Lagrangian and Eulerian Representations of Fluid Flow: Part I, Kinematics and the Equations of Motion

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Summary: This essay introduces the two methods that are commonly used to describe fluid flow, by observing the trajectories of parcels that are carried along with the flow or by observing the fluid velocity at fixed positions. These yield what are commonly termed Lagrangian and Eulerian descriptions. Lagrangian methods are often the most efficient way to sample a fluid domain and the physical conservation laws are inherently Lagrangian since they apply to specific material parcels rather than points in space. It happens, though, that the Lagrangian equations of motion applied to a continuum are quite difficult, and thus almost all of the theory (forward calculation) in fluid dynamics is developed within the Eulerian system. Eulerian solutions may be used to calculate Lagrangian properties, e.g., parcel trajectories, which is often a valuable step in the description of an Eulerian solution. Transformation to and from Lagrangian and Eulerian systems — the central theme of this essay — is thus the foundation of most theory in fluid dynamics and is a routine part of many investigations.

The transformation of the Lagrangian conservation laws into the Eulerian equations of motion requires three key results. (1) The first is dubbed the Fundamental Principle of Kinematics; the velocity at a given position and time (the Eulerian velocity) is identically the velocity of the parcel (the Lagrangian velocity) that occupies that position at that time. (2) The material time derivative relates the time rate of change observed following a moving parcel to the time rate of change and advective rate of change observed at a fixed position; \( D(\cdot)/Dt = \partial(\cdot)/\partial t + V \cdot \nabla(\cdot) \). (3) And finally, the time derivative of an integral over a moving fluid volume can be transformed into field coordinates by means of the Reynolds Transport Theorem.
Once an Eulerian velocity field has been found, it is more or less straightforward to compute Lagrangian properties, e.g., parcel trajectories, which are often of practical interest. The FPK assures that the instantaneous Eulerian and Lagrangian velocities are identically equal. However, when averaging or integrating takes place, then the Eulerian mean velocity and the Lagrangian mean velocity are not equal, except in the degenerate case of spatially uniform flows. If the dominant flow phenomenon is wavelike, then their difference may be understood as Stokes drift, a correlation between displacement and velocity differences.

In an Eulerian system the local (at a point) effect of transport by the fluid flow is represented by the advective rate of change, $V \cdot \nabla ()$, the product of an unknown velocity and the first partial derivative of an unknown field variable. This nonlinearity leads to much of the interesting and most of the challenging phenomenon of fluid flows. We can put some useful bounds upon what advection alone can do. For variables that can be written in conservation form, e.g., mass and momentum, advection alone can not be a net source or sink when integrated over a closed or infinite domain. Advection represents the transport of fluid properties at a definite rate and direction, that of the fluid velocity, so that parcel trajectories are the characteristics of the advection equation. Advection by a nonuniform velocity may cause important deformation of a fluid parcel, and it may also cause rotation, an analog of angular momentum, and that follows a particularly simple and useful conservation law.

Cover page graphic: SOFAR float trajectories (green worms) and horizontal velocity measured by a current meter (black vector) during the Local Dynamics Experiment conducted in the Sargasso Sea. Click on the figure to start an animation. The float trajectories are five-day segments, and the current vector is scaled similarly. The northeast to southwest oscillation seen here appears to be a barotropic Rossby wave; see Price, J. F. and H. T. Rossby, 'Observations of a barotropic planetary wave in the western North Atlantic', J. Marine Res., 40, 543-558, 1982. An analysis of the potential vorticity balance of this motion is in Section 6.4.4. These data and much more are available online from http://ortelius.whoi.edu/ Some animations of the extensive float data archive from the North Atlantic are at http://www.phys.ocean.dal.ca/ lukeman/projects/argo/.

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1 Kinematics of Fluid Flow.

The broad aim of this essay is to introduce and exercise a few of the concepts and mathematical tools that make up the foundation of fluid mechanics. Fluid dynamics is a vast subject, encompassing widely diverse materials and phenomenon. This essay emphasizes aspects of fluid dynamics that are relevant to the geophysical flows of what one might term ordinary fluids, air and water, that make up the Earth’s fluid environment. The physics that govern geophysical flow is codified by the conservation laws of classical mechanics; conservation of mass, (linear) momentum, angular momentum and energy. The Lagrangian/Eulerian theme of this essay follows from the question, How can we apply these conservation laws to the analysis or prediction fluid flow?

In principle the answer is straightforward; first we erect a coordinate system that is suitable for describing a fluid flow, and then we derive the mathematical form of the conservation laws that correspond to that system. The definition of a coordinate system is a matter of choice, and the issues to be considered are more in the realm of kinematics than of physics. However, as we will describe in this introductory section, the kinematics of a fluid flow are certainly dependent upon the physical properties of the fluid (reviewed in Section 1.1), and, the kinematics of even the smallest and simplest fluid flow is likely to be quite complex; fully three-dimensional and time-dependent flows are common rather than exceptional (Section 1.2). Thus kinematics is at the nub of what makes fluid mechanics challenging, and specifically, requires that the description of fluid flows be in terms of fields (beginning in Section 1.3).1,2

1.1 Physical properties of material; how are fluids different from solids?

For most purposes of classical fluid dynamics, fluids such as air and water can be idealized as an infinitely divisible continuum within which the pressure, \( P \), and the velocity, \( V \), temperature, \( T \), are in principle definable at every point in space.3 The molecular makeup of the fluid will be studiously ignored, and the crucially important physical properties of a fluid, e.g., its mass density, \( \rho \), its heat capacity, \( C_p \), among others, must be provided from outside of a continuum theory, Table (1).

The space occupied by the material will be called the domain. Solids are materials that have a more or less definite or intrinsic shape, and will not conform to their domain under normal conditions. Fluids (gases and liquids) have no intrinsic shape or preferred configuration. Gases are fluids that will completely fill their domain (or container) and liquids are fluids that form a free surface in the presence of an acceleration field, i.e., gravity.

An important property of any material is its response to an applied force, Fig. (1). If the force on the

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1Footnotes provide references, extensions or qualifications of material discussed in the main text, along with a few homework assignments. They may be skipped on first reading.
2An excellent web page that surveys the wide range of fluid mechanics is http://physics.about.com/cs/fluiddynamics/
3Readers are presumed to have a college-level background in physics and multivariable calculus and to be familiar with basic physical concepts such as pressure and velocity, Newton’s laws of mechanics and the ideal gas laws. We will review the definitions when we require an especially sharp or distinct meaning.
Some physical properties of air, sea and land (granite)

<table>
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<tr>
<th></th>
<th>density $\rho$, $\text{kg m}^{-3}$</th>
<th>heat capacity $C_p$, $\text{J kg}^{-1} \text{C}^{-1}$</th>
<th>bulk modulus $B$, $\text{Pa}$</th>
<th>sound speed $c$, $\text{m s}^{-1}$</th>
<th>shear modulus $K$, $\text{Pa}$</th>
<th>viscosity $\nu$, $\text{Pa s}$</th>
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<tr>
<td>air</td>
<td>1.2</td>
<td>1000</td>
<td>$1.3 \times 10^5$</td>
<td>330</td>
<td>na</td>
<td>$18 \times 10^{-6}$</td>
</tr>
<tr>
<td>sea water</td>
<td>1025</td>
<td>4000</td>
<td>$2.2 \times 10^9$</td>
<td>1500</td>
<td>na</td>
<td>$1 \times 10^{-3}$</td>
</tr>
<tr>
<td>granite</td>
<td>2800</td>
<td>2800</td>
<td>$4 \times 10^{10}$</td>
<td>5950</td>
<td>$2 \times 10^{10}$</td>
<td>$\geq 10^{18}$</td>
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Table 1: Approximate, nominal value of some thermodynamic variables that are required to characterize materials that might be described by a continuum theory. These important data must be derived from laboratory studies. For air, the values are at standard temperature and pressure, 0°C and nominal atmospheric pressure. The bulk modulus shown here is for adiabatic compression; under an isothermal compression the value for air is about 30% smaller; the values are nearly identical for liquids and solids. na is not applicable. The viscosity of granite is highly uncertain.

Figure 1: An orthogonal triad of Cartesian unit vectors and a small cube of material. The surrounding material is presumed to exert a stress, $\mathbf{S}$, upon the face of the cube that is normal to the $z$ axis. The outward-directed unit normal of this face is $\mathbf{n} = e_z$. To manipulate the stress vector it will usually be necessary to resolve it into components: $S_{zz}$ is the projection of $\mathbf{S}$ onto the $e_z$ unit vector and is negative, and $S_{xz}$ is the projection of $\mathbf{S}$ onto the $e_x$ unit vector and is positive. Thus the first subscript on $S$ indicates the direction of the stress component and the second subscript indicates the orientation of the face upon which it acts. This ordering of the subscripts is a convention, and it is not uncommon to see this reversed.
face of a cube, say, is proportional to the area of the face, as will often be the case, then it is appropriate to consider the force per unit area, called the stress, is represented by the symbol $S$; $S$ is a three component vector and $\mathbf{S}$ is a nine component tensor that we will discuss in some detail in this and in Section 2.2. The SI units of stress are thus Newtons per meter squared, which is commonly represented by a derived unit, the Pascal, or Pa. Why there is a stress and how the stress is related to the physical properties and the motion of the material are questions of first importance that we will begin to consider in this section. For now we can take the stress as given.

### 1.1.1 Deformation

**Linear Deformation:** The component of stress that is normal to the upper surface of the material in Fig. (1) is denoted $S_{zz}$. A normal stress can be either a tension, when $S_{zz} \geq 0$, or a compression if $S_{zz} \leq 0$, as implied in Fig. (1). The most important compressive normal stress is almost always due to pressure rather than to viscous effects, and when the discussion is limited to compressive normal stress only we will identify $S_{zz}$ with the pressure.

Every material will contract or expand as the ambient pressure is increased or decreased, though the amount varies quite widely from gases to liquids and solids. To make a quantitative measure of the compression, let $V_0$ be the initial volume at the nominal pressure $P_0$, and denote the pressure change by $\delta P$ and the resulting thickness change by $\delta h$. The normalized change in thickness, $\delta h / h_0$, is often termed the linear deformation, and the volume change of the material in this configuration is then $\delta V = V_0(\delta h / h_0)$. A convenient and general measure of the stiffness or inverse compressibility of the material is called the bulk modulus. Notice that $B$ has the units of stress or pressure, Pa, and the numerical value is the pressure increase required to compress the volume by 100%. Of course, a complete compression of that sort does not happen outside of black holes, and the bulk modulus should be treated as the first derivative accurate for small pressure changes around the ambient pressure, $P_0$. Gases are readily compressed; a 10% increase of pressure above nominal atmospheric will cause an air sample to condense by about 7% under adiabatic conditions. Most liquids are quite resistant to compressive stress, e.g., for water, $B = 2.2 \times 10^9$ Pa, which is less than but comparable to the bulk modulus of a very stiff solid, granite (Table 1). Thus the otherwise crushing pressure in the abyssal ocean, up to about 400 times atmospheric pressure at sea level, has a rather small effect upon sea water, compressing it by only a few percent. This important physical fact (as we approach it) regarding water leads to a useful simplification: water is sufficiently stiff that the comparatively small pressure variations associated with geophysical flows do not cause appreciable variations in water density. Water may thus be idealized for some purposes as an incompressible fluid, i.e., as if $B$ were infinite. It is a little surprising that the same holds very often for the geophysical motions of air, despite that air is relatively compressible, because the pressure variations associated with normal weather are small in the present context (though absolutely vital for causing air flow in the first place).

The first several physical properties of water listed in Table 1 seem to have more in common with
granite than with air, our other fluid. The character of fluids becomes most evident in their response to any other direction or sense of a stress. A tensile normal stress, i.e., $S_{zz} \geq 0$, is resisted by many solid materials, especially metals, with almost the same strength that they exhibit to compression, while some composite solid materials, e.g., concrete, may be markedly weaker in tension (measured by Young’s modulus, which is generally not equal to the bulk modulus). Gases do not resist tensile stress at all, while liquids do so only very, very weakly when compared with their resistance to compression.

