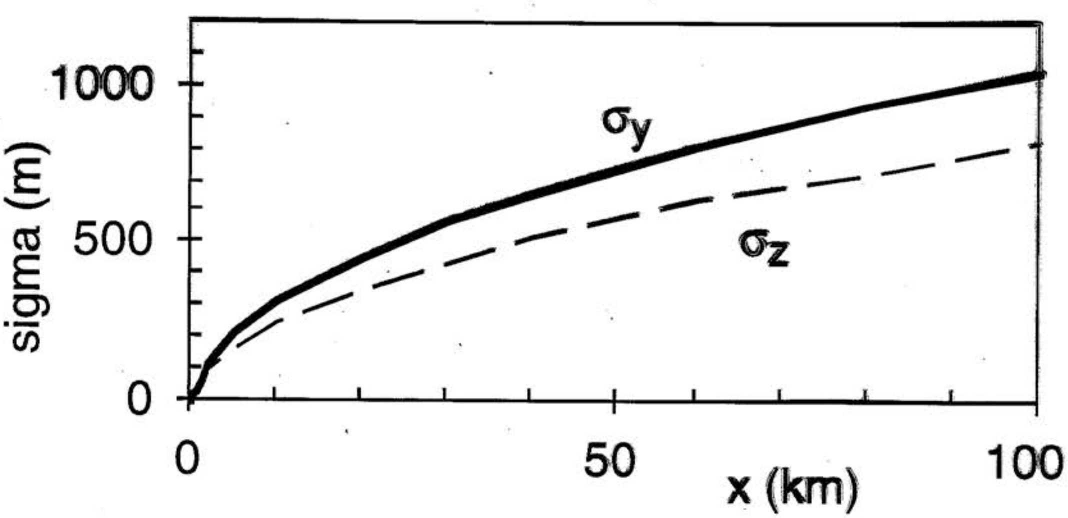


Then: $\int_0^t R \, d\xi =$

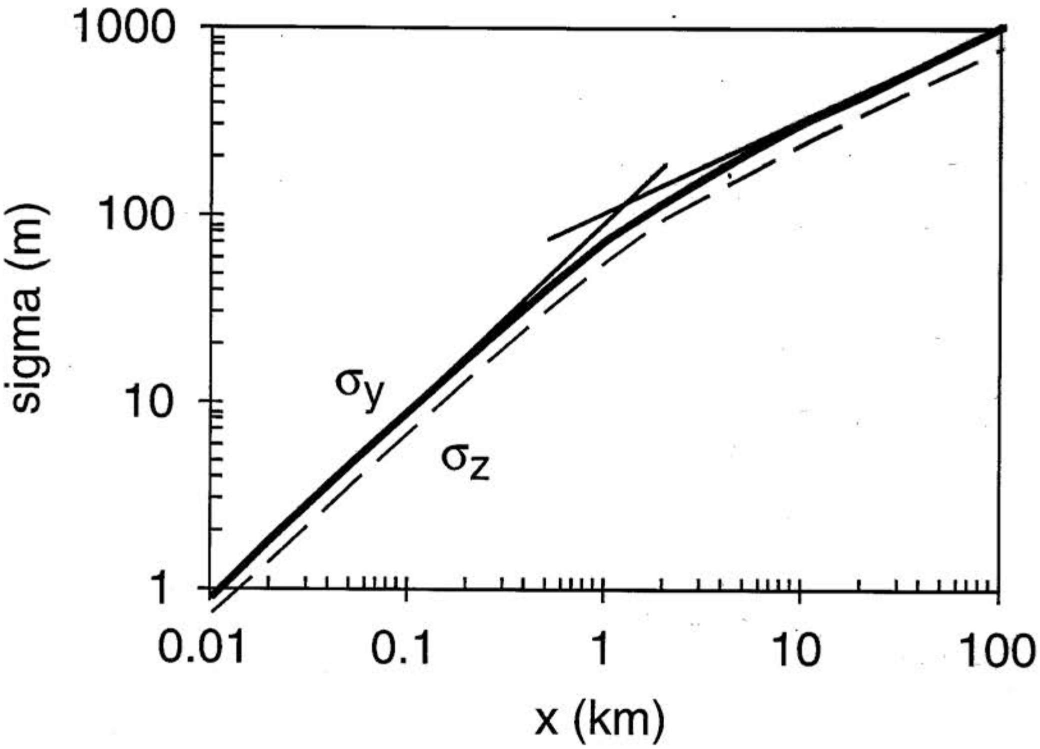


And. $\int_0^T \left(\int_0^t R \, d\xi \right) dt =$





1000
500
0
sigma (m)
σ_y
σ_z
0
50
100
x (km)



