THE MPAS MESH

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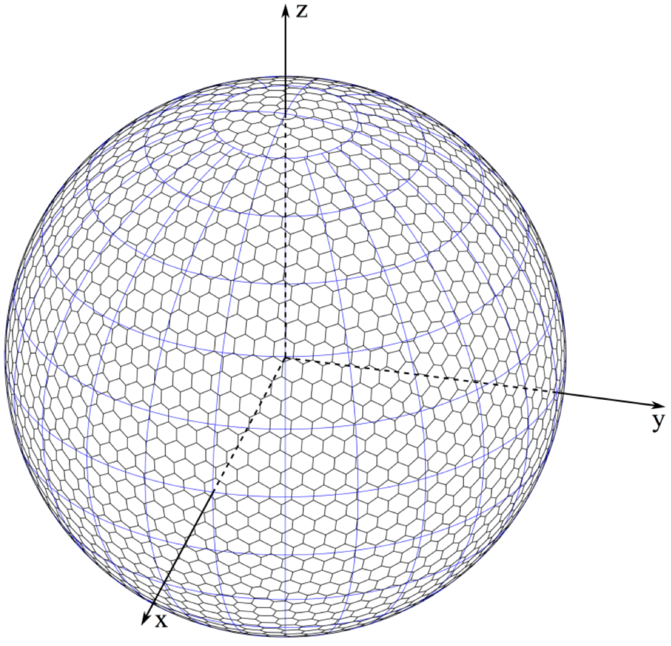
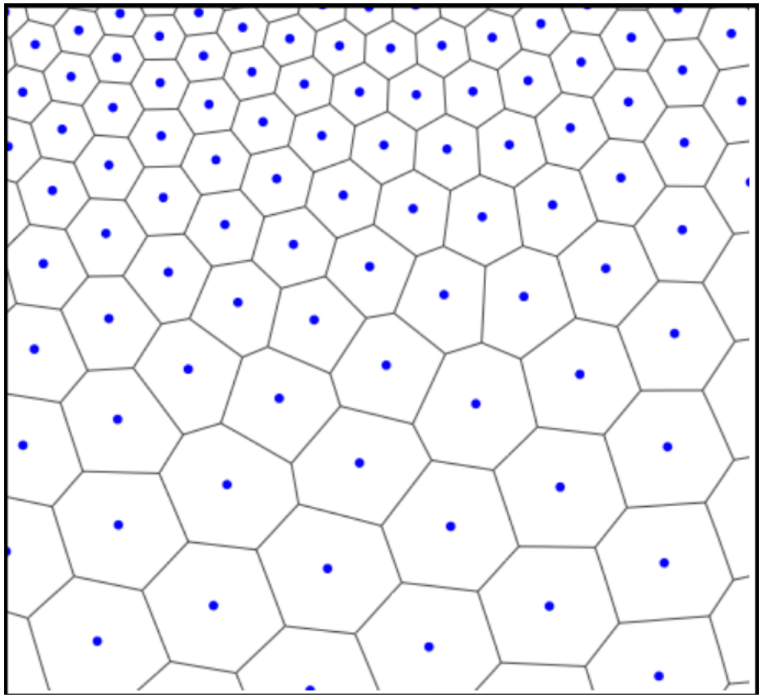
http://kisspng.com



http://mpas-dev.github.io/atmosphere/tutorial.html

Overview of seminar series

- Introduction to MPAS
- **The MPAS Mesh**
- Introduction to Finite-Volume Methods
- MPAS Numerics and Dynamics
- **MPAS Workshop



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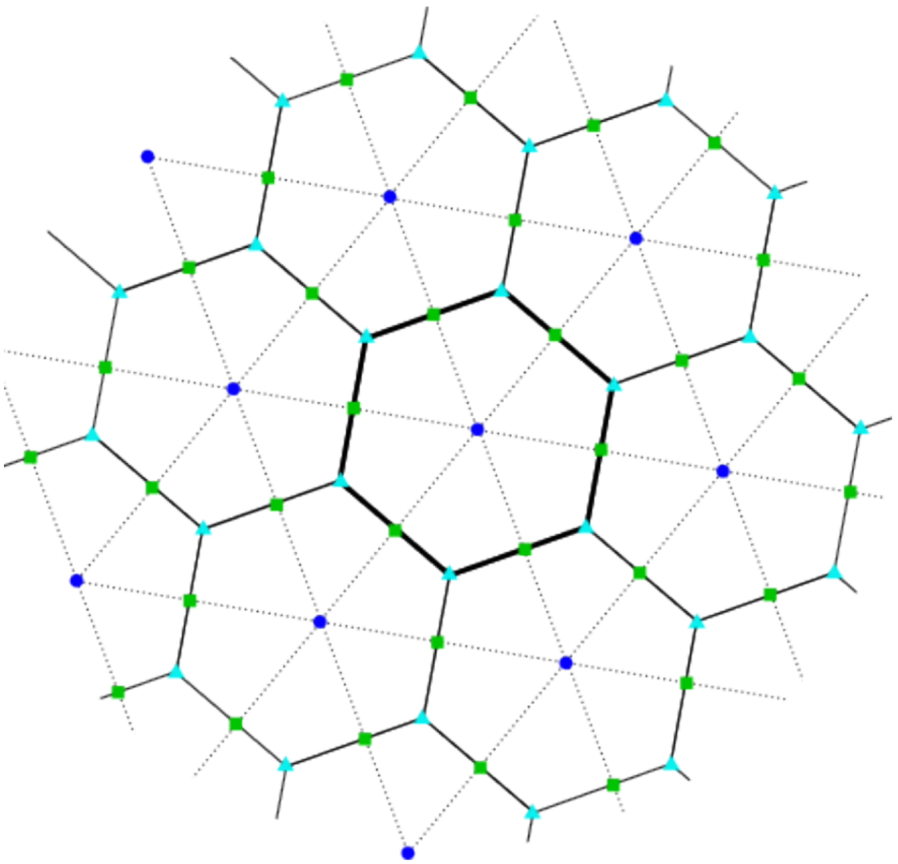

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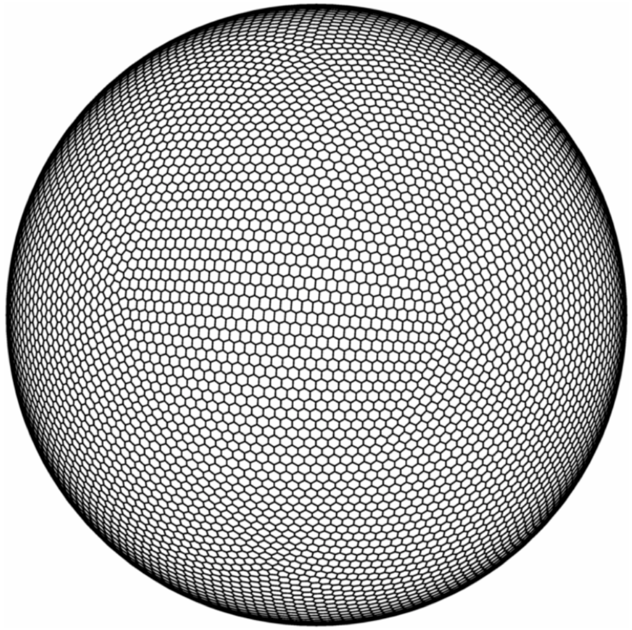
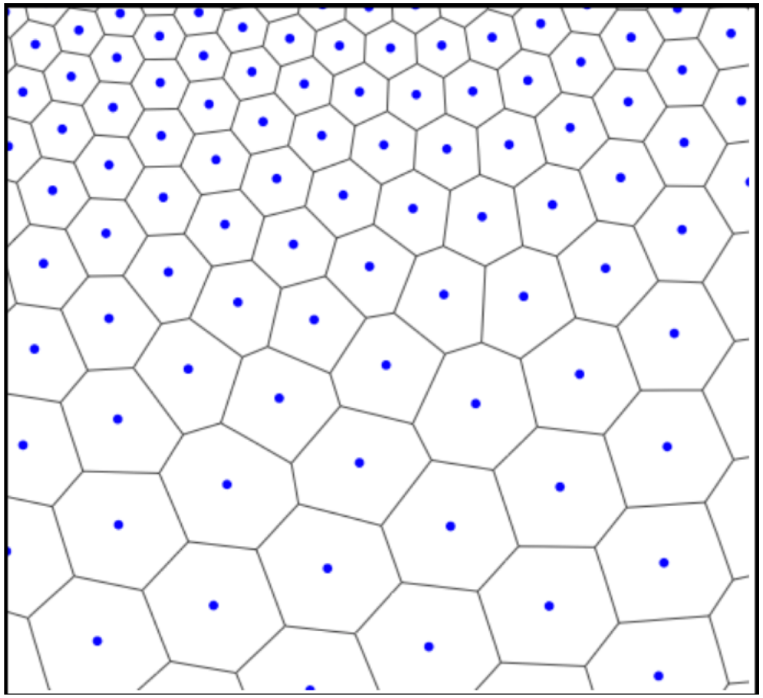
Spherical Centroidal Voronoi Tessellation



