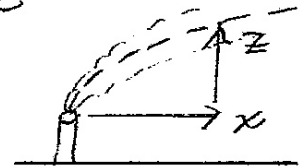


C. Solution of Eqs for Special Cases

ie, how to find z vs. x
centerline

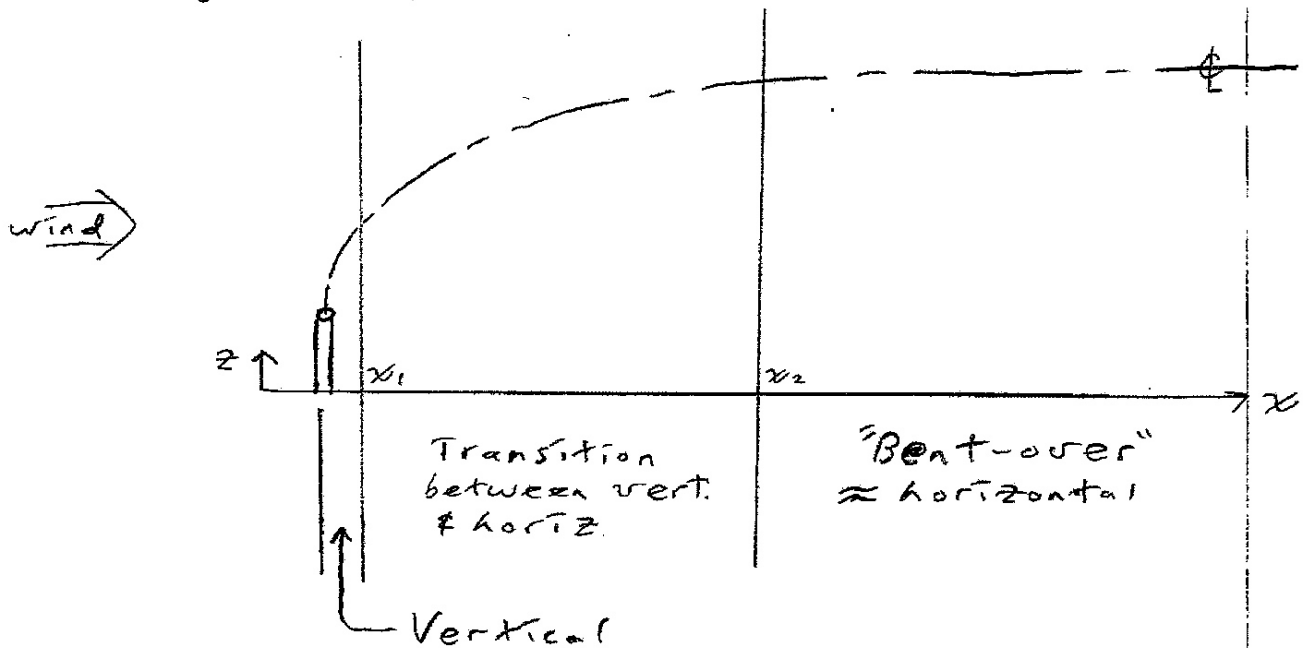


1. Definitions of Special Cases -

a) "Jet" = momentum dominates

"Buoyant Plume" = buoyancy dominates

b) Stages of plume rise



c) Typical initial conditions: vertical

- $r = r_0$ plume radius = stack radius
- $w = w_0$ = velocity out of top of stack
- $U_{sc} = w$ plume is initially vertical
- "jet" ie, initially even buoyant plumes act like a jet.

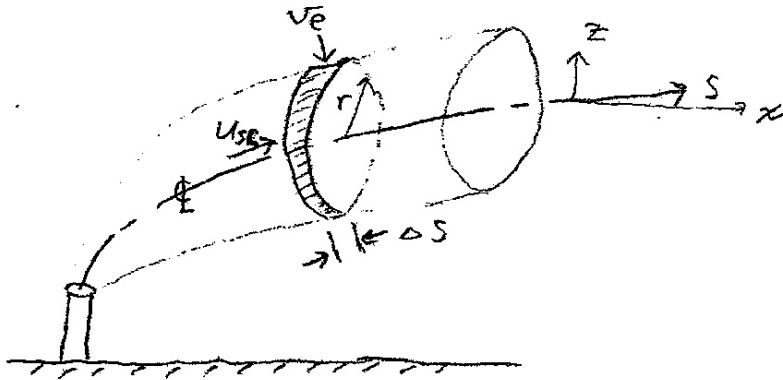
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A. Derivation of Conservation Eqs.