**Shear Deformation:** The stress component that is in the plane of the upward-looking face in Fig. (1), $S_{xz}$, is often termed a shear stress.$^4$ A measure of a material’s response to shear stress is the shear deformation, $\delta l / h$, where $\delta l$ is the sideways displacement of the face that receives the shear stress and $h$ is the column thickness (Fig. 3a and note that the cube of material is presumed to be stuck to the lower surface). The corresponding ’stiffness’ for shear stress, or shear modulus, is then defined as

$$K = \frac{S_{xz}}{\delta l / h},$$

which has units of pressure. The magnitude of $S$ is the shear stress required to achieve a shear deformation of 1 (or 45 degrees), which is past the breaking point of most materials. For many solids the shear modulus is comparable to the bulk modulus (Table 1).$^5$

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Figure 2: A solid or fluid sample confined within a piston has a thickness $h_0$ at the ambient pressure $P_0$. If the pressure is increased by an amount $\delta P$, the material will be compressed by the amount $\delta h$. If the fluid is a gas, compression may heat the sample significantly, and we have to specify whether the heat remains trapped in the material (adiabatic compression) or is free to flow into the surroundings (isothermal compression); $B$ in Table 1 is the former.

$^4$The word *shear* has an origin in the Middle English *scheren*, which means to cut with a pair of sliding blades (as in ’ Why are you scheren those sheep in the kitchen? If I’ve told you once I’ve told you a hundred times .. blah, blah, blah…’) Fluid mechanics usage brings to mind sliding planes; a shear stress is a force per unit area that is parallel to a specified surface, e.g., $S_{zx}$ is a shear stress in the $x$ direction on the upward face of the cube in Fig. (1). $S_{zy}$ is another. A velocity shear is a spatial variation of the velocity in a direction that is perpendicular to the velocity vector.

$^5$The distinction between solid and fluid seems clear enough when one considers ordinary times and forces. But on very long times, say geological times, materials that may appear unequivocally ’solid’ when observed for a few minutes may indeed be observed to flow, albeit very slowly, when observed over many days or millenia. Glaciers are an important example, and see the ’pitch drop’ experiment that can be found at the web site noted in footnote 2.
1.1.2 Shear deformation rate

We have already noted that ordinary fluids such as air and water have no intrinsic configuration, and hence there is no elastic restoring force comparable to that acting within a solid that can provide a static balance to a shear stress. Hence, there is no meaningful shear modulus for fluids. Indeed, the distinguishing property of a fluid is that it will move or flow in response to a shear stress, and moreover, a fluid will continue to flow so long as a shear stress is present. If the flow depicted in Fig. 3b is set up carefully, it may happen that the velocity $U$ varies linearly with $z$ and the deformation rate is steady and $d(\delta l/h)/dt = \partial U/\partial z$. The viscosity of a fluid, $\nu$, is defined by

$$\nu = \frac{S_{xz}}{\partial U/\partial z}$$

in analogy with the shear modulus. This relationship may be used to measure fluid viscosity in a laboratory setting, provided that the flow must be maintained laminar (the fluid velocity lies in smooth layers or lamina) and steady. It is found empirically that this requires that a nondimensional parameter called the Reynolds number, $Re$, must satisfy

$$Re = \frac{\rho U h}{\nu} \leq 400,$$

where $U$ is the speed of the upper (moving) surface relative to the lower, fixed, no-slip surface. Fluids for which the viscosity is a physical property of the fluid (and not also of the shear or stress) are often dubbed 'Newtonian', in recognition of Isaac Newton’s insightful analysis of friction in fluid flows; air and water are found to be Newtonian fluids to an excellent approximation.

Assuming that we know the fluid viscosity and it’s dependence upon temperature, density, etc., then the relationship Eq. (3) between viscosity, stress and velocity shear may just as well be turned around and used to estimate the viscous shear stress from a given velocity shear. This is the way that viscous stresses will be incorporated into the momentum balance of a fluid parcel (Section 2.2.3). It is important to remember, though, that Eq. (3) is not an identity, but rather a contingent experimental law that applies only for laminar, steady flow; if instead the fluid velocity is unsteady and two- or three dimensional, i.e., turbulent, then for a given upper surface speed $U(h)$, the shear stress will be larger than the laminar value predicted by Eq. (3) (Figure 3c). Evidently then, Eq. (3) has to be accompanied by Eq. (4) along with a description of the geometry of the flow, i.e., that $h$ is the distance between parallel planes (and not the diameter of a pipe, for example). In most geophysical flows the Reynolds number is enormously larger than the upper limit allowed by Eq. (4) and consequently geophysical flows are seldom laminar and steady, but are instead turbulent and unsteady. Thus it frequently happens that properties of the flow, rather than physical properties of the fluid, determine the stress.

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6While fluids have no intrinsic restoring forces or equilibrium configuration, nevertheless, there are very important restoring forces set up within fluids in the presence of an acceleration field. Most notably, gravity will tend to restore a displaced free surface back towards level. Earth’s rotation also endows the atmosphere and oceans with something closely akin to angular momentum that provides a restoring tendency for horizontal displacements; the oscillatory wave motion seen in the cover graphic is an example.

7There are about a dozen boxed equations in this essay, beginning with Eq. (3), that you will encounter over and over again in a study of fluid dynamics. These boxed equations are sufficiently important that they should be memorized, and you should be able to explain in detail what each term and each symbol means.
1.2 Fluid flow

Up to now we have confined the fluid sample within a piston or have assumed that the lower face was stuck to a no-slip surface. Suppose that the parcel is free to move in response to an applied force; we presume that an applied force will cause a fluid parcel to accelerate exactly as expected from Newton’s laws of mechanics. In this fundamental respect, a fluid parcel is not different from a solid particle.

But before we decide that fluids are indeed just like solids, let’s try the simplest fluid flow experiment. Some day your fluid domain will be grand and important, the Earth’s atmosphere or perhaps an ocean basin, but you can follow along with this discussion by observing the fluid flow in a domain that is small and accessible; the fluid flow in a teacup is entirely adequate because the fundamentals of kinematics are the same for flows big and small. To investigate momentum conservation in a tea cup we need only apply a gentle (linear) push with a spoon, say. The fluid moves and no doubt momentum is conserved, but the subsequent motion of the fluid bears little resemblance to what was intended to be the simplest possible forcing. The fluid that is directly pushed by the spoon can not simply continue on both because water is effectively incompressible for such gentle motions and because the inertia of the fluid that would have to be displaced is appreciable. Instead, the fluid flows mainly around the spoon from front to back, forming swirling coherent features called vortices that are clearly two-dimensional, despite that the forcing was intended to be one-dimensional. This vortex pair then moves slowly through the fluid, and careful observation will reveal that most of the linear (one-dimensional) momentum imposed by the normal stress is contained mainly within their translational motion. If the initial push is made a little more vigorous, then the...
resulting fluid motion will quickly and spontaneously become three-dimensional and turbulent. After a short time, only a few seconds, the smallest spatial scales of flow will be damped by viscosity, and the last surviving motion is likely to be a vortex that fills the entire tea cup — all of this from a simple, linear push on the fluid. Momentum, angular momentum and energy are surely conserved in the tea cup, but by way of a complex and nonlocal response that involves fluid flow over the entire domain.

What we can infer from these observations is that a parcel in fluid flow is literally pushed and pulled by the surrounding fluid parcels, via shear stress and normal stress, with the consequence that we can not predict the motion of a given parcel in isolation from its surroundings, rather we have to predict the motion of the surrounding fluid parcels as well. How extensive are the so-called surroundings? It depends upon how far backward or forward in time we may care to go, and also upon how rapidly signals (taken broadly) are propagated within the fluid. If we follow a parcel long enough, or if we need to know the history in detail, then every parcel will have a dependence upon the entire domain occupied by the fluid. In other words, even if our goal was limited to calculating the motion of just one parcel or at just one place, we would nevertheless have to solve for the fluid motion over the entire domain at all times of interest. As we have remarked and you have seen (if you have studied your teacup) fluid flows may spontaneously develop variance on all accessible spatial scales, from the scale of the domain down to a scale set by viscous or diffusive properties of the fluid, typically a few millimeters in water. Thus even though the physics of a fluid flow may be straightforward classical physics, nevertheless the kinematics of fluid flow may be remarkably complex — fully three-dimensional and time-dependent fluid flows are the rule rather than the exception.  

If we had to summarize fluid mechanics with one motto we might say that fluid mechanics emphasizes above all, 'fluid flow', and moreover, our motto is meant to imply an entire fluid domain in motion, rather than a single parcel in isolation. The measurement and description of a fluid flow is thus an important and substantial task to which we now turn.

1.3 Two ways to observe fluid flow and the Fundamental Principle of Kinematics

Let’s suppose that our goal is to observe (or measure) the fluid flow within some three-dimensional domain that we will denote by $\mathbb{R}^3$. There are two quite different ways to accomplish this, either by tracking individual parcels that are carried about with the flow, the Lagrangian method, or by observing the fluid velocity at locations that are fixed in space, the Eulerian method. Both methods are commonly used in the

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8How many observation points do you estimate would be required to define completely the fluid flow in a teacup? In particular, what is the smallest spatial scale on which there is a significant variation of the fluid velocity? Does the number depend upon the state of the flow, i.e., whether it is weakly or strongly stirred? Does it depend upon time in any way? The viscosity of water varies by a factor of about four as the temperature varies from 100 to 0 C. Can you infer the sense of this viscosity variation from your observations? What fundamental physical principles, e.g., conservation of momentum, second law of thermodynamics, can you infer from purely qualitative observations and experiments? Which of the fluid properties listed in Table 1 can you infer?

The fluid motion may also include waves: capillary waves have very short wavelengths, only a few centimeters, while gravity waves can have any larger wavelength, and may appear mainly as a sloshing of the entire tea cup. Waves can propagate momentum and energy much more rapidly than can the vortices. Capillary and gravity waves owe their entire existence to the free surface, and may not appear at all if the speed at which the spoon is pushed through the fluid does not exceed a certain threshold. Can you estimate roughly what that speed is? It may be helpful to investigate this within in a somewhat larger container.
1 KINEMATICS OF FLUID FLOW.

analysis of the atmosphere and oceans, and in fluid mechanics generally. Lagrangian methods are natural for many observational techniques and for the statement of the fundamental conservation theorems. On the other hand, almost all of the theory in fluid dynamics has been developed in the Eulerian system. It is for this reason that we will consider both coordinate systems, at first on a more or less equal footing, though for purposes of developing models of fluid flow we will favor the Eulerian system.

Perhaps the most natural way to observe the motion of a fluid is to observe the trajectories of individual fluid parcels. To make sure that the variables used to denote a parcel position are clearly distinct from the \((x, y, z)\) used to represent the usual field coordinates, we will use the Greek uppercase \(\xi\) for the position vector of a parcel whose Cartesian components are the lowercase \((\xi, \psi, \omega)\), i.e., \(\xi\) is the \(x\)-coordinate of a parcel, \(\psi\) is the \(y\)-coordinate of the parcel and so on. How can we identify specific parcels? If the parcels are a finite collection of floats that may be tracked more or less continuously in time, as in the cover graphic, the parcel identity may be represented by an index. But for the purpose of theory we will need a scheme that could in principle be used to define the identity of parcels in a continuum and throughout a domain. One possibility is to use the position of the parcels at some specified time, say the initial time, \(t = 0\); denote the initial position by the Greek uppercase alpha, \(A\), with Cartesian components, \((\alpha, \beta, \gamma)\). We somewhat blithely assume that we can determine the position of parcels at all later times, \(t\), to form the parcel trajectory, sometimes called the pathline,

\[
\xi = \xi(A, t) \tag{5}
\]

The trajectory \(\xi\) is the fundamental dependent variable in a Lagrangian description, and time and initial position \(A\) are the independent variables.

The velocity of a parcel, often termed the 'Lagrangian' velocity, \(V_L\), is just the time rate change of the parcel position,

\[
V_L(A, t) = \frac{d\xi}{dt} \tag{6}
\]

Since \(A\) denotes a particular parcel, and since we are holding the parcel identity fixed during this derivative, \(d/dt\) is an ordinary time derivative. In fact, this Lagrangian velocity of a fluid parcel is exactly the same thing as the velocity of a particle in classical dynamics, the only new wrinkle being that instead of using say an index to identify a specific particle in a finite collection, we have used \(A\) to denote the starting position of a parcel in a continuum. To make any use of \(V_L\) we will also need to keep the trajectory itself.