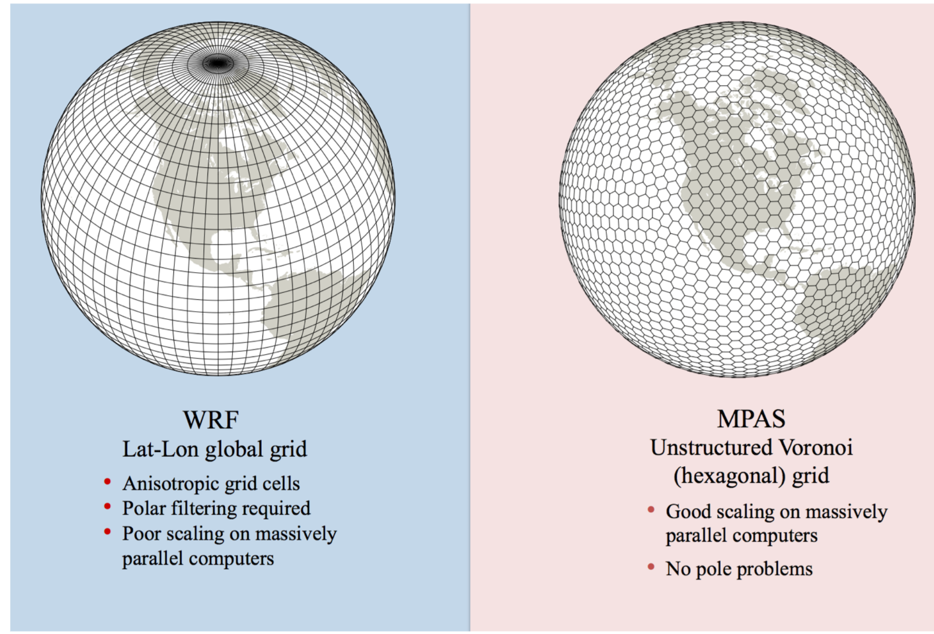


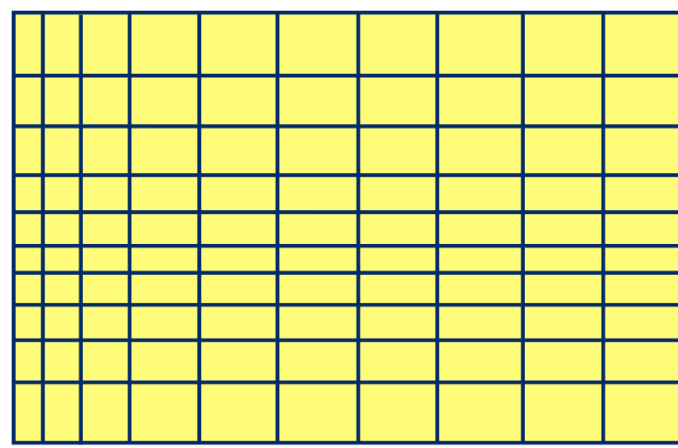
http://mpas-dev.github.io/atmosphere/tutorial.html

SCVTs and MPAS

http://mpas-dev.github.io/atmosphere/tutorial.html

- Want a multiresolution approach to global climate and weather modelling (don't want boundaries)
- Don't want any singularities (looking at you, **lat-lon**)
- Want ease of control in how you define what's "high resolution" (looking at you, **stretched-grid**)





Ringler et al. (2011)

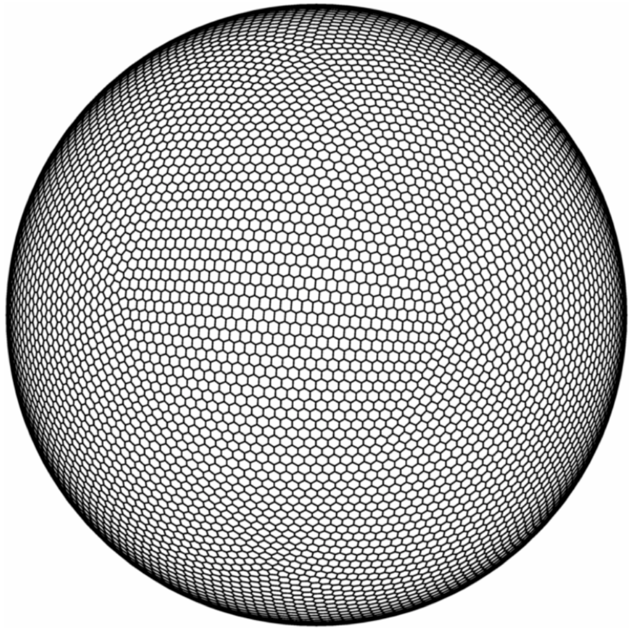
https://www3.nd.edu/~gtryggva/CFD-Course/2011-Lecture-25.pdf

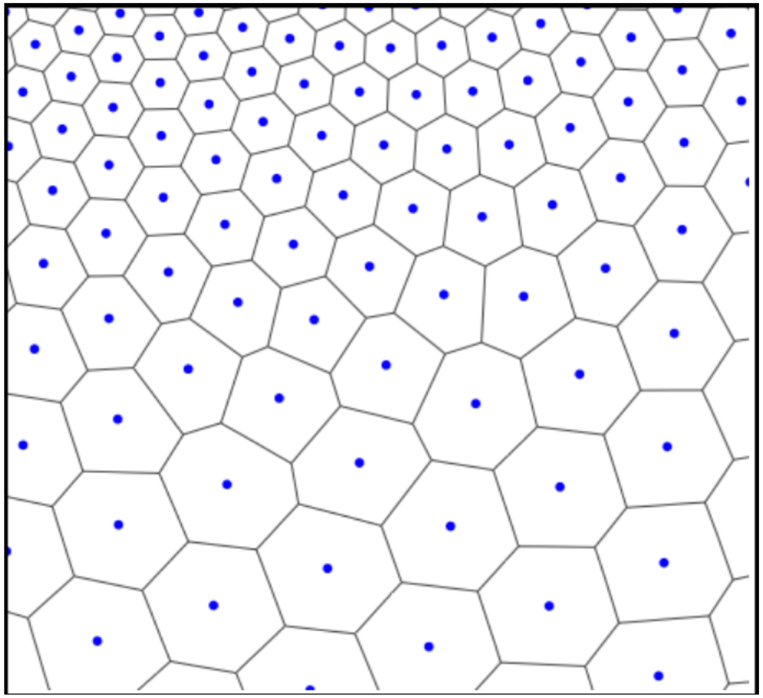
Primary Goal

- Build a mesh that covers the whole globe… **(global)**
- …that has flexible geometry… **(multiresolution)**
- …and no boundaries or discontinuities **(conformal)**

www.shutterstock.com

Spherical Centroidal Voronoi Tessellation





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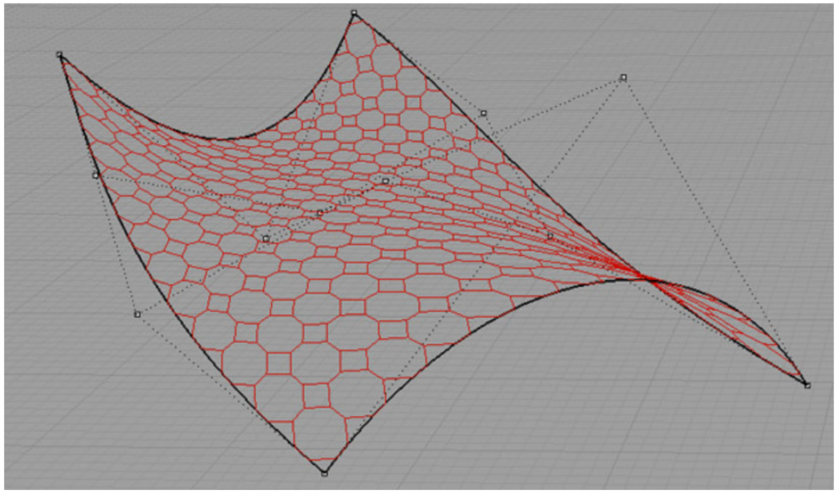
Tessellation