1. Volume Conservation

Stull

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2/6



where:

s = distance along centerline (ϕ)

U_{sc} = velocity (axial) along ϕ

v_e = lateral (radial) entrainment velocity

Volume Budget:

$$\text{Volume Flow In} = \text{Volume Flow Out}$$

$$\text{Vol. Flow Along Axis} + \text{Vol. Entrained from Sides} = \text{Vol. Out}$$

[But volume flow rate = velocity times area thru which it flows]

$$U_{sc1} \cdot \pi r_1^2 + v_e \cdot 2\pi r \cdot \Delta s = U_{sc2} \cdot \pi r_2^2$$

where 1 = in
2 = out

$$\therefore 2 r v_e = \frac{[U_{sc2} r_2^2 - U_{sc1} r_1^2]}{\Delta s}$$

Let $\Delta s \rightarrow ds$:

$$2 r v_e = \frac{d(U_{sc} \cdot r^2)}{ds}$$

Define an entrainment flux $E \equiv 2 \cdot r \cdot v_e$

$$\boxed{\frac{d(U_{sc} \cdot r^2)}{ds} = E}$$

= eq. (3.1) on p 122 of VW text

(3.1)

2. Horizontal (u) Momentum Conservation

- 5 full

A 3/10

Definitions & Concepts

Momentum \equiv mass \cdot velocity

$$\begin{aligned} \therefore \text{Mom. Flow} &= \rho \cdot \text{Volume} \cdot \text{velocity} \\ &= \rho \cdot (\text{Vol. Flow}) \cdot \text{velocity} \\ &= \rho \cdot (\text{Velocity} \cdot \text{Area}) \cdot \text{velocity} \end{aligned}$$

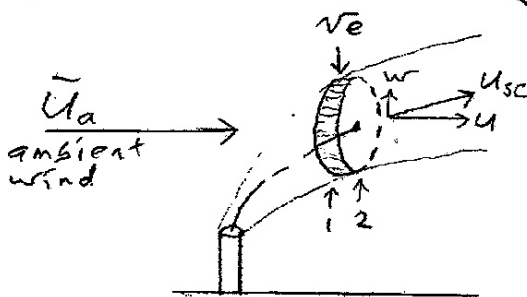
where $\rho =$ air density

Horiz. Momentum Budget:

$$\text{Mom. Flow Out} \Big|_{\Delta} - \text{Mom. Flow In} \Big|_{\Delta} = \text{Entrained Mom.}$$

$$\rho \cdot (\pi \cdot u_{sc2} \cdot r_2^2) \cdot u_2 - \rho \cdot (\pi \cdot u_{sc1} \cdot r_1^2) \cdot u_1 =$$

$$\rho \cdot (2\pi r \cdot \Delta s \cdot v_e) \cdot \bar{u}_{\text{ambient}}$$



Divide both sides by Δs :

$$\frac{u_{sc2} \cdot r_2^2 \cdot u_2 - u_{sc1} \cdot r_1^2 \cdot u_1}{\Delta s} = 2 \cdot r \cdot v_e \cdot \bar{u}_a$$

Let $\Delta s \rightarrow ds$.

$$\frac{d(u_{sc} \cdot r^2 \cdot u)}{ds} = 2 r v_e \bar{u}_a$$

Let $u \equiv \bar{u}_a + \Delta u$:

$$\frac{d(u_{sc} \cdot r^2 \cdot \Delta u)}{ds} + \frac{d(u_{sc} \cdot r^2 \cdot \bar{u}_a)}{ds} = 2 r v_e \bar{u}_a$$