If tracking fluid parcels is impractical, perhaps because the fluid is opaque, then we might choose to observe the fluid velocity by means of current meters that we could implant at fixed positions, say \(x\). The essential component of every current meter is a transducer that converts fluid motion into a readily measured signal - e.g., the rotary motion of a propeller or the Doppler shift of a sound pulse. But regardless of the mechanical details, the velocity sampled in this way, termed the 'Eulerian' velocity, \(V_E\), is intended to be the velocity of the fluid parcels that move through the (fixed) control volume sampled by the transducer, and so the Eulerian velocity is defined by

\[
V_E(x, t) = \frac{d\xi}{dt} \bigg|_{\xi=x} \tag{7}
\]
dubbed the Fundamental Principle of Kinematics, or FPK. This velocity $V_E$ is the fundamental dependent variable in an Eulerian description, and the position, $x$, and time are the independent variables (compare this with the Lagrangian counterparts noted just above).

One way to appreciate the difference between the Lagrangian velocity $V_L$ and the Eulerian velocity $V_E$ is that $\xi$ in the Lagrangian velocity of Eq. (6) is the position of a moving parcel, while $x$ in Eq. (7) is the arbitrary and fixed position of a current meter. Parcel position is a result of the fluid flow rather than our choice, aside from the initial position. As time runs, the position of any specific parcel will change, barring that the flow is static, while the velocity observed at the current meter position will be the velocity of the sequence of parcels that move through that position at later times. The float and current meter data of the cover graphic afford an opportunity to check the FPK in practice: when the flow is large scale and when the floats surround the current meter mooring, the Lagrangian (green worms) and the Eulerian (black vector) velocities appear to be indistinguishable. At other times, and especially when either velocity is changing direction rapidly, the comparison is quite difficult to make.9

Our usage ‘Eulerian’ velocity and ‘Lagrangian’ velocity is standard; if no such tag is appended, then Eulerian is almost always understood as the default. This usage can be helpful in the present context10 provided we do not infer that there are two physical fluid velocities. There is one unique fluid velocity that can be sampled in two quite different ways, by tracking specific parcels (Lagrangian) or by observing the motion of fluid parcels that flow through a fixed site (Eulerian). The formal statement of this, Eq. (7), is not impressive, and hence the imposing title. Nearly everything we have to say in this essay follows from the FPK combined with the familiar conservation laws of classical physics.

1.4 About this essay

Now that we have learned (or imagined) how to observe a fluid velocity, we can begin to think about surveying the entire domain in order to construct a representation of the complete fluid flow. This will require an important decision regarding the sampling strategy; should we make these observations by tracking a large number of fluid parcels as they wander throughout the domain, or, should we deploy additional current meters and observe the fluid velocity at many additional sites? In principle, either approach could suffice to define the flow (Fig. 4).11 Nevertheless, the observations themselves and the analysis needed to understand these observations would be quite different, as we will see in examples below. And of course, in practice, our choice of a sampling method will be decided as much by purely practical

---

9 What would we infer if the float and current meter velocities did not appear to be similar? We would not lay the blame on Eq. (7), which is, in effect, an identity. Instead we would question, probably in this order, whether $\xi = x$ as required by the FPK, since this would imply a collision between float and current meter, whether the float tracking accuracy was sufficient, and then whether the current meter had been improperly calibrated or had malfunctioned.

10 This usage may be unfortunate as historical attribution; Lamb, *Hydrodynamics*, 6th ed., (Cambridge Univ. Press, 1937), the classic tome on fluid dynamics before the age of numerical calculation, credits Leonard Euler with developing both representations.

Figure 4: An ocean circulation model sampled with Lagrangian and Eulerian methods (green worms and black vectors respectively). If you are using Acrobat 6, click on the figure to begin an animation. The domain is a square basin 2000 km by 2000 km driven by a basin-scale wind having negative curl, as if a subtropical gyre. Only the northwestern quadrant of the model domain is shown here. The main circulation features in the upper layer of the model are a thin western and northern boundary current that flows clockwise and a well-developed westward recirculation just to the south of the northern boundary current. This westward flow is (baroclinically) unstable and oscillates with a period of about 60 days. How would you characterize the Eulerian and Lagrangian representations of this circulation? In particular, do you notice any systematic differences?

This essay has been written for students who are somewhere near the beginning of a study of fluid mechanics and it is therefore pedagogical in aim and style. The material presented here is not new in any significant way and indeed most of it comes from the foundation of fluid dynamics. One notable modern source for kinematics is by Aris (1962)\textsuperscript{12} and another, less modern in appearance but no less valuable, is the

\textsuperscript{12}A rather advanced and mathematical source for fluid kinematics is Chapter 4 of Aris, R., \textit{Vectors, Tensors and the Basic Equations}
2 THE LAGRANGIAN (OR MATERIAL) COORDINATE SYSTEM.

classic by Lamb (1937). Most comprehensive fluid dynamics texts used for introductory courses include at least some discussion of Lagrangian and Eulerian representations, but not as a central theme. This essay is most appropriately used as a follow-on or supplement to a comprehensive text.

The plan is to describe further the Lagrangian and Eulerian systems in Section 2 and 3, respectively. As we will see in Section 2.3, the Lagrangian equations of motion are rather difficult when pressure gradients are included (which is nearly always necessary) and the main task of Section 3 is therefore to derive the Eulerian equations of motion, which are used almost universally for problems of continuum mechanics. It often happens that Eulerian solutions of the velocity field need to be transformed into Lagrangian properties, e.g., trajectories as in Fig. (4), a problem considered in Sections 4 and 5. In an Eulerian system the process of transport by the fluid flow is represented by advection, the nonlinear and inherently difficult part of most fluid models and considered in Section 6. Section 7 is a very brief summary.

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2 The Lagrangian (or Material) Coordinate System.

One helpful way to think of a fluid flow is that it carries or maps parcels from one position to the next, i.e., from a starting position \( A \) into the positions \( \xi \) at later times. We will assume that this mapping is unique in that adjacent parcels will never be split apart, and neither will one parcel be forced to occupy the same position as another parcel. Thus given a \( A \) and a time, we will presume that there is a unique \( \xi \). We must assume that a fluid parcel can be taken to be as small as is necessary to meet these requirements, and that a fluid is a continuum. With these conventional assumptions in place, the mapping of points from initial to of Fluid Mechanics, (Dover Pub., New York, 1962). A particularly good discussion of the Reynolds Transport Theorem (discussed here in Section 3.2) is by C. C. Lin and L. A. Segel, Mathematics Applied to Deterministic Problems in the Natural Sciences (MacMillan Pub., 1974).

Modern examples include excellent texts by P. K. Kundu and I. C. Cohen, Fluid Mechanics (Academic Press, 2001), by B. R. Munson, D. F. Young, and T. H. Okiishi, Fundamentals of Fluid Mechanics, 3rd ed. (John Wiley and Sons, NY, 1998), by D. C. Wilcox, Basic Fluid Mechanics (DCW Industries, La Canada, CA, 2000) and by D. J. Acheson, Elementary Fluid Dynamics (Clarendon Press, Oxford, 1990). A superb text that emphasizes experiment and fluid phenomena is by D. J. Tritton, Physical Fluid Dynamics (Oxford Science Pub., 1988). Two other classic references, comparable to Lamb but more modern are by Landau, L. D. and E. M. Lifshitz, 'Fluid Mechanics', (Pergamon Press, 1959) and G. K. Batchelor, 'An Introduction to Fluid Dynamics', (Cambridge U. Press, 1967). An especially good discussion of the physical properties of fluids is Ch. 1 of Batchelor’s text. I suggest that this essay is best used as a supplement rather than as a primary reference because many of the concepts or tools used here, e.g., velocity gradient tensor, Reynolds Transport Theorem, characteristics, etc., are reviewed only briefly. The contribution of this essay is (or is hoped to be) that it will help show how these concepts and tools may be understood as aspects of the Lagrangian and Eulerian representations.

subsequent positions, i.e., the trajectory, Eq. (5), can be inverted to find

\[ A = A(\xi, t), \]  

(8)

at least in principle.

### 2.1 Material coordinates

Each trajectory that we observe or construct must be tagged with its unique \( A \) and thus for a given trajectory \( A \) is a constant. Though \( A \) is constant for a given parcel, we have to keep in mind that our coordinate system is meant to describe a continuum defined over some domain, and that \( A \) varies continuously over the entire initial domain of the fluid. Thus when we need to consider the domain as a whole, \( A \) has the role of being the independent, spatial coordinate. This kind of coordinate system in which parcel position is the fundamental dependent spatial variable is often referred to as a Lagrangian coordinate system, and also and perhaps more aptly as a 'material' coordinate system.

We have noted already in Section 1 and Eq. (2) that the Lagrangian velocity is just the ordinary time derivative of the position, and the acceleration of a fluid parcel is just the second (ordinary) time derivative of the parcel position,

\[ \frac{dV_L(A, t)}{dt} = \frac{d^2\xi}{dt^2} \]

(9)

From the fact that we are differentiating \( \xi \) it should (and must, really) be understood that we are asking for the velocity of a specific parcel, and that we are holding \( A \) fixed during this differentiation. Given that we have defined and can compute the acceleration of a fluid parcel, we go on to assert that Newton’s laws of classical dynamics apply to a fluid parcel in exactly the form used in classical (solid particle) dynamics, i.e.,

\[ \frac{dV_L}{dt} = F/\rho, \]

(10)

where \( F \) is the force per unit volume imposed upon that parcel, and \( \rho \) is the density or mass per unit volume of the fluid. Thus if we observe that a fluid parcel undergoes an acceleration, then we can infer that there had to have been a force applied to that parcel. It is on this kind of diagnostic problem that the Lagrangian coordinate system is most useful, generally.

Before going much further it may be helpful to consider a very simple but concrete example of a flow represented in the Lagrangian system. Let’s assume that we have been given the trajectories of all the parcels in a one-dimensional domain \( \mathbb{R}^1 \) with spatial coordinate \( y \) by way of the explicit formula \(^{15}\)

\[ \psi(\beta, t) = \beta(1 + 2t)^{1/2} \]

(11)

where \( t_0 = 0. \) Once we specify the starting position of a parcel, \( \beta = \psi(t = 0) \), this handy little formula tells us the \( y \) position of that specific parcel at any later time. It is most unusual to have so much information

\(^{15}\)When a list of parameters and variables is separated by commas as \( \xi(\beta, t) \) on the left hand side, we mean to emphasize that \( \xi \) is a function of \( \beta \), a parameter, and \( t \), an independent variable. When variables are separated by operators, as \( \beta(1 + 2t) \) on the right hand side, we mean that the variable \( \beta \) is to be multiplied by the sum \((1 + 2t)\).
presented in such a convenient way, and in fact, this particular 'flow' has been concocted to have just enough complexity to be interesting for our purpose here, but is without physical significance. The velocity of a parcel is then

\[ V_L(\beta, t) = \frac{d\psi}{dt} = \frac{\beta}{2}(1 + 2t)^{-1/2} \tag{12} \]

and the acceleration (and also the force per unit volume) is just

\[ \frac{d^2\psi}{dt^2} = -\frac{\beta}{4}(1 + 2t)^{-3/2}. \tag{13} \]

Given the initial positions of four parcels, let’s say \( \beta = (0.1, 0.3, 0.5, 0.7) \) we can readily compute the trajectories and velocities from Eqs. (11) and (12), Fig. 1b and 1c. Note that the velocity depends upon the parcel initial position, \( \beta \). If \( V_L \) did not depend upon \( \beta \), then the flow would necessarily be spatially uniform, i.e., all the fluid parcels in the domain would have exactly the same velocity. The flow shown here has the following form: all parcels shown (and we could say all of the fluid in \( \beta > 0 \)) are moving in the direction of positive \( \psi \); parcels that are at larger \( \beta \) move faster; all of the parcels in \( \beta > 0 \) are decelerating in the sense that their speed decreases with time, and the magnitude of the deceleration increases with \( \beta \).

