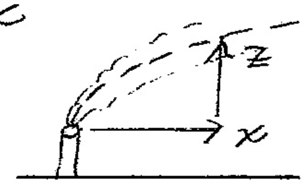
C. Solution of Eqs for Special Cases

ie, how to find $z_{\text{centerline}}$ vs. x

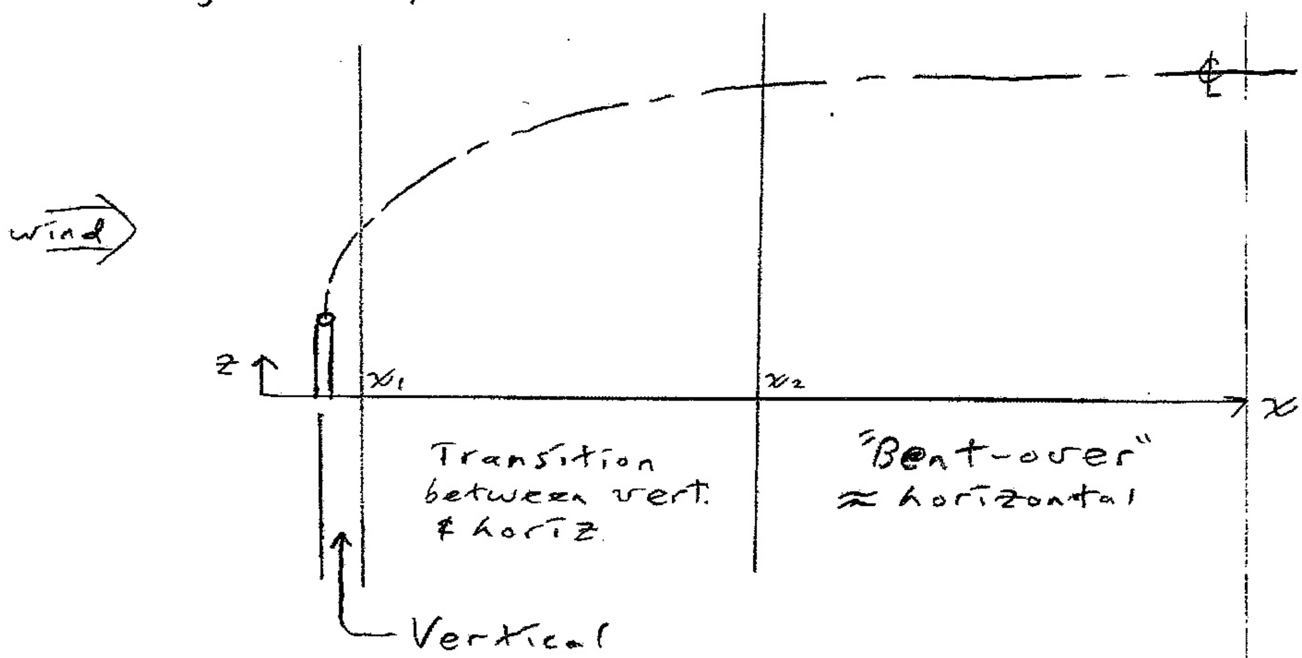


1. Definitions of Special Cases -

a) "Jet" = momentum dominates

"Buoyant Plume" = buoyancy dominates

b) Stages of plume rise



c) Typical initial conditions: vertical

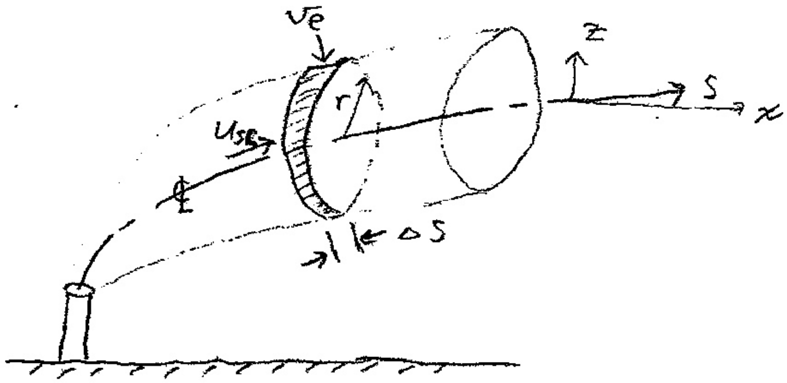
$r = r_o$	plume radius = stack radius
$w = w_o$	= velocity out of top of stack
$U_{sc} = w$	plume is initially vertical
"jet"	i.e., initially even buoyant plumes act like a jet.