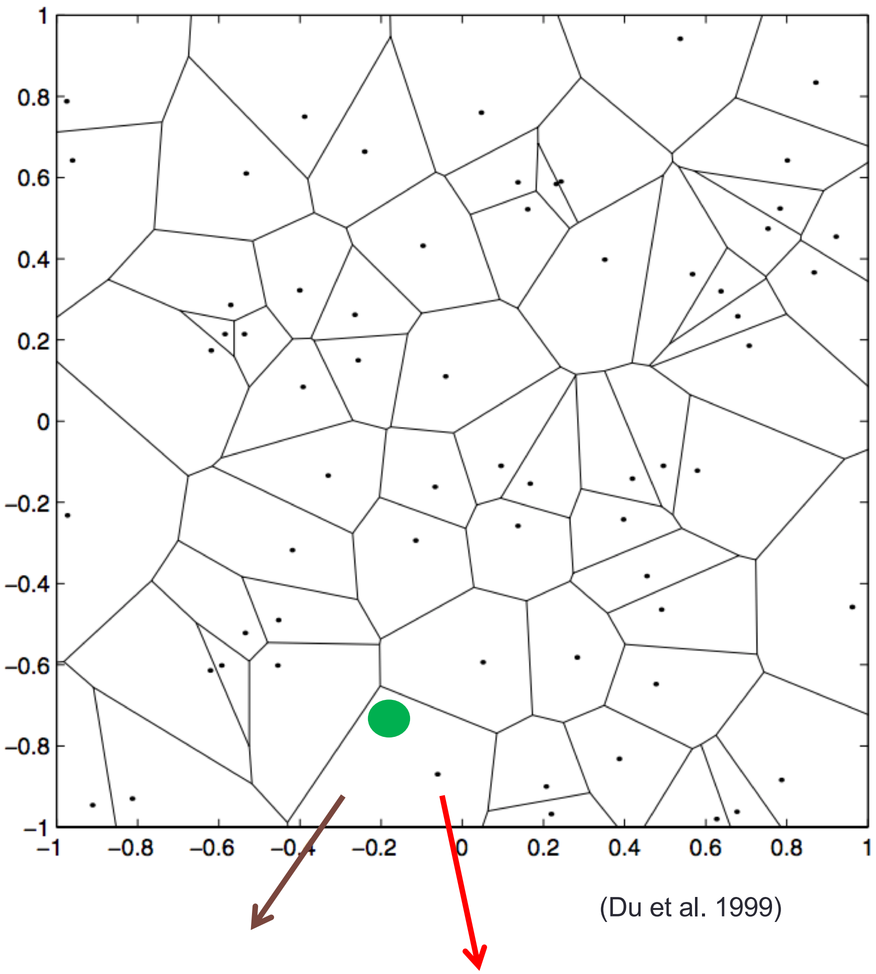
Study of Regular Division of the Plane with Reptiles (1939) – M.C. Escher

- Tiling of surface using geometric shapes without overlaps or gaps



http://www.designcoding.net/semi-regular-tesselation/

Voronoi Tessellation



(Du et al. 1999)

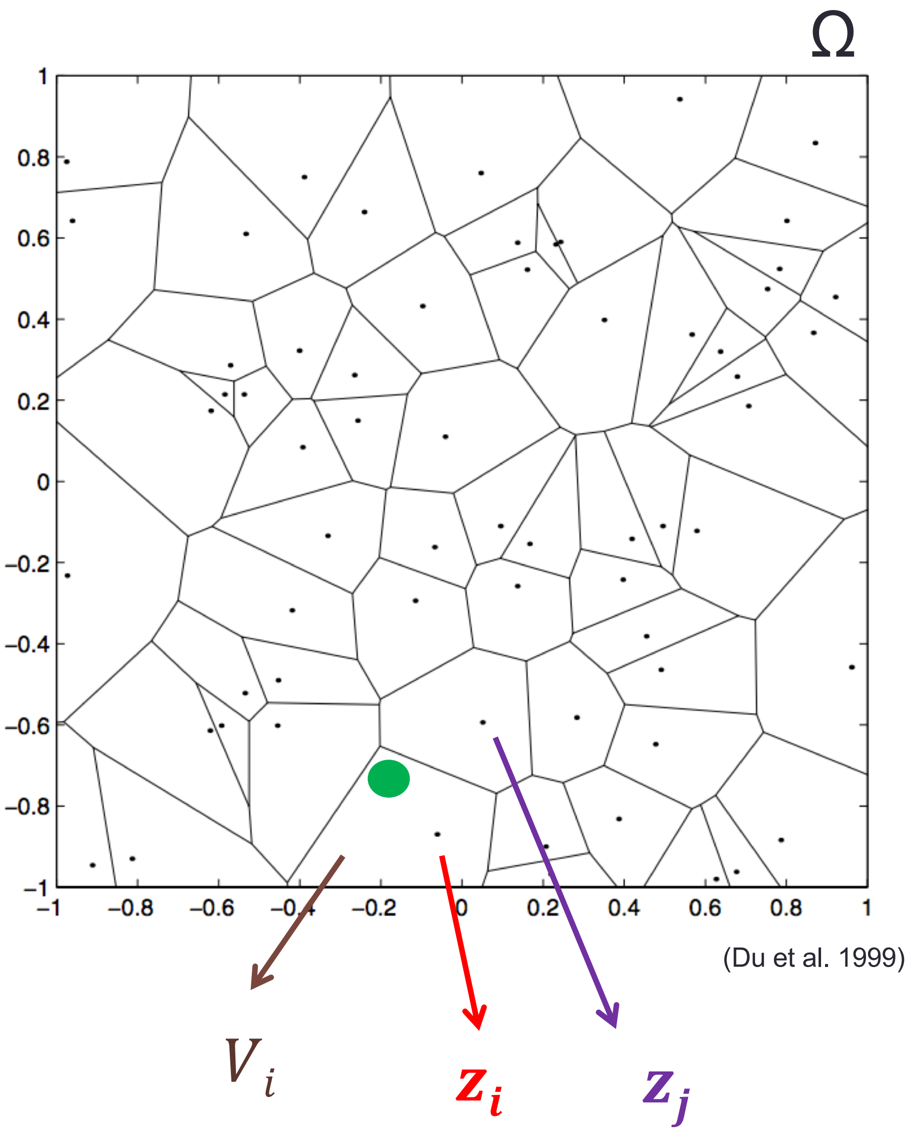
Region *i*

Generating point *i*

- Tessellation of surface with tiles (**regions**) that are polygons each associated with a **generating point**
- All **points** in each region are closer to the region's generating point than to any other generating point

http://kisspng.com

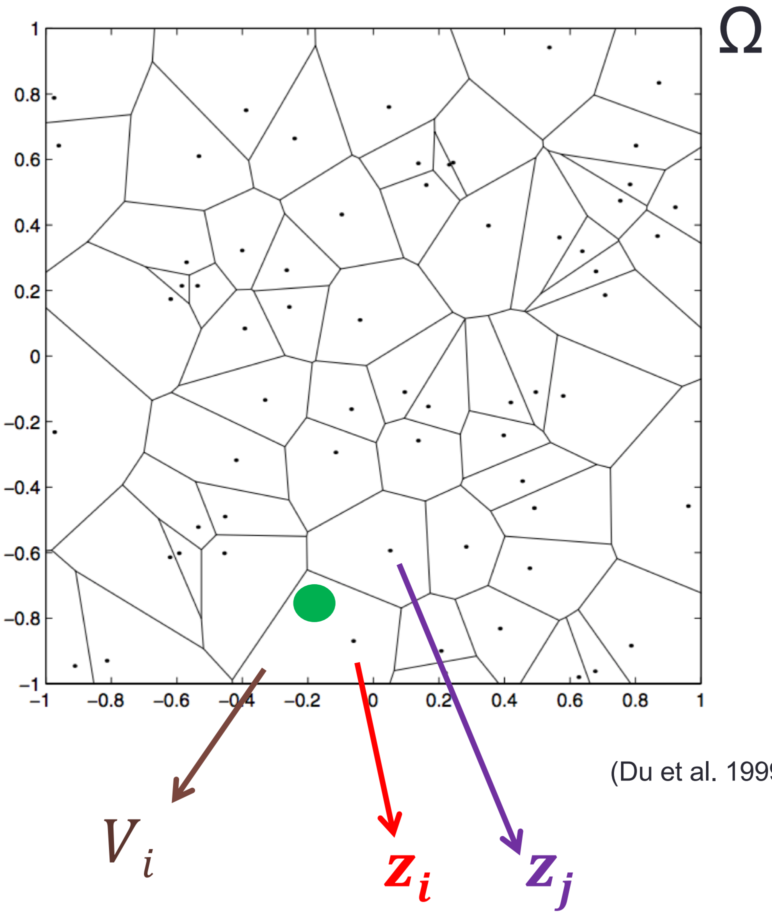
Voronoi Tessellation - Definition



(Du et al. 1999)

- Tessellation of domain $\Omega \subseteq \mathbb{R}^N$ with *k* regions V_i that are each associated with a z_i, $1 \leq i \leq k$
- All points x in V_i are closer to generating point z_i than to any other z_j, $j \neq i$

Voronoi Tessellation - Definition



(Du et al. 1999)