Use chain rule on 2nd term:

$$\frac{d(u_{sc} \cdot r^2 \cdot \Delta u)}{ds} + u_{sc} \cdot r^2 \frac{d\bar{u}_a}{ds} + \underbrace{\bar{u}_a \frac{d(u_{sc} \cdot r^2)}{ds}}_{=0 \text{ from eq. (3.1)}} = 2 r v_e \bar{u}_a$$

$$\text{But: } u_{sc} \frac{\partial(\quad)}{\partial s} = u \cdot \frac{\partial(\quad)}{\partial x} + w \cdot \frac{\partial(\quad)}{\partial z}$$

$$\frac{d(u_{sc} \cdot r^2 \cdot \Delta u)}{ds} + u \cdot r^2 \frac{\partial \Delta u}{\partial x} + w \cdot r^2 \frac{\partial \Delta u}{\partial z} = 0$$

0 for horizontal homogeneity

$$\boxed{\frac{d(u_{sc} \cdot r^2 \cdot \Delta u)}{ds} = -r^2 \cdot w \cdot \frac{\partial \bar{u}_a}{\partial z}} \quad (3.2)$$

(w)

3. Vertical Momentum Conservation

- Still

A
4/6

Momentum Flow Budget:

$$\text{Mom. Flow Out} \Big|_z - \text{Mom. Flow In} \Big|_z = \text{Entrained Mom.} + \text{Body Forces} \cdot \Delta S$$

$$\rho \cdot (\pi u_{sc2} \cdot r_2^2) \cdot w_2 - \rho \cdot (\pi u_{sc1} \cdot r_1^2) \cdot w_1 = \rho (2\pi r \cdot \Delta S) \cdot \bar{w}_a + \left(\frac{B.F.}{m}\right) \cdot \rho \pi r^2 \cdot \Delta S$$

Assume ambient mean vert. velocity is neglig.ble; $\bar{w}_a \approx 0$.

$$\frac{[u_{sc2} \cdot r_2^2 \cdot w_2 - u_{sc1} \cdot r_1^2 \cdot w_1]}{\Delta S} = r^2 \cdot \left(\frac{B.F.}{m}\right)$$

Let $\Delta S \rightarrow ds$.

Also, body force/mass = buoyancy = $g \cdot \frac{(\theta_{r,plume} - \theta_{r,amb})}{\theta_{r,amb}}$
 $= g (\Delta\theta_r / \theta_{ra})$

$$\therefore \boxed{\frac{d(u_{sc} \cdot r^2 \cdot w)}{ds} = g \cdot r^2 \left(\frac{\Delta\theta_r}{\theta_{ra}}\right)} \quad (3.3)$$

4. Heat (θ) Conservation

Derivation identical to horiz. momentum (3.2)

$$\boxed{\frac{d(u_{sc} \cdot r^2 \cdot \Delta\theta)}{ds} = -r^2 \cdot w \cdot \frac{\partial \bar{\theta}_a}{\partial z}} \quad (3.5)$$

↑ (Note: error in textbook, missing negative sign)

12, 361, 100 SHEETS 3 SQUARE
 22, 318, 200 SHEETS 3 SQUARE
 NATURAL

5. Definitions of Plume "Fluxes" (actually they are flow rates)Volume "Flux" (V):

$$V \equiv u_{sc} \cdot r^2 \quad (m^3/s)$$

$$= \bar{u}_a \cdot r^2 \quad \text{when "bent-over" horizontal}$$

$$= w \cdot r^2 \quad \text{when plume is vertical (initial condition)}$$

Momentum (upward) "Flux" (M):

$$M \equiv w \cdot V \quad (m^4/s^2)$$

$$= w \cdot u_{sc} \cdot r^2$$

$$= w_0^2 \cdot r_0^2 \quad \text{initially, when plume is vertical}$$

$$= F_m \quad \text{in notation of Weil}$$

Buoyancy "Flux" (F):

$$F = u_{sc} \cdot r^2 \cdot g \cdot \frac{\Delta \theta_v}{\theta_a} \quad (m^4/s^3)$$

$$= F_b \quad \text{in notation of Weil}$$

$$= w_0 r_0^2 \cdot g \cdot \frac{\Delta \theta_v}{\theta_a} \quad \text{initially, when plume is vertical}$$