If, as presumed in this example, we are able to track parcels at will, then we can sample as much of the domain and any part of the domain that we may care to investigate. In a real, physical experiment (the cover graphic) this could be a bit problematic; we could not be assured that any specific portion of the domain will be sampled unless we launch a parcel there. Even then, the parcels may spend most of their time in regions we are not particularly interested in sampling, a hazard of Lagrangian experimentation. This is, of course, a practical, logistical matter. It often happens that the major cost of a Lagrangian measurement scheme lies in the tracking apparatus, with additional floats or trackable parcels being relatively cheap (Particle Imaging Velocimetry noted in the next section is a prime example). In that circumstance there may be almost no limit to the number of Lagrangian measurements that can be made.

Consider the information that the Lagrangian representation Eq. (11) provides; in the most straightforward way possible it shows where fluid parcels released into a flow at a given time and position will be found at some later time. If our goal was to observe how a fluid flow carried a pollutant, say, from a source (the initial position) into the rest of the domain, then this Lagrangian representation would be ideal. We could simply release (or tag) parcels over and over again at the source position, and then observe where the parcels were subsequently carried by the flow. By releasing a cluster of parcels we could observe how the flow deformed or strained the fluid (e.g., the float cluster shown on the cover page and taken up in detail in Section 6.4.4). Similarly, if our goal was to measure the force applied to the fluid, then by tracking parcels in time and by observing their acceleration we could estimate the force directly via the Lagrangian equation of motion, Eq. (4) (the comparable thing is considerably more difficult in an Eulerian system, next section). These are important and common uses of the Lagrangian coordinate system but note that they are all related in one way or another to the observation of fluid flow rather than to the calculation of fluid flow.
2 THE LAGRANGIAN (OR MATERIAL) COORDINATE SYSTEM.

Figure 5: Lagrangian and Eulerian representations of the one-dimensional, time-dependent flow defined by Eq. (11). (a) The solid lines are the trajectories $\psi(\beta, t)$ of four parcels whose initial positions were $\beta = (0.1, 0.3, 0.5, 0.7)$. (b) The Lagrangian velocity as a function of initial position and time, $d\psi/dt(\beta, t)$. The lines plotted here are contours of constant velocity, not trajectories. (c) The Eulerian velocity field computed in Section 3 by solving for $V_E(y, t)$, and again the lines are of constant velocity.

2.2 Forces on a parcel

If our goal was to carry out a forward calculation, i.e., to predict rather than to observe fluid motion, then we would have to specify the net force, $F$ of Eq. (9), acting on a parcel in fluid flow, something we began to consider in Section 1.1. For the purpose of evaluating the Lagrangian equation of motion, we could limit the present discussion to pressure forces only. But once started down this road for the second time, it is appropriate to take account of viscous forces as well. We should note that the results of this section are essential also for the Eulerian equations of motion taken up in Section 3.

Specification of the force on fluid parcels proceeds in two steps; the first is a kind of mathematical bookkeeping in which we construct and learn to manipulate the stress tensor, essentially a $3 \times 3$ matrix of stress components. The second step is to specify the stress components. Forces within a fluid are generated in a way that depends upon the physical properties of the fluid and so this involves something more than kinematics and bookkeeping.
2.2.1 Stress tensor

Fluid mechanics (or any theory of continuum mechanics) requires specification of a stress tensor, a 3x3 matrix, rather than a stress vector, a 1x3 object, because the medium and the stress are both 3-dimensional. Thus, on any given face of a fluid parcel there may be a stress vector, defined by three components, that is dependant upon the orientation of the face, which is defined by the unit normal vector, and thus another three components. Thus a full representation of the stress within a fluid requires nine components at each point in the fluid, and it is very helpful to group these into a 3x3 matrix called the stress tensor,

\[
\mathbf{S} = \begin{pmatrix}
S_{xx} & S_{xy} & S_{xz} \\
S_{yx} & S_{yy} & S_{yz} \\
S_{zx} & S_{zy} & S_{zz}
\end{pmatrix},
\]

(14)

that we can add and multiply with the same rules that apply to matrices. By our convention (Section 1.1), the first subscript on a stress component indicates the direction of the stress (the projection of the stress vector on to the \(\mathbf{e}_x\) unit vectors), and the second subscript indicates the orientation of the surface via the direction of a unit normal vector. We will see later that three pairs of these components are equal, \(S_{yx} = S_{xy}, S_{xz} = S_{zx}\), and \(S_{zy} = S_{yz}\), at least for all ordinary fluids such as air and water, leaving six unique components. Thus the stress tensor is symmetric, \(\mathbf{S} = \mathbf{S}'\).

The stress tensor has to be defined at every point in a fluid and thus a tensor stress field is coincident with the velocity field. At first this seems a bit daunting; we had the goal of solving for the pressure and the velocity fields, and now it seems that we have to solve for a tensor field as well. However, as will see shortly, the stress tensor depends upon the velocity and the pressure in a straightforward way, so that nothing beyond the velocity and pressure fields will be required.

A handy property of the stress tensor is that the matrix product \(\mathbf{S} \cdot \mathbf{n}\) picks out the components of the stress vector acting on the face whose unit normal vector is \(\mathbf{n}\). For example, the unit normal for the upper face, \(\mathbf{n}_u\), has components \((0; 0; 1)\) (the semicolon delimiters indicate that these elements are arranged in a column vector), and so \(\mathbf{S} \cdot \mathbf{n}_u = (S_{xz}; S_{yz}; S_{zz})\), is a three element (column vector) of the components of the stress vector acting upon the upper face of the parcel. The differential force associated with this stress acting upon a differential area of the surface, \(da\), is then

\[
d\mathbf{F} = \mathbf{S} \cdot \mathbf{n} \, da,
\]

(15)

where the stress tensor is evaluated at the position of the differential area. The unit normal of the lower (downward looking) face of the parcel of Fig. (1) has components \((0; 0; -1)\) and so the stress vector on that face has components \(\mathbf{S} \cdot \mathbf{n}_l = -(S_{xz}; S_{yz}; S_{zz})\), where these components are evaluated at the position of the lower face. If if happened that the stress components were the same on the upper and lower faces, then the stress exerted by the overlying fluid on the upper face of the parcel would be equal in direction and magnitude to the stress exerted by the parcel on the underlying fluid, in which case the net stress on the parcel, \(\mathbf{S} \cdot \mathbf{n}_u + \mathbf{S} \cdot \mathbf{n}_l = \mathbf{S} \cdot (\mathbf{n}_u + \mathbf{n}_l) = 0\), would vanish.

Given Eq. (15) for the differential force, the net force on the parcel may be computed by summing over
the entire surface,

\[ F = \iiint_S \mathbf{S} \cdot \mathbf{n} \, da, \]

where \( da \) is the differential area associated with the unit normal \( \mathbf{n} \). Both \( \mathbf{S} \) and \( \mathbf{n} \) will, in general, vary over the surface. It is often desirable to express the net force as an integral over the volume of the object, rather than its surface, and this transformation is made by application of the tensor form of the divergence theorem,\(^{16}\)

\[ F = \iiint_S \mathbf{S} \cdot \mathbf{n} \, da = \iiint_S \nabla \cdot \mathbf{S}' \, dv. \]

### 2.2.2 Stress components in an ideal fluid

It remains to specify the components of the stress tensor in a way that gives, as we intend, the stress exerted on the parcel by the fluid or boundary that is outside of the parcel. We noted already in Section 1.1 that the pressure-induced stress vector on a parcel is just \(-P \mathbf{n}\). Hence the stress tensor for a fluid that sustains only pressure forces, often called an ideal fluid or an Euler fluid, is

\[ \mathbf{P} = \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix}, \]

an isotropic tensor (same in all directions) and \( \mathbf{P} = \mathbf{P}' \). The net pressure force on a parcel is then

\[ \mathbf{F}_{\text{pressure}} = -\iiint \mathbf{P} \cdot \mathbf{n} \, da = \iiint -\nabla P \, dv \]

where in the last step the divergence theorem was applied to convert the surface integral into a volume integral. The pressure force per unit volume is then

\[ \mathbf{F}_{\text{pressure}} / \text{Vol} = -\nabla P \]

\[ = -\frac{\partial P}{\partial x} \mathbf{e}_x - \frac{\partial P}{\partial y} \mathbf{e}_y - \frac{\partial P}{\partial z} \mathbf{e}_z. \]

a result that we will use repeatedly.

Admittedly, this doesn’t seem like real progress; we had something simple, pressure, and have turned it into something esoteric, a stress tensor. There is no need for the tensor formalism to this stage, and it is worthwhile to work through to this important result again using vectors only. Let the \( x, y, z \) dimensions of a

\(^{16}\)This is probably not familiar, so I’ll try to make it plausible. Each row of the stress tensor may be considered to be a vector, \( S_{1j}, S_{2j}, S_{3j} \), so that there are three vectors whose components are numbered by the index \( j \). Each of these vectors can be treated by way of the familiar divergence theorem for vectors, \( \oint S_{ij} \, da = \oint \frac{\partial S_{ij}}{\partial x_i} \, dv \) with \( i = 1, 2 \) or 3. Thus each row (or vector) of the tensor goes through separately. When we represent the divergence operator using symbolic notation, with \( \nabla \) a row vector operator that will be multiplied on the left of \( \mathbf{S} \), then to operate on rows (each vector separately) we must operate on the transpose of \( \mathbf{S} \), \( \mathbf{S}' \). Later it will turn out that \( \mathbf{S}' = \mathbf{S} \).
parcell be $w, l, h$ and we can presume that the lower face is at $z = 0$ (Fig. 1). The pressure vector force on the upper face of the parcel at $z = h$ is just $-P(z = h)n_u w l$, where the unit normal of the upper face is just $n_u = (0e_x + 1e_y + 0e_z)$. The $z$-component of the force is the only non-zero component and is just $-P(z = h)w l$, or downwards, as we would expect. The pressure force on the lower face is similarly $-P(z = 0)n_l w l$, and since $n_l = (0e_x - 1e_y + 0e_z)$, the $z$-component is just $P(z = 0)w l$, which is thus positive, or upwards (notice that the direction or sense of the pressure force is accounted properly by the normal vectors). The net pressure force in the $z$ direction is then the sum, $-(P(z = h) - P(z = 0))w l$. If we divide this by the volume, $w l h$, and let the column height $h$ go to zero, the pressure force per unit volume is readily seen to be $-\partial P/\partial z$, the $z$-component of $-\nabla P$ (Eq. 21).

2.2.3 Stress components in a viscous fluid

In Section 1.1 we claimed that the viscous shear stress in a simple shear flow in which the $u$ component of velocity varied in the $z$-direction only (Figure 3) was just $S_{xz} = \nu \partial u/\partial z$. There is nothing special about the $z$ direction and if the $u$ component of velocity varied in the $y$-direction, then there should arise a viscous stress in the $x$-direction that is exerted on the $y$-face of the parcel in exactly the same way. That viscous shear stress component would then be labelled

$$\tau_{xy} = \nu \partial u/\partial y,$$

where we have switched from $S$, indicating a stress generally, to $\tau$ to indicate a viscous stress in particular (you should make a sketch that shows this kind of shear flow and check whether the sense of the stress is given correctly by this equation). An $x$-directed stress acting upon the $x$-face might be different, however, since this would involve a linear rather than a shear deformation rate. For some fluids it is found experimentally that the viscosity for a linear deformation rate is not equal to the viscosity associated with shearing deformation, in much the same way that the bulk modulus of a solid is generally not equal to the shear modulus (Table 1). While acknowledging that this is plausible, we will nevertheless presume that there is only one viscosity, $\nu$. Thus a linear deformation rate will produce or require a normal viscous stress,

$$\tau_{xx} = \nu \partial u/\partial x.$$

In most cases this normal viscous stress will be very much less than the normal stress associated with pressure.17 With these three examples in hand, we are ready to write down the viscous stress tensor for Newtonian fluids18 in which the stress and rate of deformation are related linearly (Eq. 3),

$$T = \nu \mathcal{G}$$

for a Newtonian fluid: (22)

17A viscous normal stress is somewhat harder to envision than is a viscous shear stress. But imagine a fluid column that is falling under the influence of gravity while restrained by a normal viscous (tensile) stress associated with the linear deformation rate of the elongating fluid column, e.g., the pitch drop experiment, again! (footnote 3). We are making the implicit assumption that the average of the three normal viscous stresses is very much less than the pressure, and hence that viscous stresses and pressure are decoupled. This is exact if the fluid is incompressible, and is a very, very good (accurate) assumption for geophysical flows.

