Von Neumann Method for Determining Numerical Stability of a Finite-Difference Scheme - R. Stull, Feb 2025

1. Write the finite difference eq. for 1-D linear advection of temperature T

   $\partial T/\partial t = -U_o \cdot \partial T/\partial x$

   where U_o is constant wind speed. Use j as grid index, n as time index.

2. Assume the following eigenmode solution, & plug into the finite diff. eq.

   $T = [A(k)]^n \cdot \exp(i\, k\, j\, \Delta x)$

   where A is amplitude, k is wavenumber, i is sqrt(–1), Δx is grid spacing

3. Divide the resulting eq. by

   $[A(k)]^n \cdot \exp(i\, k\, j\, \Delta x)$

4. Solve the resulting equation for amplitude A(k).

5. Use Euler's notation to convert into sines and/or cosines

   $e^{iy} - e^{-iy} = 2\, i \sin(y)$ and $e^{iy} + e^{-iy} = 2\cos(y)$

   $e^{iy} = \cos(y) + i\sin(y)$ and $e^{-iy} = \cos(y) - i\sin(y)$

6. Replace $U_o\, \Delta t / \Delta x$ with the Courant Number C_R.

7. Simplify the result where possible, and collect the real terms together and the imaginary terms together. These are the real and imaginary parts of the Amplitude, where $A(k) = A_R(k) + i\, A_I(k)$.

8. Find the amplitude modulus: $|A(k)| = \text{sqrt}\{ [A_R(k)]^2 + [A_I(k)]^2 \}$.

9. Replace k with m, using: $k = 2\pi / L = 2\pi / (m\, \Delta x)$

10. Find the worst-case m that gives the worst (most unstable) amplitude modulus, and assume that this wave exists in the atmosphere.
    (Utilize the handout of sine and cosine values for various m values.)

11. Make stability conclusions:
    Stable, but damped, if $|A(m)| < 1$.
    Stable if $|A(m)| = 1$.
    Unstable (blows up) if $|A(m)| > 1$.

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