Example of use of "Fluxes" in the vertical mom. flux eq (3.3):

Given eq (3.3)

$$\frac{d(u_{sc} \cdot r^2 \cdot w)}{ds} = g \cdot r^2 \left(\frac{\Delta \theta_v}{\theta_{va}} \right)$$

Substitute in M & F

$$\boxed{\frac{dM}{ds} = \frac{F}{u_{sc}}}$$

For special case of vertical plume:

$$ds \rightarrow dz$$

$$u_{sc} \rightarrow w$$

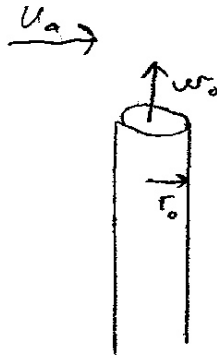
$$\therefore \boxed{\frac{dM}{dz} \approx \frac{F}{w}} \quad \text{initially}$$

Length Scales:

Momentum

$$l_m \equiv \left(\frac{\rho_0}{\rho_a} \right)^{1/2} \frac{w_0 r_0}{u_a}$$

$$\equiv \left(\frac{\rho_0}{\rho_a} \right)^{1/2} R \cdot r_0$$



ρ_0 = initial plume density
 ρ_a = ambient density
 $\left(\frac{\rho_0}{\rho_a} \right)^{1/2} \approx 1$

where $R \equiv \frac{w_0}{u_a}$

Buoyancy

$$l_b \equiv \frac{F_b}{u_a^3}$$

$$\equiv \frac{w_0 r_0^2 g \frac{\Delta \theta}{\theta_a}}{u_a^3}$$

$$\equiv \left(\frac{r_0}{u_a} \right)^2 R g \frac{\Delta \theta}{\theta_a}$$

(Note: downwash avoided when $R > 1.5$)

Fluxes (weil)

Momentum

$$F_m \equiv \frac{\rho_0 w_0^2 r_0^2}{\rho_a}$$

$$F_m \approx w_0^2 r_0^2$$

(= M_0 from Briggs)

Buoyancy

$$F_b \equiv w_0 r_0^2 g \frac{\Delta \rho_0}{\rho_a}$$

$$\approx w_0 r_0^2 g \frac{\Delta \theta_r}{\theta_{r, plume}}$$

should be ρ_{plume} ?

where $\Delta \rho = \rho_{ambient} - \rho_{plume}$

where $\Delta \theta_r = \theta_r - \theta_{r, ambient}$

should be $\theta_{r, ambient}$

B. Closure (Entrainment) Assumptions

$$V_e = \beta \cdot w$$

lob.
based on tank experiments
of Morton, Taylor & Turner.

$\therefore r = \beta \cdot z$ initially.
equivalent to:

$$E = \beta \cdot 2 \cdot r \cdot w$$

(3.4)

Physically, this means that plume-induced entrainment occurs only when the plume rises ($w \neq 0$) relative to environment.

After the plume stops rising, there continues to be environmentally-induced entrainment due to ambient turbulence. However, we are concerned only with the initial plume-rise part here.

Summary

Mass (Volume)

$$\frac{d(v_s r^2)}{ds} = 2r v_e$$

$$\frac{d u_{sc} r^2}{ds} = E \quad (w3,1)$$

$$\frac{dV}{dz} = 2r v_e \quad (B2,7\frac{1}{2})$$

Horiz. Momentum

$$\frac{d(v_s r^2 \Delta u)}{ds} = -r^2 w \frac{d u_a}{dz}$$

Entrainment Assumption

$$v_e \propto \beta w \quad E = 2r\beta w \quad (w3,4) \quad (B2,7\frac{1}{2})$$

Vertical Momentum

$$\frac{d(v_s r^2 w)}{ds} = r^2 g \frac{\Delta \theta}{\theta}$$

$$\frac{dM}{ds} = \frac{F}{v_s} \quad \text{general}$$

$$\frac{dM}{dz} = \frac{F}{w} \quad \text{vertical} \quad (B2,12)$$

Buoyancy

$$\frac{d(v_s r^2 \Delta \theta)}{ds} = -r^2 w \frac{d \theta_a}{dz}$$

$$\frac{dF}{ds} = -r^2 w N^2 \quad (w3,7)$$

$$\frac{dF}{dz} = -V N^2 \quad (B2,8) \quad \text{vertical}$$

Plume Rise: $\frac{dz}{ds} \equiv \frac{w}{v_s}$

Translation: $\frac{dx}{ds} = \frac{u_a + \Delta u}{v_s}$

where:

- $v_s = u$ - horiz. plume
- $v_s = w$ - vertical plume
- $ds = dz$ - vertical plume
- $u_{sc} = v_s$ in Weil's notation.

$$E \equiv 2r v_e$$

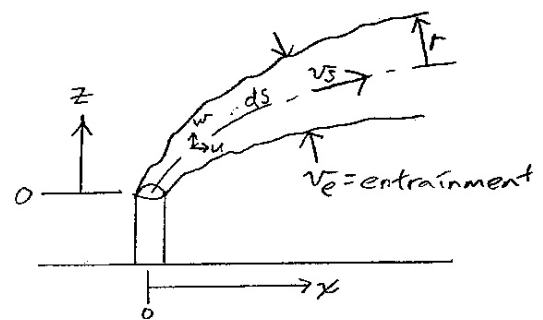
$$V \equiv v_s r^2$$

$$V = u_a r^2 \quad \text{gent area}$$

$$V = w r^2 \quad \text{vertical}$$

u_a = ambient wind

$$\Delta u = u_{\text{plume}} - u_{\text{ambient}}$$



$$M = w V = w v_s r^2$$

θ_a = ambient potential temperature

β = entrainment coef.

w = vert. vel. of plume

$$\Delta \theta = \theta_{\text{plume}} - \theta_{\text{ambient}}$$

g = gravity accel.

$$F = v_s r^2 g \frac{\Delta \theta}{\theta}$$

$$N^2 = \frac{g}{\theta} \frac{d \theta_a}{dz}$$

$ds \equiv dl$ - distance along centerline

also $\frac{dz}{dt} = w$

$$\frac{dx}{dt} = u_{\text{plume}}$$

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42,382 100 SHEETS 3 SQUARE
42,383 100 SHEETS 5 SQUARE
42,384 100 SHEETS 5 SQUARE
MPTC 004

Sample Method of Solution

In neutral environment

- 1) Vert. momentum flux is conserved (if no buoyancy)
- 2) Buoyancy flux is conserved.
- 3) Volume (mass) budget.

Examples: Vertical jet (near source) \approx vertical in neutral environ., no buoyancy.

Momentum

$$\frac{dM}{dz} = \frac{F}{w}$$

Momentum

(B 2.12)

Buoyancy

But $F=0$

Buoyancy

$$\frac{dM}{dz} = 0$$

$\therefore M = \text{constant with } z$

$$M(z) = M_0$$

but $M = w \cdot V = w \cdot v_s \cdot r^2$ but $v_s = w$ for vert. plume

$$w^2 r^2 = w_0^2 r_0^2$$

$\therefore M = w^2 r^2$ for vert.

$$\therefore w = \frac{w_0 r_0}{r}$$

Mass

But

$$\frac{d(w r^2)}{dz} = 2 r v_e$$

mass of Volume

and $v_e \approx \beta w$

Closure Assumption

$$\therefore \frac{d(w r^2)}{dz} = 2\beta r w$$

$$\frac{w_0 r_0}{dz} \frac{dr}{dz} = 2\beta r \frac{w_0 r_0}{r}$$

Integrate:

$$r = r_0 + 2\beta z$$

(B 3.15)