18Air and water are the archetypical Newtonian fluids and our main concern, but there are interesting varieties of non-Newtonian fluids that are of considerable practical importance. Many high molecular weight polymers such as paint and mayonnaise are said to be 'shear-thinning'. Under a very small stress they may not flow, and so behave like very weak solids. When subjected to a larger stress, these fluids may begin to flow, though usually with a comparatively high viscosity. 'Shear-thickening' fluids are less common,
where

\[
G = \begin{pmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\
\frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}
\end{pmatrix}
\]  

(23)

The tensor \(G\), called appropriately enough the velocity gradient tensor, is of fundamental importance and arises again on two separate occasions later in this essay.

To find the force per unit volume we can apply the tensor equivalent of the divergence theorem, Eq. (17),

\[
\mathbf{F}_{\text{viscous}} = \iiint \mathbf{T} \cdot \mathbf{n} \, da = \iiint \nabla \cdot \mathbf{T'} \, dv.
\]  

(24)

The force per unit volume, written out in components is

\[
\mathbf{F}_{\text{viscous}} / \text{Vol} = \nu \left( \begin{array}{c}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \\
\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \\
\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}
\end{array} \right)
\]  

(25)

or in vector notation,

\[
\text{for a Newtonian fluid: } \mathbf{F}_{\text{viscous}} / \text{Vol} = \nu \nabla^2 \mathbf{V}
\]  

(26)

(Note that \(\nabla^2\) operating on a vector yields another vector, whose components are the Laplacian of the components of the original vector.) In all of the above it has been assumed that the viscosity is a constant.

The expression (25) for the viscous force per unit volume may look formidable because of the second derivatives and the large number of terms involved. We will see some truly difficult equations in this essay, but this is not one of them. The viscous force term is linear, and it does not couple the components together, i.e., in the \(x\)-component equation there appears only the \(x\)-component of velocity. The viscous force per unit volume may be familiar to you as the diffusion term of the elementary heat diffusion equation, and in fact momentum components are diffused through a fluid in laminar flow just the way that thermal energy is diffused through a solid.\(^{19}\) Diffusion, or viscosity in the case of fluid momentum, acts to smooth out lumps and bumps in the velocity profile (in any direction) with the rate given by the viscosity times the Laplacian.\(^{20}\)

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\(^{19}\)It bears emphasis that we considering molecular diffusion in a laminar flow in which Eq. (3) applies. Turbulent motion may cause mixing in a fluid that is often parameterized as a diffusion process, Fig. (3.c), in which case the equivalent viscosity (stress/shear) is likely to be a complex function of the flow, and not just a physical property of the fluid.

\(^{20}\)A problem for you: after deriving the vector form of the pressure gradient force per unit volume we went back and derived the...
2.3 The Lagrangian equations of motion

Recall that our goal at the start of Section 2.2 was to write the equation of motion in the Lagrangian system. Having defined a means to specify pressure and viscous forces on a parcel we can continue on with this development. But first a warning is in order: the Lagrangian equations of motion turn out to be nearly intractable for most purposes of continuum mechanics, and we will not use them in anything approaching full form. Hence there is little practical motive for writing them out. Nevertheless, if you are curious to see in what way the Lagrangian equations of motion are difficult, and the corollary of this, why most theory in fluid mechanics is developed in the Eulerian system (Section 3) instead, then read on. Otherwise, you can skim the last paragraph of this section and continue to Section 3 with no real loss.

For this purpose we can take the density to be constant, and for that matter we can ignore viscosity, as if an ideal fluid. The vector form of the equation of motion is just

\[
\frac{d^2 \xi}{dt^2} = -\frac{1}{\rho} \nabla P - g e_z. \tag{27}
\]

where \( \nabla \) is the usual gradient operator and \( \xi \) is the position vector of the parcel.

On first sight the Lagrangian momentum equations may seem acceptable, but on second sight there is a serious problem lurking in the pressure gradient, and if we had included viscous stress terms, even worse there. The gradient of pressure is necessarily taken with respect to the usual spatial (field) coordinates \((x, y, z)\), which in this Lagrangian system are dependent variables, i.e., they are equivalent to the \((\xi, \psi, \omega)\). So we have a kind of mixed notation here. When rewritten in the notation used for Lagrangian variables the equation of motion is then

\[
\frac{d^2 \xi}{dt^2} = \frac{1}{\rho} \frac{\partial P}{\partial \xi}, \quad \frac{d^2 \psi}{dt^2} = \frac{1}{\rho} \frac{\partial P}{\partial \psi} \quad \text{and} \quad \frac{d^2 \omega}{dt^2} = \frac{1}{\rho} \frac{\partial P}{\partial \omega} - g. \tag{28}
\]

The derivative of one unknown with respect to another unknown is generally extraordinarily difficult, and the derivative we would much prefer is with respect to the independent variable, the initial position, i.e., the \((\alpha, \beta, \gamma)\). In general, we have to allow that the present position of a parcel will be dependent upon all three components of the initial position of that parcel, and hence by the usual chain rule:

\[
\frac{\partial P}{\partial \alpha} = \frac{\partial P}{\partial \xi} \frac{\partial \xi}{\partial \alpha} + \frac{\partial P}{\partial \psi} \frac{\partial \psi}{\partial \alpha} + \frac{\partial P}{\partial \omega} \frac{\partial \omega}{\partial \alpha}.
\]

Solving this for the pressure gradient \( \frac{\partial P}{\partial \xi} \) and substitution into the \( \xi \)-component of the Lagrangian

---

*z-component of the pressure force without the use of vector or tensor notation. You should do the same for the viscous force per unit volume in the case shown in Fig. 3.b, a flow in the x-direction only, and with shear in the z-direction only. Find the viscous force on the upper and lower faces of a parcel, then the sum, and allow the dimensions to shrink to infinitesimal. Locate the resulting term in the full 3-dimensional equation, 25. Suppose that the flow has reached a steady state, as in Fig. 3.b. What is the profile of viscous stress throughout the fluid? Now imagine that the stress has just been imposed at the surface. Qualitatively, what is the stress profile in this transient case? The lower boundary condition in Fig. (3) is presumed to be no-slip, so that \( u(z = 0) = 0 \), which is appropriate for real fluids having a finite viscosity. What is the equivalent boundary condition for a heat diffusion problem? What are the corresponding, plausible surface boundary conditions?
momentum equation gives
\[
\frac{\partial^2 \xi}{\partial t^2} = - \left( \frac{\partial P}{\partial \alpha} - \frac{\partial P}{\partial \psi} \frac{\partial \psi}{\partial \alpha} - \frac{\partial P}{\partial \omega} \frac{\partial \omega}{\partial \alpha} \right) / \rho_0 \frac{\partial \xi}{\partial \alpha},
\]
which does not look promising. To put this into a form in which only a single pressure gradient term appears in each component equation we can multiply the x-component equation of Eq. (28) by \( \partial \xi / \partial \alpha \), the y-component equation by \( \partial \psi / \partial \alpha \) and the z-component by \( \partial \omega / \partial \alpha \) and add the resulting three equations; the procedure is repeated for the other components with the result (Lamb,\(^{10}\) Article 1.13):
\[
\begin{align*}
\frac{\partial^2 \xi}{\partial t^2} \frac{\partial \xi}{\partial \alpha} + \frac{\partial^2 \psi}{\partial t^2} \frac{\partial \psi}{\partial \alpha} + \left( \frac{\partial^2 \omega}{\partial t^2} + g \right) \frac{\partial \omega}{\partial \alpha} &= \frac{1}{\rho} \frac{\partial P}{\partial \alpha}, \\
\frac{\partial^2 \xi}{\partial t^2} \frac{\partial \xi}{\partial \beta} + \frac{\partial^2 \psi}{\partial t^2} \frac{\partial \psi}{\partial \beta} + \left( \frac{\partial^2 \omega}{\partial t^2} + g \right) \frac{\partial \omega}{\partial \beta} &= \frac{1}{\rho} \frac{\partial P}{\partial \beta}, \\
\frac{\partial^2 \xi}{\partial t^2} \frac{\partial \xi}{\partial \gamma} + \frac{\partial^2 \psi}{\partial t^2} \frac{\partial \psi}{\partial \gamma} + \left( \frac{\partial^2 \omega}{\partial t^2} + g \right) \frac{\partial \omega}{\partial \gamma} &= \frac{1}{\rho} \frac{\partial P}{\partial \gamma}.
\end{align*}
\]

It appears that the Lagrangian momentum equations written for a continuum do not retain the simplicity of the corresponding equations for solid particles, and, more to the point, they are less amenable for most purposes of fluid dynamics than are the corresponding Eulerian continuum equations that we will take up in the next section. With no intent to make this situation seem hopeless, we will note two significant hurdles to their use: 1) The Lagrangian equations involve a second derivative with respect to time, where even one integration can be challenging. In effect, the solution of the Lagrangian equations requires solving for parcel trajectories in one fell swoop. In the Eulerian system we can solve for the velocity (one integration in time), and then can integrate the velocity solution separately to find parcel trajectories (Section 4.1), if they are required. In many problems they are not required, and indeed we may have no interest in knowing the trajectories of specific parcels. To say it a little differently, a Lagrangian solution may tell us much more than we may need to know. That would not be grounds for complaint, except that nothing comes free, and Lagrangian solutions are likely to require a very high price in computational or mathematical effort. 2) While the nonlinear advection terms in an Eulerian momentum or mass conservation equation have a clear-cut physical interpretation (Section 6.1), the same can not be said for the nonlinear terms of Eq. (29), e.g., \( \partial \xi / \partial \alpha, \partial \xi / \partial \beta \), etc., in cases like the flow in a stirred tea cup where parcels move a significant distance away from their starting position. Some form of nonlinearity is inevitable in a theory of fluid dynamics, and it happens that the Lagrangian form of it is awkward for most purposes. While there are exceptions to this, and we should keep an eye open for others, it is a fair generalization that the Lagrangian equations of motion, Eq. (29), are not as suitable for most theoretical problems as are the Eulerian equations.\(^{21}\) It is for that reason that the Eulerian form of the conservation laws is derived in the following section, and why most of our subsequent effort will be devoted to the Eulerian system.

\(^{21}\)A number of these exceptions are inherently interesting and important in their own right. In this course we will examine a (Lagrangian) model of a finite number of interacting free vortices. To the approximation of potential flow theory these vortices interact by advecting one another about. Some important microscale aspects of a dilute gas, e.g., diffusivity or viscosity or reaction dynamics, can be modeled using kinetic theory that treats individual gas molecules as hard spheres that interact via two-particle collisions. An accessible and highly recommended introduction to gas theory is Ch. 11 of A. J. Garcia, 'Numerical Methods for
THE LAGRANGIAN (OR MATERIAL) COORDINATE SYSTEM.  

Figure 6: Lagrangian and Eulerian representations of the one-dimensional, time-dependent flow defined by Eq. (11). (a) Positions; $\psi$ and $y$. The trajectory (green, solid line) is of the parcel defined by the initial position $\beta = 0.5$, and is $\psi(\beta = 0.5, t)$; the Eulerian position (dashed line) is just $y = 0.7$, a constant in this diagram. (b) Velocities; the Lagrangian velocity of the parcel defined by $\beta = 0.5$ and the Eulerian velocity at the fixed position, $y = 0.7$. Note that the parcel identified by $\beta = 0.5$ crosses the Eulerian observation position $y = 0.7$ at time $t = 0.48$, computed from Eq. (7). At that specific time the Lagrangian velocity of this parcel and the Eulerian velocity at this position are equal, but not otherwise. That this equality holds is at once trivial - a non-equality could only mean an error in the calculation - but also consistent with and illustrative of the FPK, Eq. (3). (c) Accelerations; The Lagrangian acceleration of the parcel (green, solid line) and the acceleration evaluated at the fixed position (dotted and dashed lines) $y = 0.7$. There are two ways to compute a time rate change of velocity at a fixed point; one of them, $DV_E/Dt$ (dotted line), is the counterpart of the Lagrangian acceleration, in the sense that at the time the parcel crosses the Eulerian observation site, $DV_E/Dt = dV_L/dt$, a crucially important point discussed in Section 3.1.
3 The Eulerian (or Field) Coordinate System.