- Points associated with a Voronoi region V_i make up a Voronoi set $\hat{V}_i$

$$\hat{V}_i = \{\boldsymbol{x} \in \Omega \ | \ |\boldsymbol{x} - \boldsymbol{z_i}| < |\boldsymbol{x} - \boldsymbol{z_j}| \ \text{for } j = 1, \dots, k, j \neq i \}$$

Voronoi Tessellation - Definition

- Set of all Voronoi regions is the Voronoi tessellation $\mathcal{V}$ of Ω

$$\mathcal{V} = \{V_i\}_{i=1}^{k}$$

- This is associated with the set of generating points **Z** of Ω

$$\boldsymbol{Z} = \{\boldsymbol{z}_i\}_{i=1}^{k}$$

Voronoi Tessellation - Properties

- Geometric entity V_i is the same as the set-based entity $\widehat{V}_i$
- Voronoi regions do not overlap (intersections = empty set)

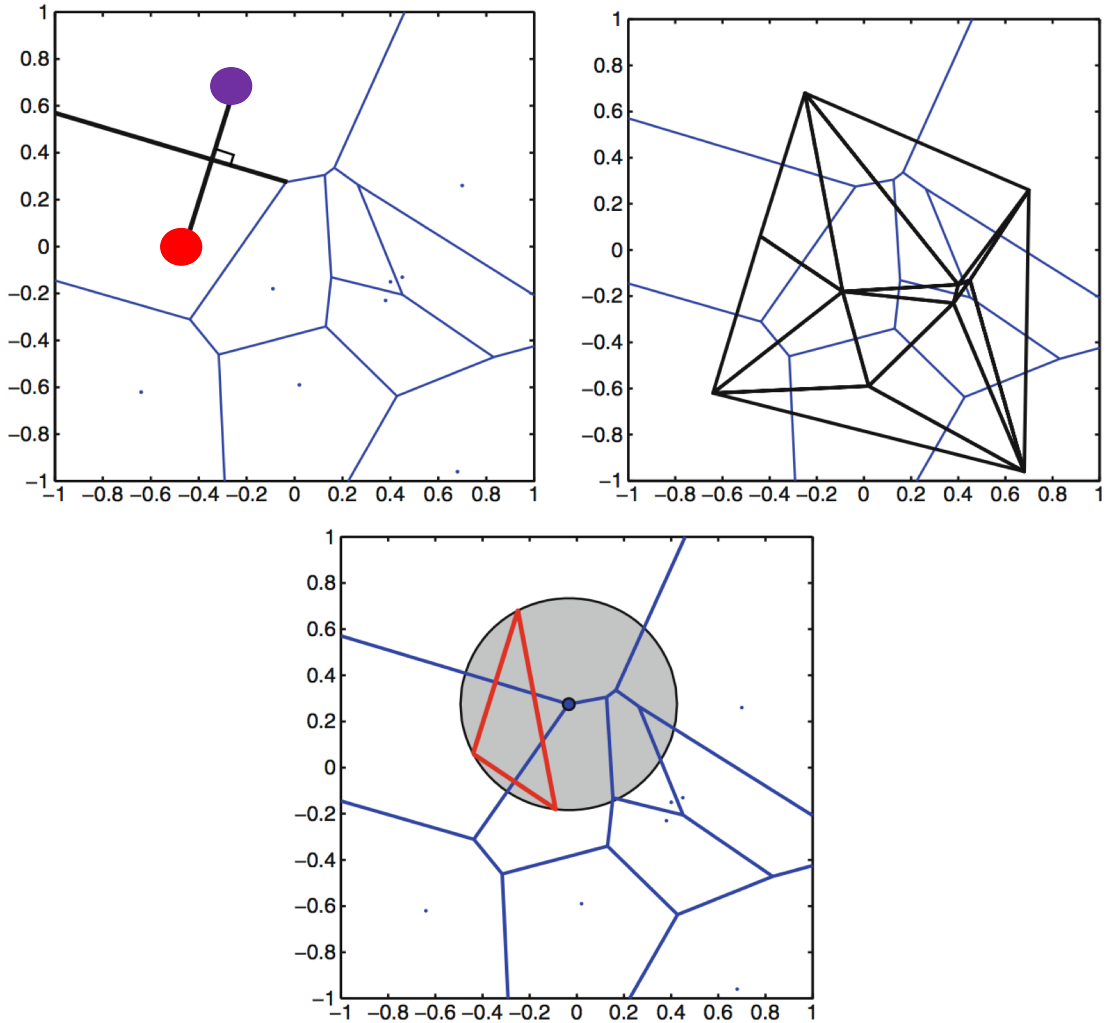
$$V_i \cap V_j = \emptyset \text{ for } i \neq j$$

- Voronoi regions and their boundaries cover Ω completely, including its boundaries

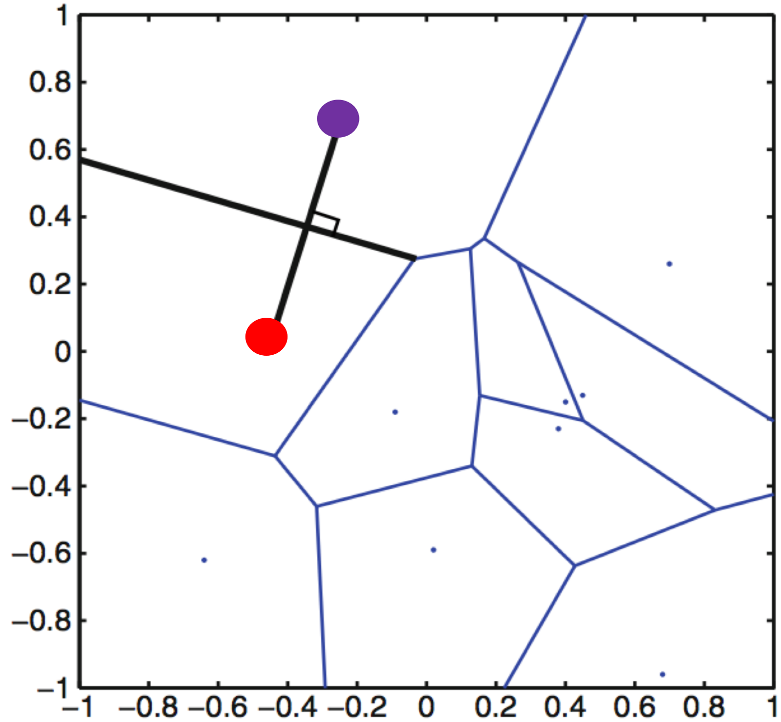
$$\bigcup_{i=1}^{k} \overline{V}_i = \overline{\Omega}$$

Voronoi Tessellation - Definition

(Ju et al. 2011)



Voronoi Tessellation - Definition

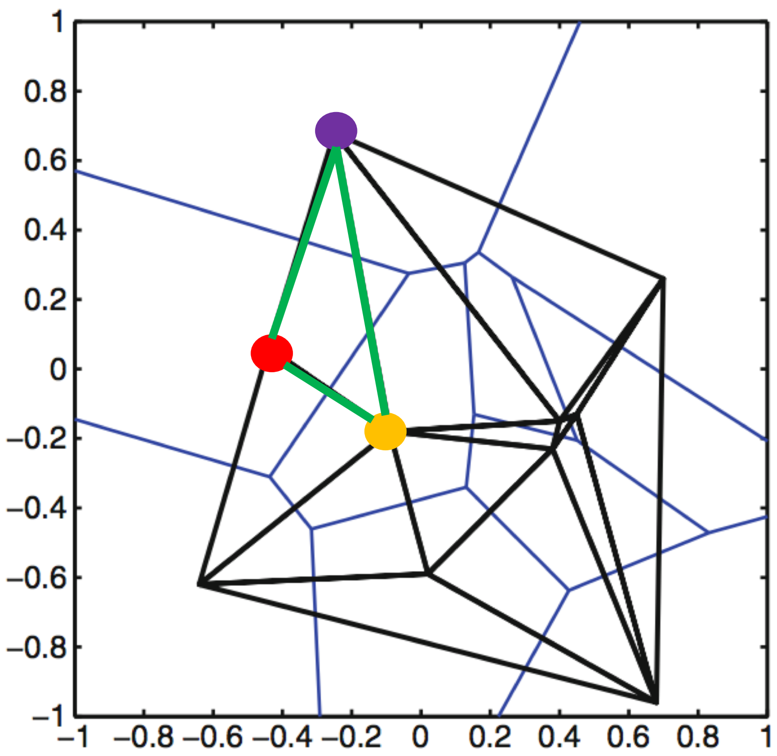


(Ju et al. 2011)

- Edges between two cells bisects (and is orthogonal to) the line joining the two generators