d) Typical final conditions: horizontal

$u_{sc} = \bar{U}_a$ blowing downwind

$w = 0$ not rising

"buoyancy" dominates, although is weak



where:
S = distance along centerline (℄)
U_{SC} = velocity (axial along ℄)
v_e = lateral (radial) entrainment velocity

Volume Budget:

$$\underbrace{\text{Volume Flow In}}_{\text{Vol. Flow Along Axis + Vol. Entrained from Sides}} = \text{Volume Flow Out}$$

Vol. Flow Along Axis + Vol. Entrained from Sides = Vol. Out

[But volume flow rate = velocity times area thru which it flows]

$$U_{SC_1} \cdot \cancel{\pi} r_1^2 + v_e \cdot 2\cancel{\pi} r \cdot \Delta S = U_{SC_2} \cdot \cancel{\pi} r_2^2$$

where 1 = in
2 = out

$$\therefore \; 2 r v_e = \frac{[U_{SC_2} r_2^2 - U_{SC_1} r_1^2]}{\Delta S}$$

Let $\Delta S \to dS$:

$$2 r v_e = \frac{d(U_{SC} \cdot r^2)}{dS}$$

Define an entrainment flux $E \equiv 2 \cdot r \cdot v_e$

$$\boxed{\frac{d(U_{SC} \cdot r^2)}{dS} = E} \qquad (3.1)$$

= eq. (3.1) on p 122 of VW text

2. Horizontal (u) Momentum Conservation - Stull

Definitions & Concepts

Momentum $\equiv$ mass $\cdot$ velocity

$= \rho \cdot$ Volume $\cdot$ velocity where ρ = air density

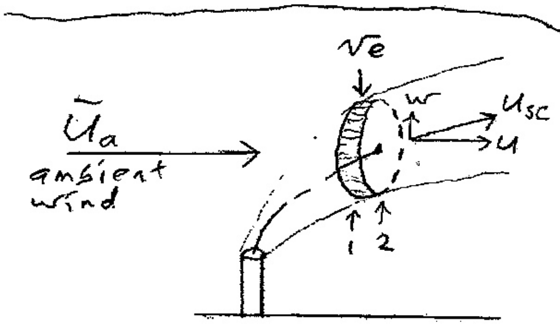
$\therefore$ Mom. Flow $= \rho \cdot$ (Vol. Flow) $\cdot$ velocity

$= \rho \cdot$ (Velocity $\cdot$ Area) $\cdot$ velocity

Horiz. Momentum Budget:

Mom. Flow Out$\big|_{ȼ}$ − Mom. Flow In$\big|_{ȼ}$ = Entrained Mom.

$$\cancel{\rho}\cdot(\cancel{\pi}\cdot u_{sc_2}\cdot r_2^2)\cdot u_2 - \cancel{\rho}\cdot(\cancel{\pi}\cdot u_{sc_1}\cdot r_1^2)\cdot u_1 = \cancel{\rho}\cdot(2\cancel{\pi}\, r\cdot\Delta s\cdot v_e)\cdot\bar{u}_{ambient}$$



Divide both sides by ΔS:

$$\frac{u_{sc_2}\cdot r_2^2\cdot u_2 - u_{sc_1}\cdot r_1^2\cdot u_1}{\Delta S} = 2\cdot r\cdot v_e\cdot\bar{u}_a$$

Let $\Delta S \to dS$.

$$\frac{d\,(u_{sc}\cdot r^2\cdot u)}{dS} = 2\, r\, v_e\, \bar{u}_a$$

Let $u \equiv \bar{u}_a + \Delta u$:

$$\frac{d(u_{sc}\cdot r^2\cdot\Delta u)}{dS} + \frac{d(u_{sc}\cdot r^2\cdot\bar{u}_a)}{dS} = 2\, r\, v_e\, \bar{u}_a$$