We have been keen on extolling the virtues of Lagrangian observation, but we should admit to some inherent problems, as well. The spatial sampling of Lagrangian data is more or less uncontrolled since the parcels will go wherever the flow takes them, and that may not be where we our interest lies. For example, if our goal was to observe the long term average flow through a channel, then we might prefer to moor a current meter directly in the channel rather than chase floats or drifters in and out. If high temporal resolution was desirable then again we might find that it was preferable to install a rapidly sampling current meter rather than resample with Lagrangian methods. Examples in which the flow conditions at a specific site are the desired goal abound, and so does the need for Eulerian observations.

An important and rapidly developing observational technique involves the generation of the Eulerian velocity field from Lagrangian data by the analysis procedure of interpolating or mapping irregularly sampled Lagrangian data $V_L(\beta, t)$ on to a spatial grid. To know where to assign the velocity we will also have to know the position, $y = \psi(\beta, t)$. This kind of procedure, which is a direct application of the FPK, is widely used to make maps of entire flow fields at once. One such method is known as Particle Imaging Velocimetry, or PIV. On the laboratory scale, the PIV technique uses successive photographic images of neutrally buoyant particles that might be illuminated by a pulsed laser source. Provided that the particles (or the pattern that they form) can be recognized from one image to the next, then it is fairly simple to differentiate parcel position with respect to time and then form a map, sometimes in three-dimensions and in great detail, of the flow throughout the domain. The same basic technique can be applied to observe winds on a planetary scale by tracking either naturally occurring features (clouds) or balloons. Satellite-tracked drifters on the ocean surface make it possible to measure directly quite detailed maps of the time-mean surface circulation, and acoustically-tracked floats may be used to observe the mid-depth circulation of the open ocean (e.g., the cover graphic).

In the contrived example of a Lagrangian flow considered in Section 2 we have the huge advantage of knowing all the parcel trajectories via Eq. (11) and so we can so make the transformation from the Lagrangian to the Eulerian system explicitly. Formally, the task is to eliminate all reference to the parcel initial position, $\beta$, in favor of the position $y = \psi$. This is readily accomplished since we can invert the trajectory Eq. (11) to find $\beta$,

$$\beta = y(1 + 2t)^{-1/2}. \quad (30)$$
where we have already substituted \( y \) for \( \psi \). Substitution into (12) and a little rearrangement gives the velocity field for this flow

\[
V_E(y, t) = y(1 + 2t)^{-1}
\]

(31)

which is plotted in Fig. (1c). Admittedly, this is not an interesting velocity field, but rather a very simple one, and partly as a consequence, this (Eulerian) velocity field looks a lot like the Lagrangian velocity of moving parcels (cf, Eq. 6 and Fig. 1b). However, the spatial coordinates in Figs. (1b, 1c) are qualitatively different - in the Lagrangian data (b), it’s \( \beta \), the initial \( y \)-coordinate of parcels, while in (c) the coordinate is the usual field coordinate, \( y \). To compare the Eulerian and the Lagrangian velocities is thus a bit like comparing apples and oranges; they are not the same things. Though different generally, nevertheless there are times and places where the two velocities are exactly the same, as evinced by the Fundamental Principle of Kinematics or FPK. By tracking a parcel around in this flow and by observing velocity at a fixed site (in Fig. 2 we have arbitrarily chosen the parcel tagged by \( \beta = 0.5 \) and the observation site \( y = 0.7 \)), we can verify that the Eulerian and the Lagrangian velocities are equal at a common \( y \) and time consistent with the FPK, Eq. (7) (Fig. 2b). Indeed, there is an exact equality since there has been no need for approximation in this transformation Lagrangian \( \rightarrow \) Eulerian.\(^{23}\)

3.1 The material derivative

The acceleration (Fig. 2c) is a little more involved. In the Eulerian system the partial derivative with respect to time represents the rate of change (of velocity, say) at a point fixed in space, and is not equal to the Lagrangian time rate change except in the somewhat degenerate case that there are no spatial variations of the flow. However, we have also emphasized that the velocity, temperature etc., of a fluid parcel is exactly the same thing as the velocity, temperature, etc. observed in an Eulerian frame at that same time and at the parcel position (if this statement sounds circular when we say it this way, then it is!). This suggests that we should be able to write the time rate of change following a parcel in terms of Eulerian (or field) variables, and this is the first of three step in deriving equations suitable for modelling most fluid flows.

To accomplish this transformation of a time derivative we have to write the time rate of change of a fluid variable, say \( s \), in terms of \( y \) and \( t \), and as before eliminate \( \beta \). Thus

\[
\frac{ds(\beta, t)}{dt} = \frac{ds(y(t), t)}{dt},
\]

(32)

where we are assuming that the parcel trajectory Eq. (1) can be used to go from \( \beta \) and \( t \) to a unique \( y \) position, \( y(t) = \psi(\beta, t) \). To compute the time derivative we apply the chain rule and find that

\[
\frac{ds(y(t), t)}{dt} = \frac{\partial s}{\partial t} + \frac{\partial s}{\partial y} \frac{\partial y}{\partial t}.
\]

(33)

\(^{23}\)Here’s one for you: assume Lagrangian trajectories \( \psi = a(e^t + 1) \) with \( a \) a constant. What is the position of parcels at \( t = 0 \)? Compute and compare the Lagrangian velocity \( V_L(\beta, t) \) and the Eulerian velocity field \( V_E(y, t) \). Suppose that two parcels have initial positions \( \beta = 2a \) and \( 2a(1 + \delta) \) with \( \delta \ll 1 \); how will the distance between these parcels change with time? How is the rate of change of this distance related to \( V_E \)? (Hint: consider the divergence of the velocity field, \( \partial V_E / \partial y \)). Suppose the trajectories are instead \( \psi = a(e^t - 1) \).
The key thing here is that \( y \) on the right hand side is meant to be the position of a moving parcel and so it is appropriate to write this in terms of the fluid velocity, \( v = \frac{\partial y}{\partial t} \) and thus

\[
\frac{ds}{dt} = \frac{\partial s}{\partial t} + v \frac{\partial s}{\partial y}. \tag{34}
\]

In case this derivative could be misinterpreted it is common to use the operator \( \frac{D}{Dt} \) and to write equations of this sort in the form

\[
\frac{D(\cdot)}{Dt} = \frac{\partial (\cdot)}{\partial t} + v \frac{\partial (\cdot)}{\partial y} \tag{35}
\]

that recurs in fluid dynamics. The operator \( \frac{D}{Dt} \) goes by a profusion of different names — the convective derivative, the substantive (or substantial) derivative, the Stokes derivative and the material derivative (our choice) — giving a clue to its great importance.

\( \frac{D}{Dt} \) is very often said to be the time derivative ‘following the flow’. This is true insofar as \( \frac{\partial y}{\partial t} \) in Eq. (33) is the fluid velocity. However, this name could be a bit misleading if it was interpreted to mean parcel tracking in the Lagrangian sense. Rather, the operator \( \frac{D}{Dt} \) gives, entirely in field coordinates (just \( y \) in this case), the time rate of change that would be observed by a parcel moving through the point \( y \) at the time \( t \). Thus \( \frac{D}{Dt} \) ‘follows the flow’ only in an instantaneous sense. Fig. (2c) showed an example, the time rate change of a velocity component following a parcel, \( d^2 \psi(\beta, t)/dt^2 \), along with the field equivalent, \( DV / Dt \) defined above. These two accelerations are exactly equal at a common position and time in precisely the same way that the corresponding Lagrangian and Eulerian velocities are equal at a common position and time. The material derivative is then the time rate of change equivalent of the FPK and is the second key step in the transformation of dynamics into the Eulerian coordinate system.

In the example above we have assumed a one-dimensional domain to minimize the algebra. The same relations hold in a three-dimensional flow having Cartesian coordinates \((x, y, z)\) and velocity \((u, v, w)\), the material derivative then being

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}, \tag{36}
\]

or using more compact vector notation,

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \tag{37}
\]

Notice that the subscript \( E \) has been dropped from the velocity vector, since ‘Eulerian’ should be clear. The gradient operator expanded in Cartesian coordinates is

\[
\nabla = e_x \frac{\partial}{\partial x} + e_y \frac{\partial}{\partial y} + e_z \frac{\partial}{\partial z},
\]

with \( e \) the unit vector. Notice that the subscript \( E \) has been dropped from the velocity vector, since it should be clear that the velocity is Eulerian. The advective derivative term \( \mathbf{V} \cdot \nabla \) of (37) is, in effect, a scalar that is multiplied into the variable being differentiated. To emphasize this property, the advective derivative is sometimes written \((\mathbf{V} \cdot \nabla)\). \(^{24}\)

\(^{24}\)Care has to be taken when \( \frac{D}{Dt} \) is applied to velocity (or any vector) and then expanded in other than Cartesian coordinates.
3 THE EULERIAN (OR FIELD) COORDINATE SYSTEM.

3.2 Reynolds Transport Theorem

We have noted that the conservation principles of classical physics apply to specific parcels or chunks of material, and not to volumes fixed in space, but we have also indicated in Section 2.2 that the development of models of fluid flow is generally best done in an Eulerian or field coordinate system rather than Lagrangian or material coordinates. Thus the need to know the field form equivalent of the time derivative of an integral taken over a moving fluid volume leads to the third key piece of the transformation of dynamics from Lagrangian to Eulerian form, the Reynolds Transport Theorem, or RTT.

Consider the integral in \( R^1 \) of an intensive, fluid property, let’s say, \( s \), over a moving ‘volume’ of fluid (the results are easily extended to \( R^3 \)):

\[
S(t) = \int_{\psi_1}^{\psi_2} s d\psi.
\]

(38)

The volume integral \( S \) is thus an extensive property which may follow a conservation law. We can here assume that \( S \) is conserved aside from a possible source, \( -\nabla \cdot Q \). It is far preferable to specify the source in field coordinates rather than material coordinates, which is part of the motivation for the present development.

The integral Eq. (38) looks a little exotic because the independent coordinates are material coordinates and the limits on the integral are the positions of moving parcels, e.g., \( \psi_1(\beta_1, t) \) is the \( y \)-position of the parcel tagged by the initial location \( y = \beta_1 \). Nevertheless it is no more than the sum of an integrand, in this case, \( s \), multiplied by a length, \( d\psi \), that happens to be embedded in a moving fluid. Thus the length may change with time. To transform this integral to field variables we have to transform the time derivative of \( s \), which we have already done (Section 3.1), and the time derivative of a material length,

\[
L(t) = \int_{\psi_1}^{\psi_2} d\psi = \psi_2 - \psi_1
\]

(39)

which is just the distance between the two moving parcels. The time derivative of this length is then just the

In the Cartesian example we find that the \( D/Dt \) produces four terms for each component of the vector being differentiated, one term being the partial with respect to time, and three terms arising from the advective term. However, when \( D/Dt \) is expanded in cylindrical polar coordinates we get five terms, with an additional advective term arising from the \( \theta \) dependence of the \( r \) and \( \theta \) unit vectors (in the Cartesian system the unit vectors are constant). In spherical polar coordinates the \( D/Dt \) operator expands to six terms. It is sometimes convenient to represent the advective derivative of velocity by the following vector identity:

\[
\nabla \cdot (V \times \nabla) = \nabla V \cdot (V^2) + (\nabla \times V) \times V,
\]

which is less likely to be misinterpreted. This form is especially useful applied to an ideal fluid, wherein frictional effects vanish and so too may the curl of the velocity, also called the vorticity (Section 6.4).