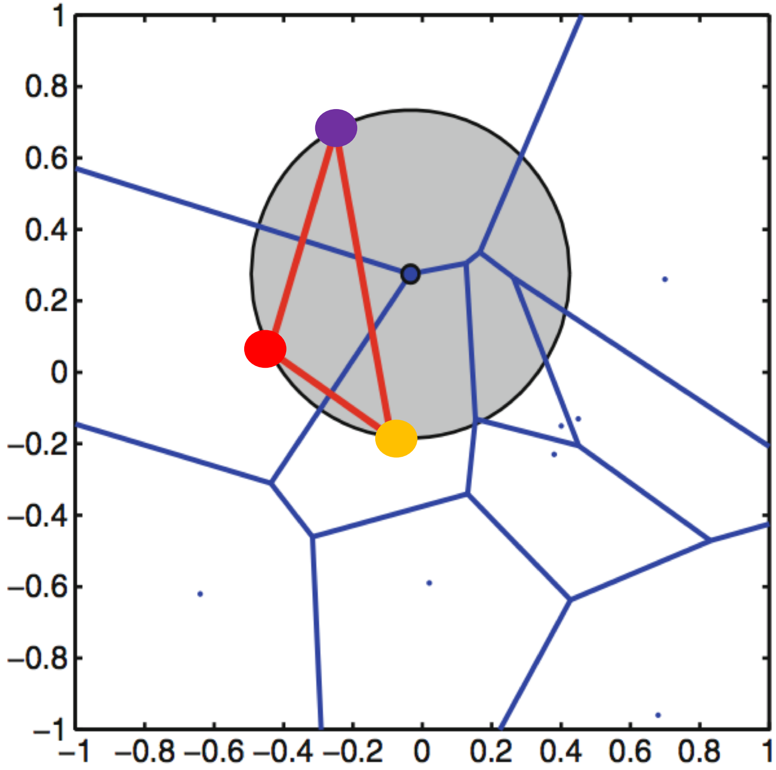
Voronoi Tessellation - Definition

(Ju et al. 2011)



- Connecting the generators of the cells forms the **Delaunay triangulation** of the mesh

Voronoi Tessellation - Definition

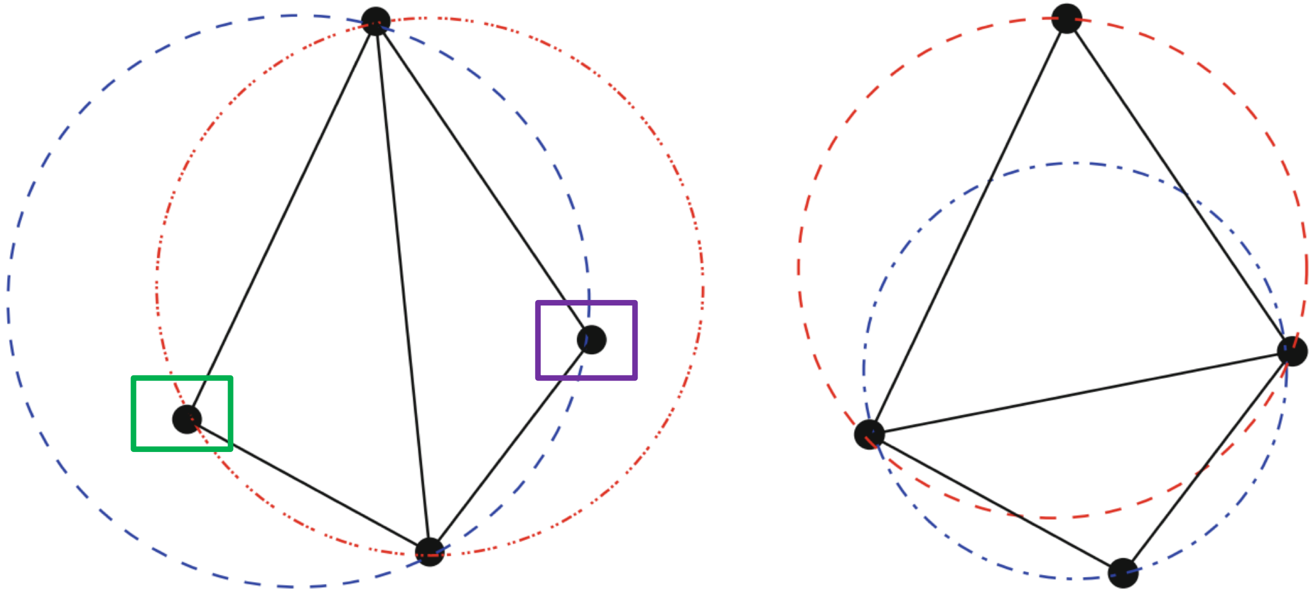


(Ju et al. 2011)

- Circle centered at the **vertex** shared by the three cells should touch the three generators; no generators inside the circle (circumscription property/**Delaunay condition**)

Voronoi Tessellation - Definition

(Ju et al. 2011)



- Left: bad (green rectangle within blue circle; purple rectangle within red circle)
- Right: good (meets Delaunay condition)

Voronoi Tessellation - Energy

- Energy functional (cost function/variance/inertia)

$$\mathcal{E}\left(\{\boldsymbol{z}_i\}_{i=1}^{k}, \mathcal{V}(\{\boldsymbol{z}_i\}_{i=1}^{k})\right) = \sum_{i=1}^{k} \mathcal{E}_i(\boldsymbol{z}_i, \mathcal{V}_i(\boldsymbol{z}_i)) = \sum_{i=1}^{k} \int_{V_i} \rho(\boldsymbol{x}) \left|\boldsymbol{x} - \boldsymbol{z}_i\right|^2 d\boldsymbol{x}$$

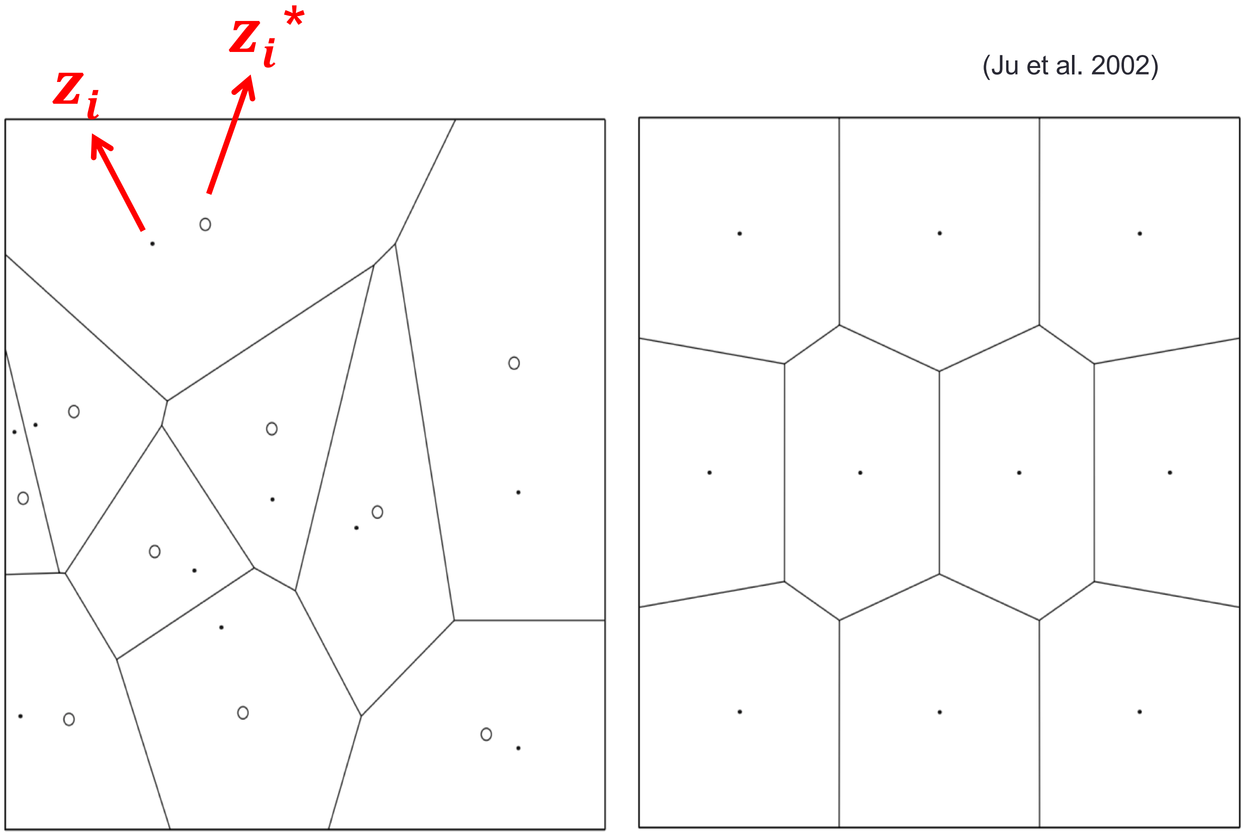
- Total energy is equal to the sum of the energies of each region within domain

- $\rho(\boldsymbol{x}) > 0$ is a differentiable density function defined in $\overline{\Omega}$ (important for MPAS later!)