Velocity \rightarrow Distance

$$\frac{dz}{dt} \equiv w = \frac{w_0 r_0}{r}$$

$$\frac{dz}{dt} = \frac{w_0 r_0}{r_0 + 2\beta z}$$

$$\approx \frac{w_0 r_0}{2\beta z}$$

for $z \gg r_0$

Rearrange

$$z dz = \frac{w_0 r_0}{2\beta} dt$$

Integrate:

$$\frac{1}{2} z^2 = \left(\frac{1}{2\beta}\right) w_0 r_0 dt$$

Answers:
$$z = \left(\frac{1}{\beta} w_0 r_0 t\right)^{1/2}$$

\rightarrow If gentle wind $x = u_g t$

$$z = \left(\frac{w_0 r_0 x}{u_g \beta}\right)^{1/2} \quad (\text{over})$$

$$z = \left(\frac{R \cdot r_0 x}{\beta}\right)^{1/2} \quad \text{where } R = \frac{w_0}{u_g}$$

$$\frac{z}{L_m} = \left(\frac{1}{\beta}\right)^{1/2} \left(\frac{x}{L_m}\right)^{1/2} \quad (3.12)$$

$$z = \left(\frac{w_0 t_0}{u_0 \beta} \right)^{1/2} x^{1/2}$$

$$z = \left(\frac{1}{\beta} \right)^{1/2} \left(\frac{w_0 t_0}{u_0} \right)^{1/2} x^{1/2}$$

But $\left(\frac{w_0 t_0}{u_0} \right)^{1/2}$ has units of length,
so define

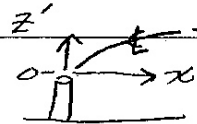
$$= l_m$$

$$z = \left(\frac{1}{\beta} \right)^{1/2} l_m^{1/2} x^{1/2}$$

Divide by $l_m \Rightarrow$ dimensionless

$$\frac{z}{l_m} = \left(\frac{1}{\beta} \right)^{1/2} \left(\frac{x}{l_m} \right)^{1/2}$$

For initial, near-
vertical, rise
ONLY



Neutral BL

Vertical plume (for $x < x_1$)

$$\frac{z'}{l_m} = \left(\frac{R}{\alpha R + \beta} \right)^{1/2} \left(\frac{x}{l_m} \right)^{1/2}$$

for $R \equiv \frac{w_0}{u_0}$
 $\alpha = 0.11$
 $\beta = 0.6$

Bent-over plume

Momentum Only (for $x_1 < x < x_3$)

$$\frac{z'}{l_m} = \left(\frac{3}{\beta^2} \right)^{1/3} \left(\frac{x}{l_m} \right)^{1/3}$$

Momentum + Buoyancy

$$\frac{z'}{l_b} = \left[\frac{3}{\beta^2} \left(\frac{l_m}{l_b} \right)^2 \frac{x}{l_b} + \frac{3}{2\beta^2} \left(\frac{x}{l_b} \right)^2 \right]^{1/3}$$

Buoyancy Only (for $x > x_3$)

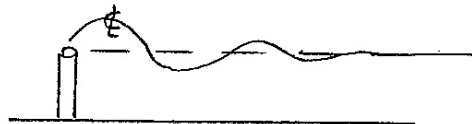
$$\frac{z'}{l_b} = \left(\frac{3}{2\beta^2} \right)^{1/3} \left(\frac{x}{l_b} \right)^{2/3}$$

Where $\frac{x_1}{l_b} = \left(\frac{3}{2\beta} \right)^2 \left(\frac{\alpha R + \beta}{R} \right)^3 \frac{l_m}{l_b}$

$$\frac{x_3}{l_b} = 2 \left(\frac{l_m}{l_b} \right)^2$$

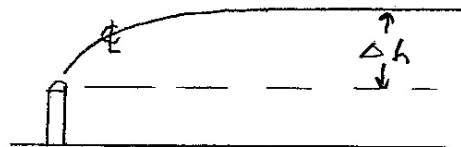
Stable BL

Momentum only:



Buoyancy:

$$\Delta h = 2.6 \left(\frac{F_b}{u_a N^2} \right)^{1/3}$$



Wolfe, J.C., 1988: Plume Rise. Chapt. 3 In "Lectures on Air Pollution Modeling", Edited by Venkatram & Wyngaard, Amer. Meteor. Soc. p119-166.

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42-284 100 SHEETS EYEGLASS SQUARE
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MADE IN U.S.A.