25There are three important distinctions between intensive and extensive fluid properties. (1) An intensive property of a fluid is measurable at a point, while an extensive property of a fluid is the integral of an intensive property over a finite volume and so is probably not directly measurable. (2) Imagine two volumes of fluid, \( V_1 \) and \( V_2 \), having temperatures \( T_1 \) and \( T_2 \). Now suppose that the volumes are added together. The new volume will be \( V_3 = V_1 + V_2 \), while the new temperature will be \( T_3 = (T_1 V_1 + T_2 V_2) / V_3 \), aside from nonlinear effects in the equation of state. Thus extensive properties add up when volumes are combined, while intensive properties are an average. (3) Extensive properties such as mass, and internal energy, may be subject to a conservation law, while the corresponding intensive properties, density and temperature, almost certainly will not be. Nevertheless, as we will see in this section, conservation laws for extensive properties will lead to useful differential balance equations for the intensive properties.
velocity difference at the location of the two parcels that mark it’s endpoints, i.e.,

\[
\frac{dL}{dt} = \frac{d}{dt}(\psi_2 - \psi_1) = v_2 - v_1,
\]

where \( v_1 \) is the fluid velocity at \( y = \xi_1 \). When the length \( L \) is infinitesimal the velocity difference may be written \( v_2 - v_1 = \frac{\partial v}{\partial y} L \) and hence

\[
\frac{dL}{dt} = \frac{\partial v}{\partial y} L.
\]

The normalized rate of change of length, \( \frac{1}{L} \frac{dL}{dt} = \frac{\partial v}{\partial y} \) is called the linear strain rate, something we have seen already in Section 1.1.1 and will see again in Section 7.4. The time derivative of an infinitesimal material length thus transforms to field coordinates as

\[
\frac{d}{dt} dy_{\text{material}} = \frac{\partial v}{\partial y} dy_{\text{field}}.
\]

where the derivative \( \frac{\partial v}{\partial y} \) is in field coordinates and \( dy_{\text{field}} = dy_{\text{material}} \) at the time that the transformation is made. This accounts for the possible change in the length of a material line. If instead of a line segment we had transformed a differential volume, then the corresponding result would be

\[
\frac{1}{\text{Vol} \frac{d}{dt} (\text{Vol})} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla \cdot V
\]

\( \nabla \cdot V \) is the divergence of the fluid velocity and is the volumetric (or normalized) time rate change of the volume.

With these results in hand, we are ready to state the Reynolds Transport Theorem (or RTT) which relates the time derivative over a material volume to the equivalent field quantities. For this one-dimensional case:

\[
\frac{dS}{dt} = \frac{d}{dt} \int_{\psi_1}^{\psi_2} s \, d\psi = \int_{y_1}^{y_2} \left( \frac{DS}{Dt} + s \frac{\partial v}{\partial y} \right) dy.
\]

where \( y_1 = \psi_1 \), etc. at the time the transformation is made. Notice that the integral on the left side of the equation is over material coordinates, while the integral on the right side is over field coordinates (in case there could be any ambiguity the type of coordinate will be written out). In three dimensions the RTT applied to a scalar \( s \) that has a body source \( -\nabla \cdot Q \) is just

\[
\frac{d}{dt} \int_{\text{material}} s \, d\text{Vol} = \int_{\text{field}} \left( \frac{DS}{Dt} + s \nabla \cdot V \right) d\text{Vol} = -\int_{\text{field}} \nabla \cdot Q \, d\text{Vol}
\]

The RTT is an exact, kinematic relationship that holds for any intensive fluid property, i.e., \( s \) could be mass density, \( \rho \), momentum density, \( \rho V \), energy density, etc. The appropriate source term \( \nabla Q \) will, of course, be different for each of these.
3 THE EULERIAN (OR FIELD) COORDINATE SYSTEM.

3.3 The Eulerian equations of motion

3.3.1 Mass conservation

An important application of the RTT is to the mass of a moving, three-dimensional volume of fluid,

\[ M = \iiint_{\text{material}} \rho \, d\text{Vol}, \]  

(45)

which is thus an extensive property defined on a specific material volume. There is no source or sink for mass in the classical physics that we presume holds, and thus we can assert that the \( M \) of a specific material volume must remain exactly constant,

\[ \frac{dM}{dt} = 0, \]  

(46)

for all flow conditions. It is crucial to understand that we could make no such general assertion for the intensive property, \( \rho \), nor could we assert this conservation property for a volume that was fixed in space.

The mass conservation Eq. (46) has a very clear and precise physical meaning, but it is not in and of itself directly useful as a means to predict mass or density in most models of fluid flow. By application of the Reynolds Transport Theorem, Eq. (44) and use of Eq. (45) we can write this in a form that will be,

\[ \frac{dM}{dt} = \frac{d}{dt} \iiint_{\text{material}} \rho \, d\text{Vol} = \iiint_{\text{field}} \frac{D\rho}{Dt} + \rho \nabla \cdot V \, d\text{Vol} = 0, \]  

(47)

where the triple integral is over the spatial position of the volume. If this integral relation holds at all times and for all positions within a domain, and if the integrand is smooth (no discontinuities), then the integrand must vanish at all times and positions in that domain\(^{26}\) yielding the differential form of the mass conservation relation,

\[ \frac{D\rho}{Dt} + \rho \nabla \cdot V = 0 \]  

(48)

The meaning of Eq. (48) is also very clear: if the density of a fluid parcel changes in time, say it decreases, \( \frac{D\rho}{Dt} < 0 \), then we can conclude that there must have been an a divergence of the fluid velocity, \( \nabla \cdot V > 0 \), and thus an increase in the volume occupied by the fluid. This is a kinematic relationship that holds regardless of the immediate cause of the density change, i.e., whether due to a pressure variation or heat exchange with the surroundings, or even a phase change of the fluid material. Thus Eq. (48) is not sufficient by itself sufficient to predict or understand why density might change, which comes instead from a thermodynamic state equation for the fluid, as we will discuss further below. However, it is worth noting that Eq. (48) does not hold if the parcel exchanges material with its surroundings, e.g., salt in the case of sea water or water vapor if air, in which case it would not be a constant material.

\(^{26}\)We know that integral \( \int_{x_1}^{x_2} \Psi(x) \, dx = 0 \) vanishes for any \( x_1, x_2 \) and that \( \Psi \) is smooth. If we now suppose that \( \Psi(x_a) > 0 \), we can show that this leads to a contradiction. If \( \Psi \) is indeed smooth, then there will be some neighborhood around \( x_a \) where \( \Psi(x) > 0 \). Choose \( x_1 \) and \( x_2 \) to lie within this neighborhood, and apply the mean value theorem to the integral to find that \( \Psi(\bar{x})(x_2 - x_1) \geq 0 \) since \( \Psi(\bar{x}) \geq 0 \). This contradicts what we know about this integral, hence \( \Psi(x) \) must be zero at every point. In other words, the only smooth function whose integral is zero on every interval is the zero function.
3.1 THE EULERIAN (OR FIELD) COORDINATE SYSTEM.

The mass conservation relation may be written in one of two forms that emphasizes the local time rate of change of density,

\[ \frac{\partial \rho}{\partial t} + V \cdot \nabla \rho + \rho \nabla \cdot V = \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho V) = 0. \]  

(49)

Expanding this first version in Cartesian components yields

\[ \frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z} + \rho \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \]

The meaning of Eqs. (49) is that if the density changes at a fixed position, \( \frac{\partial \rho}{\partial t} > 0 \), say, then the sum of the divergence and advective terms must have been negative (more on this in Section 6.1).

**Incompressible flow:** The divergence term is clearly necessary in the mass conservation equation, and yet for most phenomenon of the atmosphere or ocean and for many engineering flows the fluid velocity associated with this divergence is very, very small compared to other fluid velocities and may be ignored with no appreciable error.27 Under this so-called incompressibility assumption, the velocity is assumed to follow

\[ \nabla \cdot V = 0, \]

(50)

which in effect says that the volume (rather than the mass) of fluid parcels is exactly constant. Even with the incompressibility assumption in place, it is not inconsistent to have a model in which density may nevertheless change, and indeed density changes may be of primary importance in causing the flow of an incompressible fluid. Under the incompressibility assumption, density may be computed from an equation of state given the pressure, temperature, salinity, etc., with no reference made to the divergence of the fluid velocity. That we can ignore fluid divergence is a very useful mathematical approximation that is contingent upon the physical phenomenon under consideration. One important class of phenomena, acoustic waves, owe their entire existence to velocity divergence and associated pressure changes. Once a few more pieces are in place, we will be able to appreciate that the incompressibility assumption for the velocity field, Eq. (50), can be made with negligible error provided that the fluid velocity is much less than the speed of sound, which holds well for most natural flows of the atmosphere and ocean.

3.3.2 Momentum conservation; Newton’s Second Law.

A second important application of the Reynolds Transport Theorem arises on consideration of momentum balance. The momentum of a moving, three-dimensional volume of fluid can be written

\[ B = \iiint_{\text{material}} \rho V \, dVol. \]

(51)

Because this is a material volume we can assert Newton’s Second Law, that the momentum of this specific volume of fluid can change only if there is a net force,

\[ \frac{dB}{dt} = \iiint_{\text{material}} F \, dVol, \]

(52)

27A question for you: Suppose that a volume of air, 10 km on a side by 1 km thick, is heated at constant pressure by 5 C in a period of an hour (as might occur in a vigorous sea breeze circulation). What is the magnitude of the associated (divergent) velocity, assuming that it appears on one vertical face only of the volume?
3. THE EULERIAN (OR FIELD) COORDINATE SYSTEM.

where $F$ is the sum of all forces acting on the fluid. These forces could include inertial forces, such as gravitational mass attraction or the Coriolis force that act throughout the volume, and stresses (Sections 1.1 and 2.3) that act on the surface of the volume. By means of the Reynolds Transport Theorem and the mass conservation relation we can write the left side of Eq. (52) in field coordinates as

$$
\frac{d}{dt} \int V \rho dV = \int \rho \frac{DV}{Dt} dV.
$$

A term that might have been expected, $\frac{D}{Dt} V$, has dropped out by application of the mass conservation requirement. Thus the momentum of a fluid parcel (or marked fluid volume) can change only because of a change of the velocity.\(^{28}\) The right side of Eq. (52) is a volume integral over field coordinates that is performed over the volume occupied instantaneously by the moving fluid,

$$
\int F dV = \int F dV,
$$

and thus from Eqs. (53) and (54),

$$
\int \rho \frac{DV}{Dt} dV = \int F dV.
$$

The volume considered here is arbitrary, and so the differential form of the momentum balance for a fluid continuum is

$$
\frac{DV}{Dt} = \frac{\partial V}{\partial t} + (V \cdot \nabla) V = F / \rho \quad \text{(56)}
$$

We developed a specification of pressure and viscous forces in Sections 1.1 and 2.3, and repeat the main result here,

$$
F = -\nabla P + \nu \nabla^2 V.
$$

The Eulerian form of the momentum balance equation is then, including the body force due to gravity,

$$
\frac{\partial V}{\partial t} + (V \cdot \nabla) V = \frac{-1}{\rho} \nabla P + \frac{\nu}{\rho} \nabla^2 V - g e_z
$$

(57)

3.3.3 Energy conservation; the First Law of Thermodynamics.

As a final application we will consider the balance of thermal energy in a compressible fluid. The First Law of Thermodynamics keeps track of several kinds of energy storage and energy exchange that may occur between a given fluid volume and the environment, schematically,

$$
\frac{dE}{dt} = \text{Work Rate} + \text{Heat Source}
$$

(58)

where $E = \rho C_v T$, is the internal energy, and $C_v$ is the heat capacity of the fluid in an isovolume process, taken to be a known physical property of the fluid, and a constant for this purpose. \(\text{Work Rate}\) is the rate at

\(^{28}\) You should fill in the steps of Eq. (53) to verify that this is true whenever the extensive property, say $H$, is the volume integral of $\rho h$, where $\rho$ is mass density and $h$ is the corresponding intensive property.
which mechanical work is done by the environment on the fluid volume. In the case of an Euler fluid, this can be due only to the rate of work by pressure, \(-P \mathbf{n} \cdot \mathbf{V}\), evaluated over the surface of the volume. 