- "Measure of how well information in Ω is spread out in $\mathcal{V}$ for a given information density $\rho(\boldsymbol{x})$"

Centroidal Voronoi Tessellation

(Ju et al. 2002)



$\mathbf{z}_i \neq \mathbf{z}_i^*$, $i = 1,\ldots, k$

$\mathbf{z}_i = \mathbf{z}_i^*$, $i = 1,\ldots, k$

- In general, the generating point z_i of each Voronoi region is not the same as the mass centroid z_i^* of the region

CVT – Definition through $\mathcal{E}$

- CVT = generating points are collocated with centers-of-mass of Voronoi regions:

$$\boldsymbol{z_i} = \frac{\int_{V_i} \boldsymbol{x}\rho(\boldsymbol{x})d\boldsymbol{x}}{\int_{V_i} \rho(\boldsymbol{x})d\boldsymbol{x}} = \boldsymbol{z_i^*}$$

- $\boldsymbol{z_i} = \boldsymbol{z_i^*}$ results in a critical point of energy $\mathcal{E}$

CVT – Definition through $\mathcal{E}$

- Can show by (homework):

- **Variational approach**
  - Perturb generating point $\boldsymbol{z_i}$ with some small $\alpha\boldsymbol{v}$, where $\boldsymbol{v}$ is some arbitrary vector in $\mathbb{R}^N$ and let $\alpha \to 0$

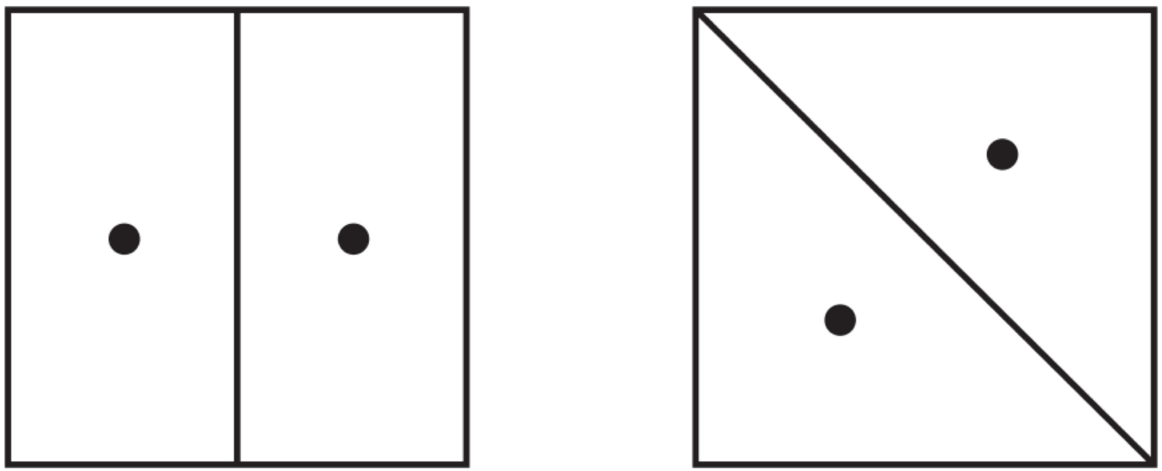
- **Gradient approach**
  - Find $\boldsymbol{z_i}$ such that $\nabla\mathcal{E} = 0$

$$\boldsymbol{z_i} = \frac{\int_{V_i} \boldsymbol{x}\rho(\boldsymbol{x})d\boldsymbol{x}}{\int_{V_i} \rho(\boldsymbol{x})d\boldsymbol{x}}$$

CVT – Uniqueness

Uniqueness?

$\rho(\boldsymbol{x}) = 1$ in unit square

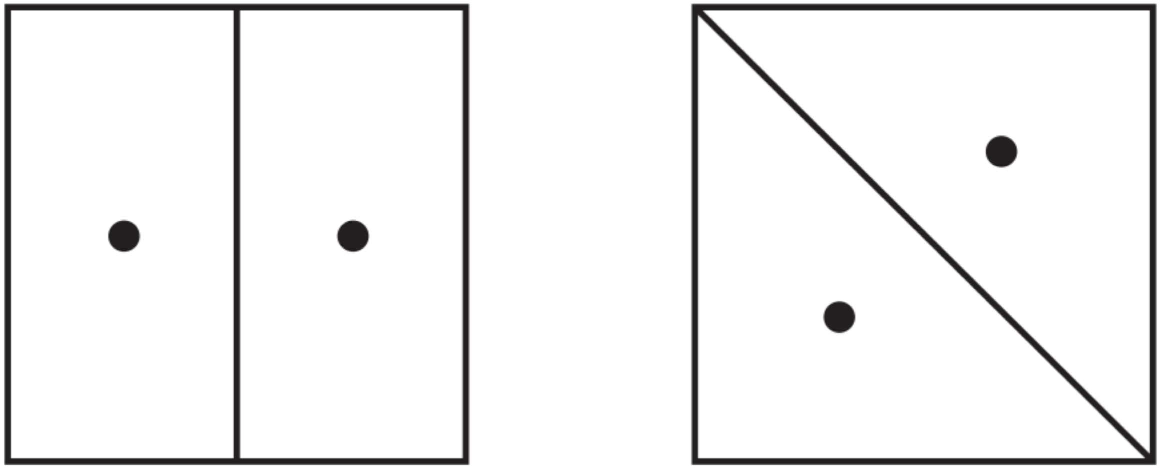


(Ju et al. 2002)

Energy functional reaches a critical point with a CVT, but **not necessarily** an absolute minimum

CVT – Uniqueness

More precisely:



(Ju et al. 2002)

For a given set of generating points $\boldsymbol{Z} = \{\boldsymbol{z}_i\}_{i=1}^{k}$ there is one unique $\mathcal{V} = \{V_i\}_{i=1}^{k}$

For a given Ω, k, and/or ρ, there is no unique $\boldsymbol{Z}$ (so no unique $\mathcal{V}$)

CVT – Goal

- Best thing we can do for a given Ω, *k*, and/or ρ:

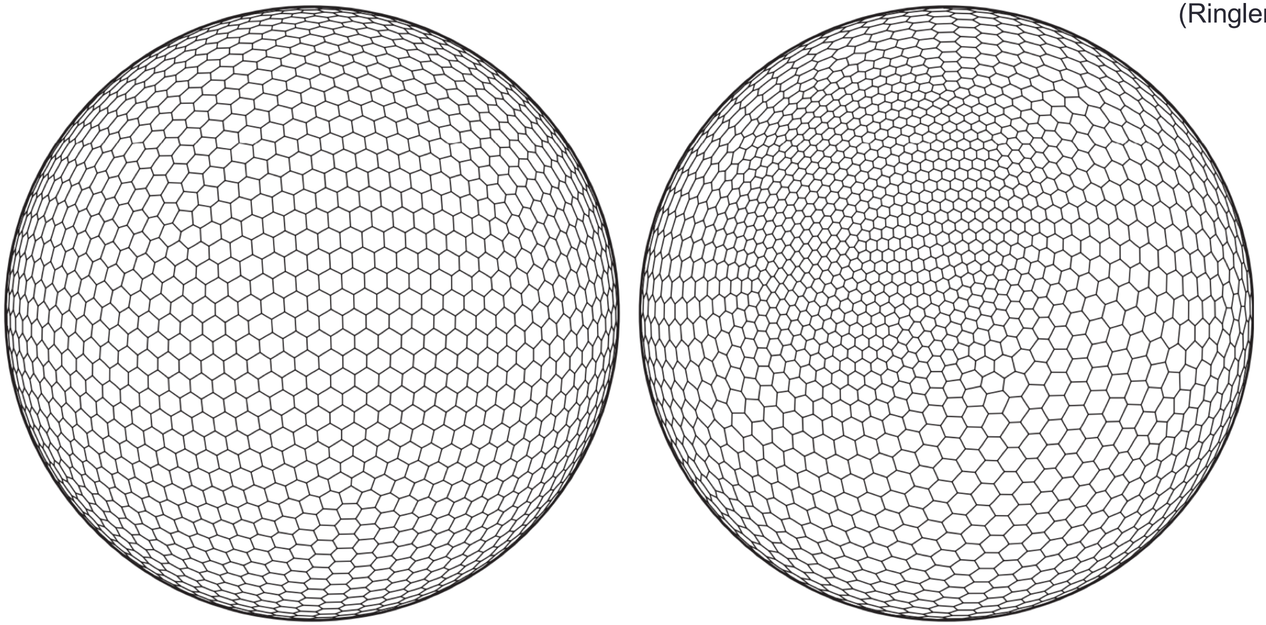
Find a tessellation $\mathcal{V}$ that minimizes $\mathcal{E}$

- Gersho's conjecture in **2D** (1979): for sufficiently large *k*, the **shapes of the Voronoi regions for an optimal tessellation is the hexagon**
  - Proved by Newman (1982)
  - No proof for 3D+

Spherical Centroidal Voronoi Tessellation

Goal: Given a spherical region $S^2 = \{\boldsymbol{x} \in \mathbb{R}^3 : |\mathbf{x}| = r\}$, a positive integer *k*, and a differentiable density function *ρ*, compute a *k*-point CVT of S^2 (with *k* regions V_i, $i = 1,\ldots,k$)

(Ringler et al. 2011)



Both meshes have 2562 regions, but have different density functions.