1000
100
10
1
sigma (m)
σ_y
σ_z
0.01
0.1
1
10
100
x (km)

b) <u>Very short time scales</u> < 30 sec.

$$R^L_{uu} \approx 1$$

$$\therefore \int_0^t R\, d\xi \approx \int_0^t d\xi = t$$

$$\therefore \left[\int_0^T \left(\int_0^t R\, d\xi\right) dt\right] = \int_0^T t\, dt = \frac{1}{2}T^2$$

Thus $$\boxed{\sigma_x^2(T) \equiv \sigma_u^2 T^2}$$ IV.B.5.7

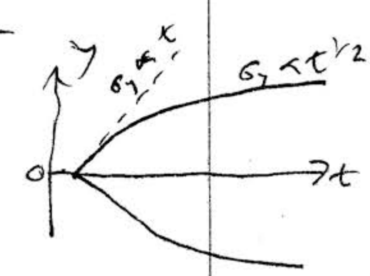
$$\begin{aligned} \sigma_x &= \sigma_u T \\ \sigma_y &= \sigma_v T \\ \sigma_z &= \sigma_w T \end{aligned}$$ linear behavior

c) <u>Very long time scales</u> > 2 min

$$R^L_{uu} \approx 0$$

$$\int_0^t R\, d\xi \approx \text{const} \equiv t_L \quad \text{for } t \to \infty$$

$$\left[\int_0^T \int_0^t R\, d\xi\, dt\right] = t_L \int_0^T dt = t_L T$$

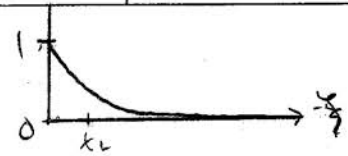


Thus $$\boxed{\sigma_x^2(T) \equiv 2\sigma_u^2 t_L T}$$ IV.B.5.8

$$\begin{aligned} \sigma_x &= (2t_L)^{1/2}\sigma_u T^{1/2} \\ \sigma_y &= (2t_L)^{1/2}\sigma_v T^{1/2} \\ \sigma_z &= (2t_L)^{1/2}\sigma_w T^{1/2} \end{aligned}$$ $\sqrt{\ }$ behavior

Transition between $\sigma_x^2 \propto T^2$ and $\propto T$ occurs at $\approx 2t_L$.

If $R(\xi) = e^{-\xi/t_L}$



Then

$$\boxed{\sigma_y^2(T) = 2\,\sigma_v^2\, t_L^2 \left[\frac{T}{t_L} - 1 + e^{-T/t_L}\right]}$$ Eq. 5.8

See eqs. (19.13) in my PrMet book

Define

$$\boxed{\frac{\sigma_y}{\sigma_v T} \equiv f_y\left(\frac{T}{t_L}\right)}$$ To be used later

$\therefore\ f_y = 2^{1/2}\,\frac{t_L}{T}\left[\frac{T}{t_L} - 1 + e^{-T/t_L}\right]^{1/2}$ using (5.8)

We can define a different time scale:

$\boxed{t_i \equiv \frac{\sigma_y}{0.5\,\sigma_v} \equiv}$ = time when $f_y = 0.5$

Thus $t_i \approx 7\, t_L$

3. Separation of 2 particles (Relative Dispersion)

Note, if instead of the average separation of particles from the release point, we are interested in the average separation of particles from each other, Then

$$\boxed{\overline{l^2} = 2\,\sigma_x^2} \qquad \text{IV, A, 3, 1}$$

where l is the separation distance betwee 2 particles.