Use chain rule on 2^{nd} term:

$$\frac{d\,(u_{sc}\cdot r^2\cdot\Delta u)}{dS} + u_{sc}\cdot r^2\frac{d\bar{u}_a}{dS} + \overbrace{\bar{u}_a\frac{d\,(u_{sc}\cdot r^2)}{dS} = \bar{u}_a\,(2\cdot r\cdot v_e)}^{=0 \text{ from eq. (3.1)}}$$

But: $u_{sc}\dfrac{\partial(\;)}{\partial s} = u\cdot\dfrac{\partial(\;)}{\partial x} + w\cdot\dfrac{\partial(\;)}{\partial z}$

$$\frac{d\,(u_{sc}\cdot r^2\cdot\Delta u)}{dS} + u\cdot r^2\cancelto{0}{\frac{\partial\bar{u}_a}{\partial x}} + w\cdot r^2\frac{\partial\bar{u}_a}{\partial z} = 0$$

0 for horizontal homogeneity

$$\boxed{\frac{d\,(u_{sc}\cdot r^2\cdot\Delta u)}{dS} = -r^2\cdot w\cdot\frac{\partial\bar{u}_a}{\partial z}} \qquad (3.2)$$

3. Vertical Momentum (w) Conservation - Stull

Momentum Flow Budget:

$$\text{Mom. Flow Out}\big|_{\not{c}} - \text{Mom. Flow In}\big|_{\not{c}} = \text{Entrained Mom.} + \text{Body Forces}$$

$$\cancel{\rho}\cdot(\cancel{\pi}\, u_{sc_2}\, r_2^2)\cdot w_2 - \cancel{\rho}\cdot(\cancel{\pi}\cdot u_{sc_1}\cdot r_1^2)\cdot w_1 = \cancel{\rho}\,(2\cancel{\pi}\, r\cdot\Delta S)\cdot v_e\cdot \overline{w}_a + \left(\frac{B.F.}{m}\right)\cdot\cancel{\rho}\,\cancel{\pi}\, r^2\cdot\Delta S$$

Assume ambient mean vert. velocity is negligible: $\overline{w}_a \approx 0$.

$$\frac{\left[u_{sc_2}\cdot r_2^2\cdot w_2 - u_{sc_1}\cdot r_1^2\cdot w_1\right]}{\Delta S} = r^2\cdot\left(\frac{B.F.}{m}\right)$$

Let $\Delta S \to dS$.

Also, body force/mass $=$ buoyancy $= g\cdot\dfrac{(\theta_{v_{plume}} - \theta_{v_{amb}})}{\theta_{v_{amb.}}}$

$= g\,(\Delta\theta_v/\theta_{v_a})$

$$\therefore\ \boxed{\frac{d\,(u_{sc}\cdot r^2\cdot w)}{dS} = g\cdot r^2\left(\frac{\Delta\theta_v}{\theta_{v_a}}\right)} \qquad (3.3)$$

4. Heat (θ) Conservation

Derivation identical to horiz. momentum (3.2).

$$\boxed{\frac{d\,(u_{sc}\cdot r^2\cdot\Delta\theta)}{dS} = -\,r^2\cdot w\cdot\frac{\partial\overline{\theta}_a}{\partial z}} \qquad (3.5)$$

(Note: error in textbook, missing negative sign)

5. Definitions of Plume "Fluxes" (actually they are flow rates)

Volume "Flux" (V):

$$V \equiv u_{sc} \cdot r^2 \qquad (m^3/s)$$

$= \bar{U}_a \cdot r^2$ when "bent-over" horizontal

$= w \cdot r^2$ when plume is vertical (initial condition)

Momentum (upward) "Flux" (M):

$$M \equiv w \cdot V \qquad (m^4/s^2)$$

$= w \cdot u_{sc} \cdot r^2$

$= w_o^2 \cdot r_o^2$ initially, when plume is vertical.