Applications of (S)CVTs

- Original papers by Gersho (1979) focused on how to cut up a signal fairly (quantization)

- Analog -> Digital stuff

- Useful for electrical engineers

CVTs for Image Compression

Original | **Monte-Carlo (random distribution)**

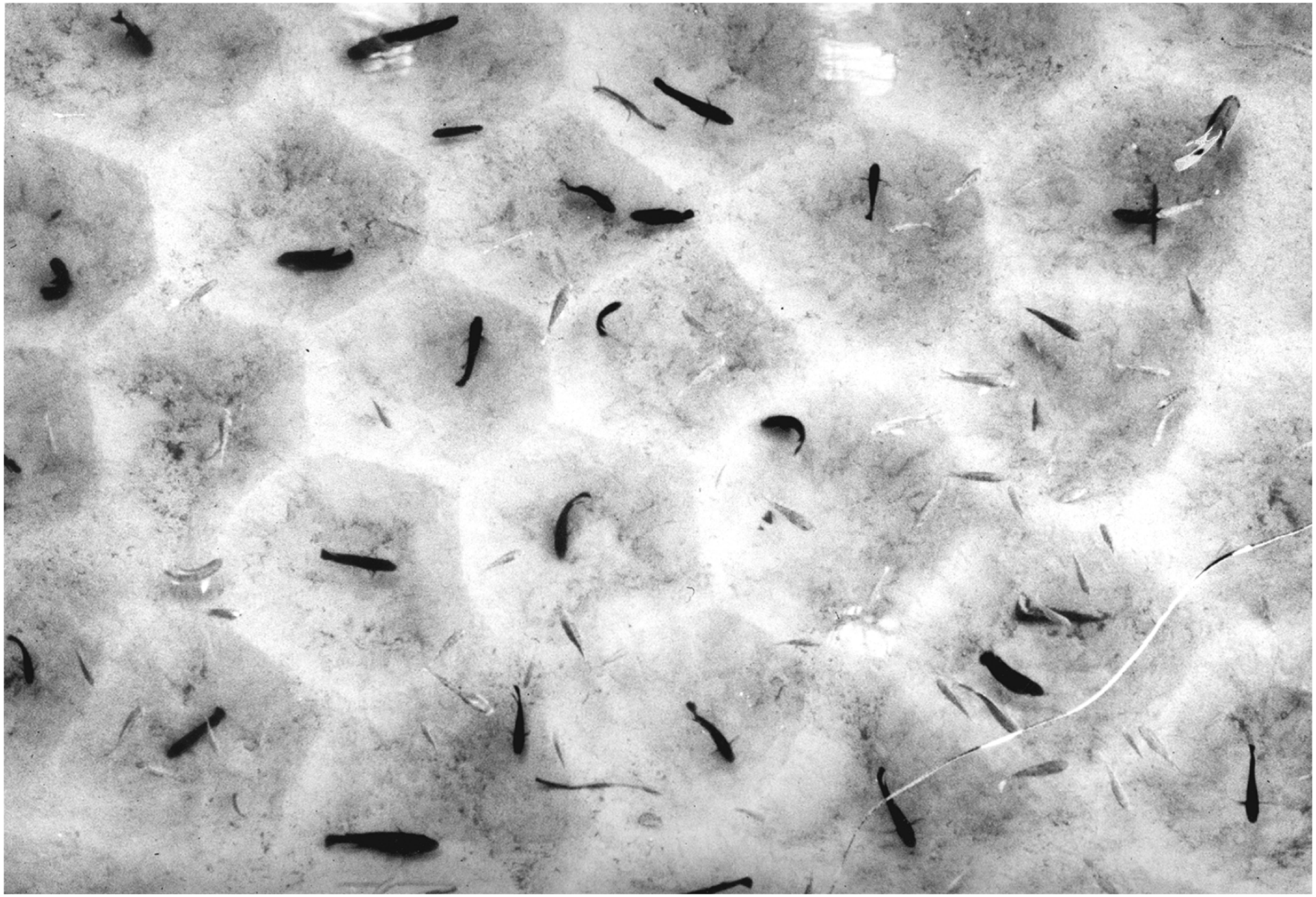
(Du et al. 1999)

CVT | **CVT + ”Dithering”**

CVTs in Biology/Zoology

Male tilapia territories (sand pits)

(Du et al. 1999)



CVTs in Geology

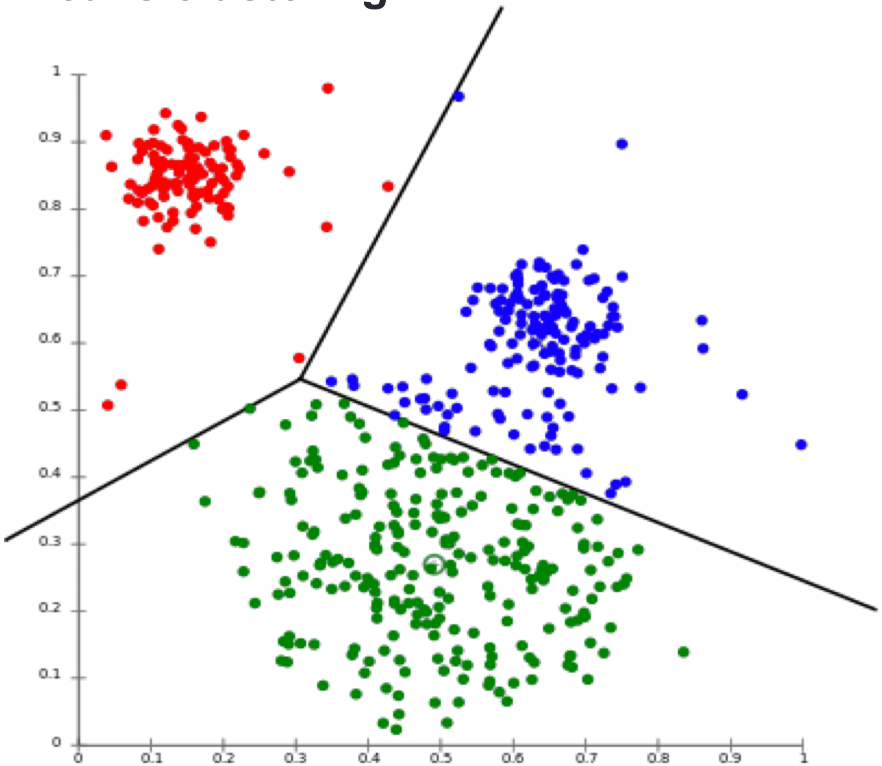
https://economictimes.indiatimes.com/magazines/panache/the-40000-odd-basalt-formations-at-the-giants-causeway-in-northern-ireland/articleshow/65454173.cms

Giant's Causeway, Northern Ireland

https://www.ireland.com/en-se/what-is-available/natural-landscapes-and-sights/articles/giants-causeway-myth/

CVTs in Statistics/Data Analysis

***k*-means clustering**



https://aws.amazon.com/blogs/machine-learning/k-means-clustering-with-amazon-sagemaker/

Mesh Generation Methods

- All methods are **iterative**
- Two main methods:
  - <u>Deterministic Methods</u>
    - Next iteration based directly on the locations of generator points/Voronoi regions from previous step
    - Slow, but generally better convergence
  - <u>Probabilistic Methods</u>
    - Next iteration affected by some random process
    - Fast, but generally worse convergence

Mesh Generation Methods

- Most common deterministic method: **Lloyd's Method** (Lloyd 1982)
- Update generators $\boldsymbol{z_i}$ and Voronoi regions $\boldsymbol{V_i}$ independently from each other
  - Fix generating points, and create Voronoi regions
  - Fix regions, and compute new centroids

Lloyd's Method

Lloyd's Method (currently used by MPAS)

- 1) Populate sphere with generating points $\mathbf{z}_i$
- 2) Create Voronoi regions with those points
- 3) Compute centroids of those regions via numerical integration
- 4) Repeat (2) and (3) until change in some criterion (like energy) reaches tolerance (i.e. "converges" to some value)