From Batchelor & Brier 1950 JM
"Statistical Theory of Turb. & the Problem of Diffusion in the Atmosphere".

$$l = x_i - x_j$$

$$l^2 = (x_i - x_j)^2$$

$$\overline{l^2} = \frac{1}{N(N-1)} \sum_{j=1}^{N} \sum_{i=1}^{N} (x_i - x_j)^2$$

Thus, remembering that

$$\sigma_x^2 = \frac{1}{(N-1)} \sum_{i=1}^{N} (x_i - \bar{x})^2$$

It can be directly shown thru mathematics that (see proof on next page)

$$\overline{l^2} = 2\,\sigma_x^2$$

Relative Dispersion

$$\overline{\ell^2} = \frac{1}{N(N-1)} \sum_{j=1}^{N} \sum_{i=1}^{N} (x_i - x_j)^2$$ = separation between 2 particles

$$\simeq \frac{1}{N^2} \sum_j \sum_i \left(x_i - \bar{x}_i - x_j + \bar{x}_i\right)^2$$

But $\bar{x}_i = \bar{x}_j$ because i & j represent the same field of particles.

$$= \frac{1}{N^2} \sum_j \sum_i \left[(x_i - \bar{x}_i) - (x_j - \bar{x}_j)\right]^2$$

$$= \frac{1}{N^2} \sum_j \sum_i \left(x_i' - x_j'\right)^2$$

$$= \frac{1}{N^2} \sum_j \sum_i \left[x_i'^2 - 2x_i' x_j' + x_j'^2\right]$$

$$= \frac{1}{N} \sum_j \left(\frac{1}{N}\sum_i x_i'^2\right) - \frac{2}{N} \sum_j x_j' \left(\frac{1}{N}\sum_i x_i'\right) + \frac{1}{N}\sum_j x_j'^2 \left(\frac{1}{N}\sum_i 1\right)$$

$$= \frac{1}{N}\sum_j \overline{x_i'^2} - 0 + \overline{x_j'^2}$$

$$= \overline{x_i'^2} + \overline{x_j'^2}$$

$$= 2\overline{x_i'^2}$$ Assuming $\overline{x_i'^2} = \overline{x_j'^2}$

$$\boxed{\overline{\ell^2} = 2\sigma_x^2}$$

4) Spectral Form : Dispersion vs Eddy Size.

Let $F(n)$ be the Lagrangian TKE energy spectrum. (similar to previous spectra)

Note that $F(n)$ is also defined as

$$F(n) = 4\int_0^\infty R(\xi)\cos(2\pi n\xi)\,d\xi$$ IV.13.5.9

where n = frequency

Note that

$$F(0) = 4\int_0^\infty R(\xi)\,d\xi$$

$$= 4\ t_L$$

where t_L = Lagrangian integral time scale.

$t < t_L$ implies particles move in "instantaneous" straight lines (Eulerian)

$t > t_L$ Gaussian distribution of particles. Fickian diffusion theories work here.

Plugging the inverse transform of IV.13.5.9 into IV.13.5.5 yields the spectral form of Taylor's theory — See proof on next page

$$\boxed{\sigma_y^2 = \sigma_v^2 T^2 \int_0^\infty F(n)\,\frac{\sin^2(\pi n T)}{(\pi n T)^2}\,dn}$$ IV.13.5.10

Spectral Form of Statistical Theory

If $F(n) = 4 \int_0^\infty R(\xi) \cos(2\pi n \xi) d\xi$

Then the inverse transform is

$$R(\xi) = \int_{n=0}^\infty F(n) \cos(2\pi n \xi)\, dn \qquad \text{for an even function}$$

Plug into Taylors eq:

$$\sigma_x^2 = 2\sigma_u^2 \int_0^T \int_0^t R(\xi)\, d\xi\, dt$$

$$= 2\sigma_u^2 \int_{t=0}^T \int_{\xi=0}^t \int_{n=0}^\infty F(n) \cos(2\pi n \xi)\, dn\, d\xi\, dt$$

$$= 2\sigma_u^2 \int_{t=0}^T \int_{n=0}^\infty F(n) \int_{\xi=0}^t \cos(2\pi n \xi)\, d\xi\, dn\, dt$$

$$= 2\sigma_u^2 \int_{t=0}^T \int_{n=0}^\infty F(n) \left[\frac{\sin(2\pi n \xi)}{2\pi n}\right]_{\xi=0}^t dn\, dt$$

$$= 2\sigma_u^2 \int_{t=0}^T \int_{n=0}^\infty F(n) \frac{\sin(2\pi n t)}{2\pi n}\, dn\, dt$$

$$= 2\sigma_u^2 \int_{n=0}^\infty F(n) \int_{t=0}^T \frac{\sin(2\pi n t)}{2\pi n}\, dt\, dn$$

$$= 2\sigma_u^2 \int_{n=0}^\infty F(n) \left[\frac{-\cos(2\pi n t)}{(2\pi n)^2}\right]_{t=0}^T dn$$

$$= 2\sigma_u^2 \int_{n=0}^\infty F(n) \left[\frac{1 - \cos(2\pi n T)}{(2\pi n)^2}\right] dn$$