$= F_m$ in notation of Weil

Buoyancy "Flux" (F):

$$F = u_{sc} \cdot r^2 \cdot g \cdot \frac{\Delta\theta_v}{\theta_a} \qquad (m^4/s^3)$$

$= F_b$ in notation of Weil

$= w_o \, r_o^2 \cdot g \cdot \frac{\Delta\theta_v}{\theta_a}$ initially, when plume is vertical

Example of use of "Fluxes" in the vertical mom. flux eq (3.3):

Given eq (3.3)

$$\frac{d\,(U_{sc}\cdot r^2\cdot w)}{ds} = g\cdot r^2\left(\frac{\Delta\theta_v}{\theta_{va}}\right)$$

Substute in M & F

$$\boxed{\frac{dM}{ds} = \frac{F}{U_{sc}}}$$

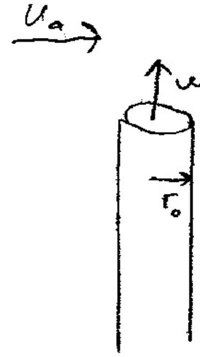
For special case of vertical plume:

$$ds \rightarrow dz$$

$$U_{sc} \rightarrow w$$

$$\therefore \quad \boxed{\frac{dM}{dz} \approx \frac{F}{w}} \quad \text{initially}$$

Length Scales:



Momentum

$$l_m \equiv \left(\frac{\rho_o}{\rho_a}\right)^{1/2} \frac{w_o}{U_a} r_o$$

$$\equiv \left(\frac{\rho_o}{\rho_a}\right)^{1/2} R \cdot r_o$$

ρ_o = initial plume density

ρ_a = ambient density

$$\left(\frac{\rho_o}{\rho_a}\right)^{1/2} \approx 1$$

where $R \equiv \frac{w_o}{U_a}$

(Note: downwash avoided when $R > 1.5$)

Buoyancy

$$l_b \equiv \frac{F_b}{U_a^3}$$

$$\equiv \frac{w_o}{U_a^3} r_o^2 \, g \, \frac{\Delta\theta}{\theta_a}$$

$$\equiv \left(\frac{r_o}{U_a}\right)^2 R \, g \, \frac{\Delta\theta}{\theta_a}$$

Fluxes (Weil)

Momentum

$$F_m \equiv \frac{\rho_o}{\rho_a} w_o^2 r_o^2$$

$$F_m \approx w_o^2 r_o^2 \qquad (= M_o \text{ from Briggs})$$

Buoyancy

$$F_b \equiv w_o \, r_o^2 \, g \frac{\Delta\rho_o}{\rho_a}$$

should be ρ_{plume} ?

where $\Delta\rho = \rho_{ambient} - \rho_{plume}$

$$\approx w_o r_o^2 \, g \frac{\Delta\theta_v}{\theta_{v_{plume}}}$$

where $\Delta\theta_v = \theta_{v_{plume}} - \theta_{v_{ambient}}$

should be $\theta_{v_{ambient}}$

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B. Closure (Entrainment) Assumptions

$$\boxed{v_e = \beta \cdot w}$$

based on lab. tank experiments of Morton, Taylor & Turner.

$\therefore$ $r = \beta \cdot z$ initially.

equivalent to:

$$\boxed{E = \beta \cdot 2 \cdot r \cdot w} \qquad (3.4)$$

Physically, this means that plume-induced entrainment occurs only when the plume rises ($w \neq 0$) relative to environment.

After the plume stops rising, there continues to be environmentally-induced entrainment due to ambient turbulence. However, we are concerned only with the initial plume-rise part here.