Lloyd's Method for MPAS

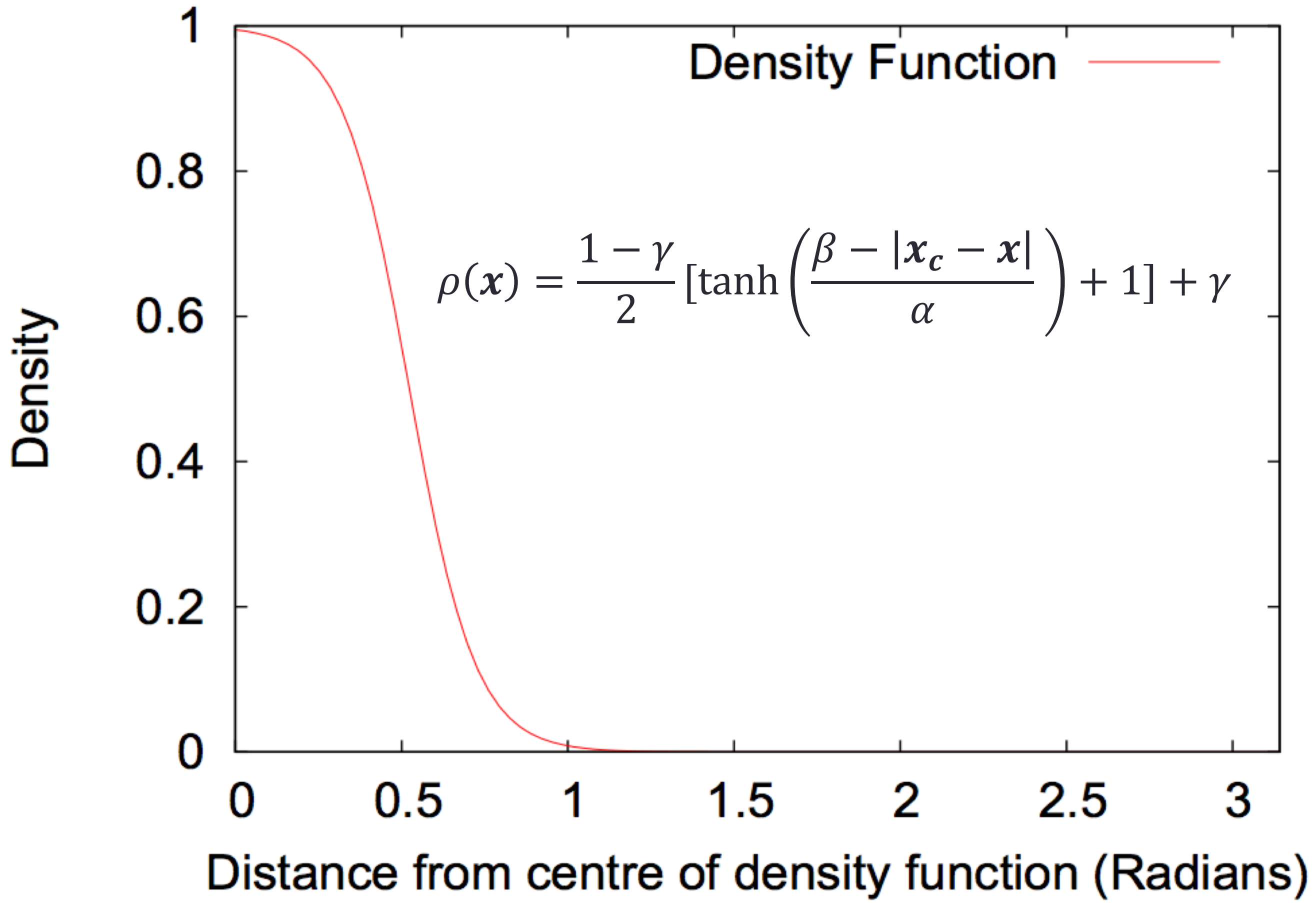
- Ju et al. (2011): for sufficiently many Voronoi cells, the diameters *h* of the cells on an SCVT are conjectured to be related by:

$$\frac{h_i}{h_j} \approx \left(\frac{\rho(\mathbf{z}_j)}{\rho(\mathbf{z}_i)}\right)^{1/4}$$

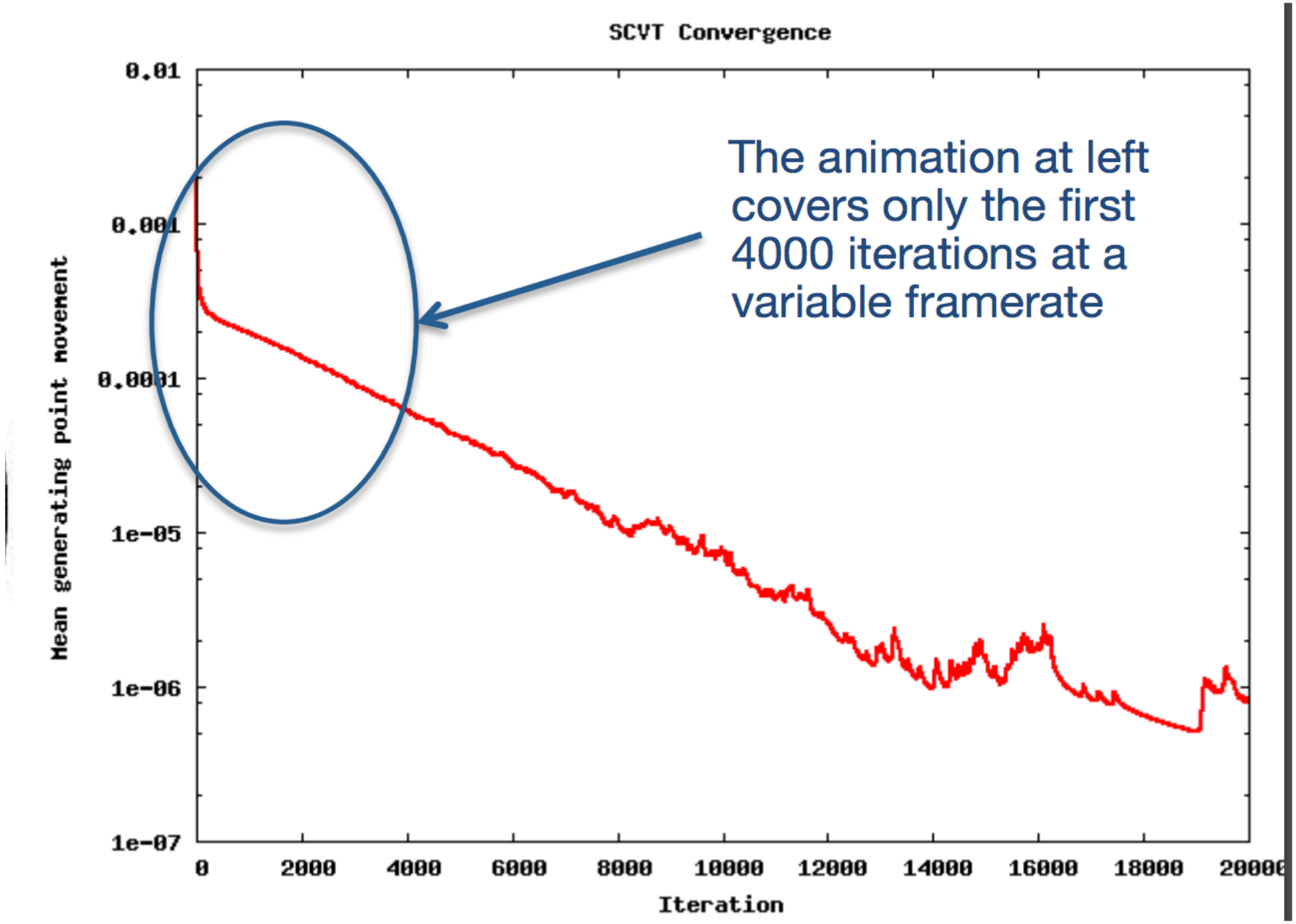
- Can use density function to control resolution of mesh at certain points!

Lloyd's Method for MPAS

Jacobsen et al. (2013): 8x difference in resolution between coarse and fine portions of mesh; $\max(\rho) = 1, \min(\rho) = \left(\frac{1}{8}\right)^4$



Lloyd's Method – Convergence for MPAS



http://www2.mmm.ucar.edu/people/duda/files/mpas/talks/mpas_mesh_generation.pdf

Summary

- CVTs and SCVTs have lots of applications in various fields, including weather and climate modelling
- CVTs/SCVTs have more desirable (energy) properties than general VTs
- Many ways to generate an SCVT (but it's a very tough problem)
- MPAS mesh currently uses Lloyd's method to generate an SCVT

Next Time on *Dragon Ball Z*….

$$
\begin{aligned}
f(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 \\
&\quad + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \frac{f''''(x_0)}{4!}(x - x_0)^4 + \cdots \\
&= \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.
\end{aligned}
$$

References

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