But $1 - \cos 2A \equiv 2\sin^2 A$ Trig. identity

$$= \cancel{2}\,\sigma_u^2 \int_{n=0}^\infty F(n) \frac{\sin^2(\pi n T)}{\cancel{2}(\pi n)^2}\, dn$$

Multiply by T^2

$$\sigma_x^2 = \sigma_u^2 T^2 \int_0^\infty F(n) \left[\sin(\pi n T)\right]^2 dn$$

Spectral Form of Statistical Theory

If $F(n) = 4 \int_0^{\infty} R(\xi) \cos(2\pi n \xi)\, d\xi$

Then the inverse transform is

$$R(\xi) = \int_{n=0}^{\infty} F(n) \cos(2\pi n \xi)\, dn$$ for an even function

Plug into Taylors eq:

$$\sigma_x^2 = 2\sigma_u^2 \int_0^T \int_0^t R(\xi)\, d\xi\, dt$$

$$= 2\sigma_u^2 \int_{t=0}^{T} \int_{\xi=0}^{t} \int_{n=0}^{\infty} F(n) \cos(2\pi n \xi)\, dn\, d\xi\, dt$$

$$= 2\sigma_u^2 \int_{t=0}^{T} \int_{n=0}^{\infty} F(n) \int_{\xi=0}^{t} \cos(2\pi n \xi)\, d\xi\, dn\, dt$$

$$= 2\sigma_u^2 \int_{t=0}^{T} \int_{n=0}^{\infty} F(n) \left[\frac{\sin(2\pi n \xi)}{2\pi n}\right]_{\xi=0}^{t} dn\, dt$$

$$= 2\sigma_u^2 \int_{t=0}^{T} \int_{n=0}^{\infty} F(n) \frac{\sin(2\pi n t)}{2\pi n}\, dn\, dt$$

$$= 2\sigma_u^2 \int_{n=0}^{\infty} F(n) \int_{t=0}^{T} \frac{\sin(2\pi n t)}{2\pi n}\, dt\, dn$$

$$= 2\sigma_u^2 \int_{n=0}^{\infty} F(n) \left[\frac{-\cos(2\pi n t)}{(2\pi n)^2}\right]_{t=0}^{T} dn$$

$$= 2\sigma_u^2 \int_{n=0}^{\infty} F(n) \left[\frac{1-\cos(2\pi n T)}{(2\pi n)^2}\right] dn$$

But $1 - \cos 2A \equiv 2\sin^2 A$ Trig. identity

$$= \cancel{2}\sigma_u^2 \int_{n=0}^{\infty} F(n) \frac{\sin^2(\pi n T)}{\cancel{2}(\pi n)^2}\, dn$$

Multiply by T^2

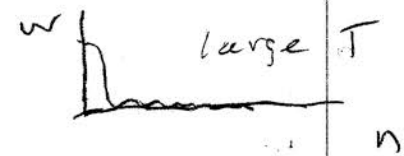
$$\sigma_x^2 = \sigma_u^2 T^2 \int_{n=0}^{\infty} F(n) \left[\frac{\sin(\pi n T)}{\pi n T}\right]^2 dn$$

We can rewrite this as

$$\sigma_y^2 = \sigma_v^2 T^2 \int_0^\infty W(n,T)\, F(n)\, dn$$

where $W(n,T)$ is a weighting function

$$W(n,T) \equiv \left(\frac{\sin \pi n T}{\pi n T}\right)^2$$



$W \uparrow$ as $nT \downarrow$

for $W \approx$ const $n \downarrow$ as $T \uparrow$

Thus, larger eddies (small n) dominate at

larger times.

Thus, W filters out the higher frequencies as T increases.

Conclusion: ~~The~~ Eddies $\approx$ Puff size contribute the most to diffusion.

Larger eddies move the whole puff. Small eddies have a smaller effect on the puff edges.

$\therefore$, As a puff grows, the most effective eddy size also gets larger.