Summary

Mass (Volume)

$$\boxed{\frac{d(v_s r^2)}{ds} = 2 r v_e}$$

$$\frac{d\, u_{sc} r^2}{ds} = E \qquad (W3.1)$$

$$\frac{dV}{dz} = 2 r v_e \qquad (B2.7\tfrac{1}{2})$$

Horiz. Momentum

$$\boxed{\frac{d(V_s r^2 \Delta u)}{ds} = -r^2 w \frac{d u_a}{dz}}$$

Entrainment Assumption

$$\boxed{v_e \propto \beta w} \qquad E = 2 r \beta w \quad (W3.4) \quad (B\,2.7\tfrac{1}{2})$$

Vertical Momentum

$$\boxed{\frac{d(v_s r^2 w)}{ds} = r^2 \cdot g \frac{\Delta\theta_v}{\theta}}$$

$$\frac{dM}{ds} = \frac{F}{V_s} \qquad \text{general}$$

$$\frac{dM}{dz} = \frac{F}{w} \qquad \text{vertical } (B\,2.12)$$

Buoyancy

$$\boxed{\frac{d(v_s r^2 \Delta\theta)}{ds} = -r^2 w \frac{d\theta_a}{dz}}$$

$$\frac{dF}{ds} = -r^2 w N^2 \qquad (W3.7)$$

$$\frac{dF}{dz} = -V N^2 \qquad (B2.8) \text{ vertical}$$

where:

$v_s = u$ – horiz. plume

$v_s = w$ – vertical plume

$ds = dz$ – vertical plume

$u_{sc} = v_s$ in Weil's notation.

$E \equiv 2 r v_e$

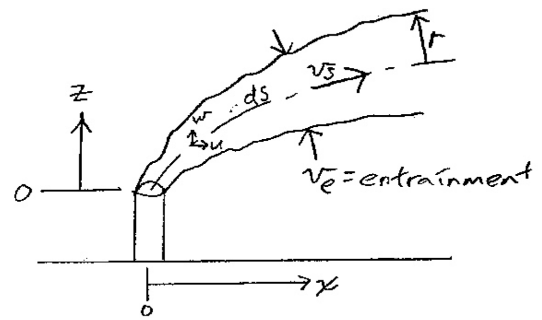
$V \equiv v_s r^2$

$V = u_a r^2$ bent over

$V = w r^2$ vertical

u_a = ambient wind

$\Delta u = u_{plume} - u_{ambient}$



$M = wV = w v_s r^2$

θ_a = ambient potential temperature

β = entrainment coef.

w = vert. vel. of plume

$\Delta\theta = \theta_{plume} - \theta_{ambient}$

g = gravity accel.

$F = v_s r^2 g \frac{\Delta\theta}{\theta}$

$N^2 = \frac{g}{\theta} \frac{\partial \theta_a}{\partial z}$

$ds \equiv d\ell$ – distance along centerline

Plume Rise: $\boxed{\frac{dz}{ds} \equiv \frac{w}{v_s}}$ also $\boxed{\frac{dz}{dt} = w}$ $\boxed{\frac{dx}{dt} = u_{plume}}$

Translation: $\boxed{\frac{dx}{ds} = \frac{u_a + \Delta u}{v_s}}$

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Sample Method of Solution

In neutral environment
1) Vert. momentum flux is conserved (if no buoyancy)
2) Buoyancy flux is conserved.
3) Volume (mass) budget.

Example: Vertical Jet (near source) z vertical in neutral environ., no buoyancy.

Momentum

$$\frac{dM}{dz} = \frac{F}{w} \qquad \boxed{\text{Momentum}} \qquad (B\ 2.12)$$

Buoyancy

$\boxed{\text{But } F=0}$ $\boxed{\text{Buoyancy}}$

$$\frac{dM}{dz} = 0 \qquad \therefore M = \text{constant with } z$$

$$M(z) = M_0$$

but $M = w \cdot V = w \cdot v_s \cdot r^2$ but $v_s = w$ for vert. plume

$$w^2 r^2 = w_0^2 r_0^2 \longleftarrow \therefore M = w^2 r^2 \text{ for vert.}$$

$$\therefore \quad w = \frac{w_0 r_0}{r}$$

Mass

But $\dfrac{d\, w r^2}{dz} = 2 r v_e$ $\boxed{\text{mass}}$ or Volume

Closure Assumption

and $\boxed{v_e \propto \beta w}$

$$\therefore \frac{d\, w r^2}{dz} = 2\beta r w$$

$$\cancel{w_0 r_0} \frac{dr}{dz} = 2\beta\, \cancel{r_0 w_0}$$

Integrate:

$$r = r_0 + 2\beta z \qquad (w3, 15)$$

Velocity → Distance

$$\boxed{\frac{dz}{dt} \equiv w} = \frac{w_0 r_0}{r}$$

$$\frac{dz}{dt} = \frac{w_0 r_0}{r_0 + 2\beta z}$$

$$\approx \frac{w_0 r_0}{2\beta z} \qquad \text{for } z \gg r_0$$

Rearrange

$$z\, dz = \frac{w_0 r_0}{2\beta}\, dt$$

Integrate:

$$\tfrac{1}{2} z^2 = \left(\tfrac{1}{2\beta}\right) w_0 r_0\, dt$$

Answers: $\boxed{z = \left(\frac{1}{\beta} w_0 r_0 t\right)^{1/2}}$

→ If gentle wind $x = u_a t$

$$\boxed{z = \left(\frac{w_0 r_0 x}{u_a \beta}\right)^{1/2}}$$

(over) →

$$\boxed{z = \left(R \cdot \frac{r_0 x}{\beta}\right)^{1/2}} \qquad \text{where } R \equiv \frac{w_0}{u_a}$$

$$\boxed{\frac{z}{\ell_m} = \left(\frac{1}{\beta}\right)^{1/2} \left(\frac{x}{\ell_m}\right)^{1/2}} \qquad (3, 12)$$

$$z = \left(\frac{w_0 r_0}{u_a \beta}\right)^{1/2} x^{1/2}$$

$$z = \left(1/\beta\right)^{1/2} \left(\frac{w_0 r_0}{u_a}\right)^{1/2} x^{1/2}$$

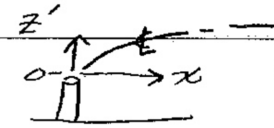
But $\left(\frac{w_0 r_0}{u_a}\right)^{1/2}$ has units of length, $\therefore$ define $\equiv l_m$

$$z = \left(1/\beta\right)^{1/2} l_m^{1/2} x^{1/2}$$

Divide by l_m, $\Rightarrow$ dimensionless

$$\boxed{\frac{z}{l_m} = \left(\frac{1}{\beta}\right)^{1/2} \left(\frac{x}{l_m}\right)^{1/2}}$$

For initial near-vertical rise ONLY.



Neutral BL

Vertical Plume (for $x < x_1$)

$$\boxed{\frac{z'}{l_m} = \left(\frac{R}{\alpha_1 R+\beta}\right)^{1/2}\left(\frac{x}{l_m}\right)^{1/2}}$$

for $R \equiv \frac{w_0}{u_a}$

$\alpha_1 = 0.11$

$\beta = 0.6$

Bent-over Plume

Momentum Only (for $x_1 < x < x_3$)

$$\boxed{\frac{z'}{l_m} = \left(\frac{3}{\beta_1^2}\right)^{1/3}\left(\frac{x}{l_m}\right)^{1/3}}$$

Momentum & Buoyancy

$$\boxed{\frac{z'}{l_b} = \left[\frac{3}{\beta^2}\left(\frac{l_m}{l_b}\right)^2\frac{x}{l_b} + \frac{3}{2\beta^2}\left(\frac{x}{l_b}\right)^2\right]^{1/3}}$$

Buoyancy Only (for $x > x_3$)

$$\boxed{\frac{z'}{l_b} = \left(\frac{3}{2\beta^2}\right)^{1/3}\left(\frac{x}{l_b}\right)^{2/3}}$$

Where $\frac{x_1}{l_b} = \left(\frac{3}{2\beta}\right)^2\left(\frac{\alpha R+\beta}{R}\right)^3\frac{l_m}{l_b}$

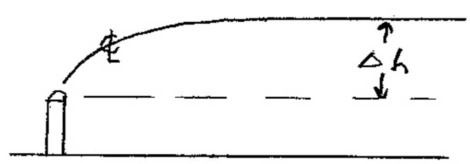
$$\frac{x_3}{l_b} = 2\left(\frac{l_m}{l_b}\right)^2$$

Stable BL

Momentum only:

Buoyancy:

$$\boxed{\Delta h = 2.6\left(\frac{F_b}{u_a N^2}\right)^{1/3}}$$



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