Goals

- Understand the non-uniqueness in geophysical interpretations
- Understand the concepts of inversion.
- Basic workflow for solving inversion problems.
- Some important aspects of magnetic data inversion
- Case history example
Superposition for Magnetics Data (GPG d5)

Magnetic field for one prism

Magnetic field for 5 prisms
Earth can be complicated

A complicated earth model

Magnetic data for a complicated earth model.

To interpret field data from a complicated earth we need to have formal inversion procedures that recognize non-uniqueness.
Geophysics using inversion

Inversion: find values for cells such that data are explained. Use optimization theory.

Physical property distributions = MODELS
A generic inversion flow chart – not a linear pathway.

Goals:
- Choose a model that can predict the measurements.

Decisions:
- Are predictions similar enough to field data?
- Does the model agree with prior knowledge and geological intuition?
Estimating a model and predicting data

- How to represent the Earth? **Discretize** into rectangular cells.

- Return to **model estimation** later.

- **Predicting data**: result must be able to cause the data. This is forward modelling.

\[ F[m] \]
Misfit: comparing predictions to measurements

Once a model is estimated …

- Calculate data caused by that model.
- Refer to survey data:
- Compare predictions to these measurements.
- Is comparison within errors?

YES: Compare

NO: Modify model and try again

Proceed to check for acceptibility
Acceptable models and non-uniqueness

- There are infinitely many models that could generate the data.

Why is this so?
1. Because there will always be more unknowns than data points.
   \# model cells > \# measurements
2. Some physically based non-uniqueness.
Inversion

Why is it difficult to “calculate” the earth’s structure based on all those measurements?
- Non-uniqueness.
  - Green’s Theorem
  - Algebraically many more variables than data

There are infinitely many configurations of $K$ that are capable of causing the data.

Solutions?
- Optimization theory …
- find the model with the “least” or “most” … of what ?????
Questions to consider:

- Consider the simple problem that involves two unknowns, x and y. We have one datum

  \[ x + y = 2 \]

- What are the values of x and y?

  \[ x=1, \ y=1 \]
  \[ x=0, \ y=2 \]
Define a ruler to measure the model. Call it $\phi_m$.

Values of a model can be plotted. What “norms” or rulers are sensible?

**Norm 1:**
Size of a model is sum of all values.
“Sum of squares” is a so-called $L_2$ norm.

$$\phi_m = \| m \|^2 = \sum_{i=1}^{4} m_i^2$$

**Norm 2:**
Smoothness of a model measures differences between adjacent model values:

$$\phi_m = \sum_{i=1}^{3} (m_{i+1} - m_i)^2$$
Model norms: useful versions

Smallest model (measure model’s “size”):

$$\phi_m = \int (m)^2 \, dx$$

Smoothest model (measure model’s “smoothness”):

$$\phi_m = \int \left( \frac{dm}{dx} \right)^2 \, dx$$

Smallest model compared to a reference:

$$\phi_m = \int (m - m_0)^2 \, dx$$

Combination; choice of “measure” is adjustable:

$$\phi_m = \alpha_s \int (m - m_0)^2 \, dx + \alpha_x \int \left( \frac{d}{dx} (m - m_0) \right)^2 \, dx$$
What is a good measure of misfit?

- If errors are Gaussian with standard deviation $\varepsilon_i$, then a good measure of misfit between predictions and field data is called $\phi_d(m)$:

$$
\phi_d(m) = \sum_{i=1}^{N} \left( \frac{d_i^{obs} - d_i^{pred}}{\varepsilon_i} \right)^2
$$

- Now, predictions can be considered OK when $\phi_d(m) < \text{tolerance}$

- From basic statistics, with Gaussian data, $\phi_d(m)$’s expected value is $E[\phi_d(m)] = N$ (N = # of data)
Combining model norms and misfit:  
*Inversion as optimization.*

- A statement of the inverse problem is:
  - Find the model $m$ that
    - minimizes the model norm $\phi_m(m)$, and
    - produces an acceptably small misfit ($\phi_d(m) < \text{tolerance}$).

- Mathematically, such problems are often re-cast as a single optimization:
  
  “Minimize $\phi = \phi_d + \beta \phi_m$”
  
  where $0 < \beta < \infty$

- Two parts $\phi_d$ and $\phi_m$ were explained above - what is $\beta$?
Essential aspects

- Each box has important implications for successful inversion.
- Ability to do forward modelling calculations is assumed.