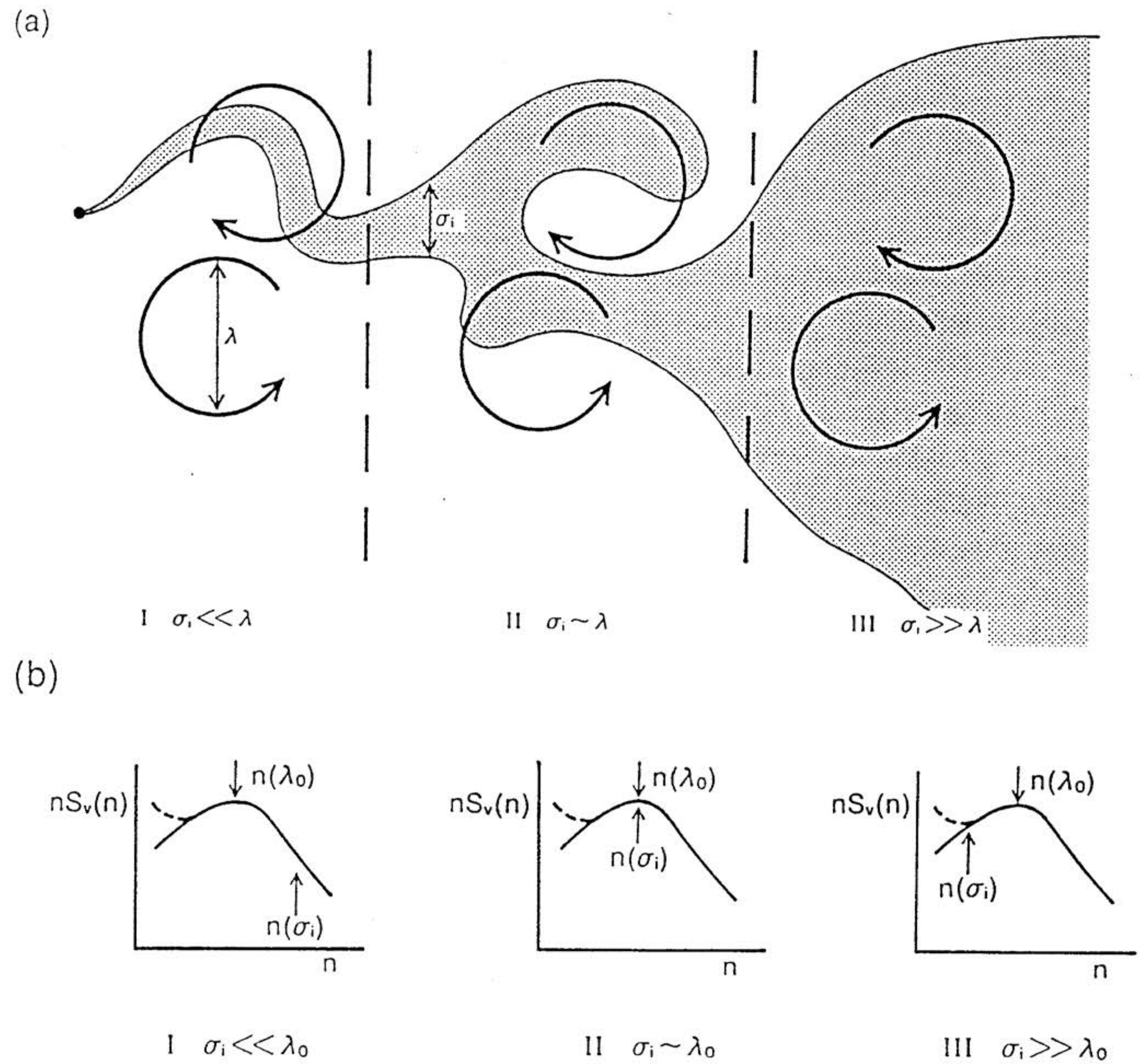


Fig. 9. An illustration of the conceptual model of plume dispersion. Part (a) shows the three different effects of eddies of length scale λ according to the relative size of the crosswind length scale of the instantaneous plume σ_i. Part (b) illustrates how the plume development depends on the form of the velocity spectra, using the example of the horizontal crosswind velocity v. Three regimes are defined similar to those defined in part (a) for the particular length scale λ_0 corresponding to the peak of the velocity spectrum $S_v(n)$, and the growth and structure of the plume as a whole depends on the relative scales of σ_i and λ_0 as described in the text. The dashed lines show a possible form of the increased energy at low-frequency in stable conditions due to large-scale two-dimensional meandering motions.