atsc507_finite_volume

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- Note: This assignment requires lots of straightforward but tedious integrals and algebra. It is strongly recommended that you use software (i.e. symbolic programming with Python, Matlab, Wolfram Mathematica/Alpha) to assist you with your derivations.

1. (/5) Show that T_i (i.e. $T(x)$ at the centroid of control-volume CV$_i$) and $\overline{T_i}$ (i.e. the control-volume averaged value of $T(x)$ in CV$_i$) are the same only to second-order accuracy.

   Hint 1: Try expanding $\overline{T_i}$ at $x = x_i$

   Hint 2: $x_i = \frac{x_{i+\frac{1}{2}} + x_{i-\frac{1}{2}}}{2}$

2. (/15) Derive the 2nd-order centred difference form for the 3-dimensional Poisson's equation using the finite-volume method:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = S$$

   where $T = T(x, y, z)$ is the temperature, and $S = S(x, y, z)$ is the source/sink term. Assume the mesh is structured and rectangular, with CV dimensions $\Delta x \times \Delta y \times \Delta z$.

   Hint 1: $\nabla^2() = \nabla \bullet \nabla()$. Knowing this, how would you express the flux $\vec{F}$ in terms of T or derivatives of T?

   Hint 2: The component of the flux integral across the two faces perpendicular to the x-axis (faces $i + \frac{1}{2}$ and $i - \frac{1}{2}$) is the net flux across the faces times the area of the faces:

$$F_{x,i+\frac{1}{2}} \Delta y \Delta z - F_{x,i-\frac{1}{2}} \Delta y \Delta z = (F_{x,i+\frac{1}{2}} - F_{x,i-\frac{1}{2}}) \Delta y \Delta z$$

   What is the total flux integral across all faces of the control volume?

   Hint 3: $F_{x,i+\frac{1}{2}}$ (i.e. the flux across the "right" face of CV$_i$ = $-$ the flux across the "left" face of CV$_{i+1}$) will need a linear combination of $\overline{T_i}$ and $\overline{T_{i+1}}$. The "centred" part of the differencing refers to the flux calculations being equally dependent on the neighbouring control-volume averages. Once you've found how to express $F_{x,i+\frac{1}{2}}$ in terms of $\overline{T_i}$ and $\overline{T_{i+1}}$, finding the fluxes across the other faces should be doable by inspection.