Given:
- Field observations
- Error estimates
- Ability to forward model
- Prior knowledge

Discretize the Earth

Choose a suitable misfit criterion

Design model objective function

Perform inversion

Evaluate results

Iterate

Interpret preferred model(s)

Icons will keep us on track
Inversion of Magnetic Data: Phase I
Discretizing the Earth

- First assume \( \tilde{m} = \kappa \tilde{H}_0 \)
  - ignore remanent magnetization
  - ignore self-demagnetization

- Discretize the earth into \( M \) cells, each with constant \( \kappa \).

- \( \mathbf{B}_i \) includes contributions from \( j = 1 \ldots M \) cells:

\[
\tilde{B}(r_i) = \sum_{j=1}^{M} \frac{\kappa_j \mu_0}{4\pi} \int_{Vol_j} \tilde{H}_0 \cdot \nabla \nabla \left( \frac{1}{|r_i - r|} \right) dv
\]
The Forward Problem:

Given a susceptibility distribution $\kappa_j$, $j = 1, M$, calculate the data $b_i$, $i=1, N$.

$$b_i = \sum_{j=1}^{M} G_{ij} \kappa_j$$

or in matrix form: $$\vec{b} = G \vec{\kappa}$$
The Inverse Problem

Given the data \( b_i, \quad i=1, \, N \)

find the susceptibilities \( \kappa_j, \quad j = 1, \, M \)

such that \( b_i = \sum_i G_{ij} \kappa_j \)
3D Inversion of Magnetic Data

Synthetic Model:

- Block top = 50 m
- 100 x 100 x 100 m
- Susceptibility = 0.05
- I = 30 degrees
- D = 45 degrees
Synthetic survey:

Measured data:
- 100 m line spacing.
- 25 station spacing.
- \( N = 176 \) (elevation = 2m).

Data as recorded.

Contoured “perfect” data

Contoured data with noise = 2 nT
Non-uniqueness

Equivalent layer models which reproduce the data:
Objective function:

\[
\phi_m(\kappa) = \alpha_S \int_{\text{Vol}} (\kappa - \kappa_{\text{ref}})^2 \, dv + \\
\alpha_X \int_{\text{Vol}} \left( \frac{d\kappa}{dx} \right)^2 \, dv + \alpha_Y \int_{\text{Vol}} \left( \frac{d\kappa}{dy} \right)^2 \, dv + \alpha_Z \int_{\text{Vol}} \left( \frac{d\kappa}{dz} \right)^2 \, dv
\]

Choose \( \kappa_{\text{ref}} = 0, \, \alpha_S = 0.0001, \, \alpha_X = \alpha_Y = \alpha_Z = 1; \) length scale is:

Misfit: \( \phi_d = \sum_{i=1}^{N} \left( (d_{\text{obs}} - d_i) / \varepsilon_i \right)^2 \)

The inverse problem is:

minimize: \( \phi = \phi_d + \beta \phi_m \)

Find beta such that \( \phi_d = \phi_d^{*} \) where \( \phi_d^{*} = N \)
Discretization:

Earth model for inversion:
- \( x = y = z = 25 \text{m} \).
- \( \text{N/S and E/W padding} = 50 \text{m} \).
- \( \text{Number of cells is} \quad M = 11,492 \)

Therefore:
- No. data is \( N = 176 \)
- No. unknowns is \( M = 11,492 \)
Compare inversions we have seen:

1. The block model
2. No depth weighting, *no* positivity
3. Depth weighting but *no* positivity
4. Depth weighting *and* positivity
Example: Raglan aeromagnetic data

Select a region of interest.

Keep data set size within reason.

Digitized the Earth – up to $10^6$ cells.
Raglan aeromagnetic data

- Estimate a model for the distribution of subsurface magnetic material.
- Model will be “smooth”, and close to pre-defined reference.
- Display result as cross sections and as isosurfaces.

- Are “sills” connected at depth? Inversion result supports this idea.
- It helped justify a 1050m drill hole.
- 330m of peridotite intersected at 650m 10m were ore grade.
- Image shows all material which has k > 0.04 SI.
Homework/Readings

- Inversion chapter in the GPG 4.a – 4.e and GPG 4.g