HeatSource is the divergence of heat fluxes due to radiation, \(-\nabla \cdot \mathbf{Q}\), and conduction, \(-\kappa \nabla T\), where \(\kappa\) is the thermal conductivity of the fluid, and to a body source due to the dissipation of kinetic energy to internal energy, \(\epsilon\). In most cases \(\epsilon\) is negligibly small compared to the heat fluxes or WorkRate and will be omitted from here on. The approximate balance of thermal energy is then:

\[
\frac{dE}{dt} = C_v \frac{d}{dt} \iiint_{\text{material}} \rho T \, dVol = -\oint P \mathbf{n} \cdot \mathbf{V} \, da - \oint \kappa \nabla T \cdot \mathbf{n} \, da + \iiint_{\text{material}} -\nabla \cdot \mathbf{Q} \, dVol. \tag{59}
\]

Applying the divergence theorem to the surface integrals and the RTT and then collecting terms under the volume integral yields the differential form of the thermal energy balance in field coordinates:

\[
\rho C_v \frac{dT}{Dt} = -P \nabla \cdot \mathbf{V} + \kappa \nabla^2 T - \nabla \cdot \mathbf{Q}. \tag{60}
\]

To simplify this a little further, the fluid will be presumed to be an ideal gas described the equation of state

\[
P = \rho RT,
\]

where \(R = C_p - C_v\) is the universal gas constant and \(C_p\) is the heat capacity at constant pressure (Table 1). The divergence of fluid velocity may be eliminated by use of the mass conservation equation (48) and then the equation of state,

\[
-P \nabla \cdot \mathbf{V} = -\frac{\rho}{\rho} \frac{D\rho}{Dt} = \frac{\rho}{\rho} \frac{\partial \rho}{\partial T} |_{\rho} \frac{DT}{Dt} = -\frac{P}{\rho} \frac{P}{RT^2} \frac{DT}{Dt} = -\rho R \frac{DT}{Dt}.
\]

When this is substituted into Eq. (60) the result is

\[
\begin{align*}
\text{for an ideal gas: } \rho C_p \frac{DT}{Dt} &= \kappa \nabla^2 T - \nabla \cdot \mathbf{Q} \\
\end{align*}
\]

where notice the heat capacity is now \(C_p\) in place of \(C_v\). For most liquids and solids the distinction between \(C_p\) and \(C_v\) is negligible under geophysical conditions, since there is usually very little volume change.

### 3.3.4 State Equation

A complete model of a fluid will require a specification of the density of the fluid. This might be as simple as the specification \(\rho = \rho_0 = \text{const}\), which would suffice in a study of gravity waves on the sea surface, for example. But in general the density of a fluid will vary with at least the pressure and the temperature, and the relation between density and the other relevant thermodynamic variables is called an equation of state. If the fluid is air having a more or less constant humidity then the ideal gas law noted above may suffice, but in the case of seawater there will be an important dependence also upon the salinity, and the equation of state is known precisely but only from empirical studies that we indicate only schematically,

\[
\rho = \rho(P, T, S).
\]

Thus a salinity budget equation will also be required. A state equation is a scalar equation; it is obviously highly dependent upon the physical properties of the fluid under study, but is not at all effected by the choice of Lagrangian or Eulerian coordinate systems. Hence it is outside the scope of this essay.
3.4 Remarks on the Eulerian balance equations

The momentum balance equations (57) have two terms that are characteristic of fluid flows, the pressure gradient term and the advection terms. Pressure is a scalar, and the gradient of the pressure appears in each of the component equations. A localized pressure perturbation in a three-dimensional flow is thus very likely to induce motion in all three components. The pressure gradient thus acts to couple the component equations, and is generally the physical process that allows for (or enforces) mass conservation; as fluid converges into a given volume the pressure will rise and so produce a compensating pressure-driven divergence, e.g., acoustic and gravity waves if high or low frequency. The pressure gradient term is linear, and does not, in and of itself, present any special mathematical difficulty.

From a physical (and Eulerian) perspective, it is the process of advection that endows many fluid flows with rich spatial structure and complexity and it is advection that most distinguishes a fluid flow from solid mechanics. The advection terms are nonlinear, in general, and hence the advective terms stymie most of the familiar PDE solution techniques that require superposition of solutions (more about advection in Section 6). It is not uncommon that the advection terms are demonstrably much, much smaller than the pressure gradient term, and so they may be omitted to leave a linear balance equation. A linearized system of that sort clearly omits some of the distinctive character of fluid flow but may be quite useful for the analysis of wave-like motions, especially.

The (Eulerian) momentum, mass and energy balance equations are impressive, and it might seem that writing them out in full would be a significant step towards the solution of a fluid dynamics problem. Well, yes and no. Understanding the origin and the meaning of these equations is certainly a vital step toward understanding fluid dynamics, generally. But it is important to understand that these equations are extremely general, and that the definition of a specific problem requires a number of additional steps and likely some important simplifications. In fact, the content of these equations is just what their name implies - that momentum, mass and energy are conserved (though we omitted the dissipation of mechanical energy to thermal energy) and, here’s the punch line, with nothing more implied than that. Whether momentum conservation occurs by virtue of wave motions or swirling vortices or random turbulence, or more likely all three at once (as in the tea cup) isn’t given by these equations alone. A complete model of a fluid flow problem will likely require all three of the Eulerian equations derived above together with careful consideration of the boundary and initial conditions that serve to define a specific problem.

The solution of such a model system is often a very substantial task requiring a great deal of time and effort and resort to numerical and sometimes experimental methods; most of your study of fluid mechanics will be aimed at this kind of task. Here and now, though, we are now going to make the generous assumption that we have generated analytic or numerical solutions of the velocity field, so that we can continue on with the development of Lagrangian/Eulerian kinematics.
4 Depictions of Fluid Flow.

It may be apparent from observing the flow in a tea cup or in an ocean circulation model that displaying or depicting a fluid flow can be a significant task in cases where the domain is multi-dimensional and the flow is time-dependent. A variety of methods are used to show the flow dependence upon one or more of the independent variables, and some of these are a direct application Lagrangian-Eulerian transformation problem considered in Sections 2 and 3. 29

4.1 Trajectories, or pathlines

One important example is the parcel trajectories, often called pathlines. In this section we will consider position and velocity in a two-dimensional space, \( \mathbb{R}^2 \), and \( x \) and \( V \) indicate vector position and velocity. From here on out we are going to drop the subscripts \( L \) and \( E \) that have been used to emphasize Lagrangian and Eulerian velocity. The kind of velocity should be clear from the context, or from the list of independent variables.

We can compute parcel trajectories from the Eulerian velocity field via

\[
\frac{dx}{dt} = V(x(t), t)
\]  

(62)

provided we recognize that \( x \) on the right side is the moving (time-dependent) parcel position. The appropriate initial condition is just

\[
x(t = 0) = \xi.
\]  

(63)

Note that (62) is in the form of the FPK, or Eq. (7). In component form this may be written out

\[
\frac{dx}{dt} = u(x, y, t); \quad \frac{dy}{dt} = v(x, y, t)
\]  

(64)

and with the initial conditions (ICs)

\[
x(t = t_0) = \alpha; \quad y(t = t_0) = \beta
\]  

(65)

which makes clear that we have two first order ODEs. On first sight these trajectory equations (64) could be deceptive; as here written they are quite general and applicable to any fluid motion in \( \mathbb{R}^2 \). Thus it should not be surprising if on most occasions they prove intractable by elementary methods. If \( u \) depends upon \( y \) or \( v \), or if \( v \) depends upon \( x \) or \( u \), then these are coupled equations that have to be solved simultaneously; if \( u \) or \( v \) are nonlinear then they are nonlinear equations. Either way their solution may have to be sought with numerical techniques. What is surprising about Eq. (64), even after several encounters, is that what can seem to be very simple velocity fields can yield complex and interesting trajectories (one example is in Section 4.2).

29 An excellent web page that shows the practical reasons and methods for computing the trajectories of air parcels using Eulerian data from large scale numerical models of the atmosphere is at http://www.arl.noaa.gov/slides/ready/conc/conc2.html
4  DEPICTIONS OF FLUID FLOW.

We can best illustrate these diagnostic quantities with a two-dimensional velocity field,

\[ \mathbf{V} = x \mathbf{e}_x + \frac{y}{1 + 2t} \mathbf{e}_y, \]

which is plotted for two times in Fig. 7. The component equations are then

\[ \frac{dx}{dt} = x; \quad \frac{dy}{dt} = \frac{y}{1 + 2t}, \]

and with ICs as above. The dependent variables are uncoupled, and moreover, within each component equation the independent variables can be readily separated,

\[ \frac{dx}{x} = dt; \quad \frac{dy}{y} = \frac{dt}{(1 + 2t)}. \]

These can then be integrated over the limits \( \alpha \) to \( x \) (\( \beta \) to \( y \)) and \( t_0 \) to \( t \) to yield the trajectory

\[ x(\alpha, t_0, t) = \alpha e^{p(t - t_0)}; \quad y(\beta, t_0, t) = \beta \frac{(1 + 2t)^{1/2}}{(1 + 2t_0)^{1/2}}. \]

Notice that the \( y \)-component is just as before, Eq. (11), except that we have retained the initial time as a parameter (we will need it below). Trajectories staring from a few different \( \alpha \), \( \beta \) are in Fig. 8. In this case the trajectories are roughly in the direction of the flow as seen in Fig. 7, and appear to bend over in time, consistent with the temporally-decreasing \( y \)-component of the velocity field. There is nothing surprising in this case, but in a later section, 5.2, we will see an example where the trajectories could not have been anticipated in advance of an integration, and the Eulerian and Lagrangian mean flows are qualitatively different.

4.2  Streaklines

Another useful characterization of the history of parcel positions is the so-called streakline, which shows the positions, at a fixed time, of all of the parcels which at some earlier time passed through a given point. An example of this would be the plume of smoke coming from a point source located at \( x_p \) and recorded, say by a photograph taken at a time, \( t_p \). The information needed to construct a streakline is contained within the trajectory, Eq. (68). To see this we will construct a streakline by releasing parcels one after the other from a fixed source. The first parcel is released at time \( t_0 = 0 \), and we let the trajectory run until \( t = t_p \), the time we make the photograph. The only data point we retain from this trajectory is the position at time \( t = t_p \), i.e., we record \( x(t_p, x_p, t_0 = 0) \). A second parcel is released a little later, say at \( t_0 = \frac{1}{4} \), and again we let the trajectory run until \( t = t_p \), where we retain only the last point, \( x(t_p, x_p, t_0 = \frac{1}{4}) \). A third parcel is released at \( t_0 = \frac{3}{4} \), and again we record it’s position at \( t = t_p \), \( x(t_p, x_p, t_0 = \frac{3}{4}) \). It appears, then, that a recipe for making streakline from a trajectory is that we treat the initial time, \( t_0 \), as a variable, while holding \( t \) constant at \( t_p \), and also the initial position at \( x = x_p \). Several streaklines are in Fig. 9. Notice that in this time-dependent flow, trajectories and streaklines are not parallel.
Figure 7: (a) Velocity field and streamlines (the family of solid lines) for Eq. (67) at $t = 0$. (b) At $t = 1$. Notice that the velocity at a given point turns clockwise with time as the $y$-component of the velocity decreases with time.

Figure 8: Trajectories of six parcels that were released into the flow given by Eq. (67) at the same time, $t_0 = 0$, and tracked until $t = 1$. The sources are shown by asterisks. Dots along the trajectories are at time intervals of 0.1
4.3 Streamlines

Still another useful method is to draw streamlines, a family of lines that are everywhere parallel to the velocity. Time is fixed, say at \( t = t_f \), and thus streamlines portray the direction field of a velocity field, with no reference to parcels or trajectories or time-dependence of any sort. There is more than one way to construct a set of streamlines, but a method that lends itself to generalization is to solve for the parametric representation of a curve, \( X(s) \) that is everywhere parallel to the velocity:

\[
\frac{dX}{ds} = v(t_f, x, y) \tag{69}
\]

or in components:

\[
\frac{dX}{ds} = u(t_f, x, y); \quad \frac{dY}{ds} = v(t_f, x, y). \tag{70}
\]

A suitable ‘initial’ condition is \( X(s_0) = X_0 \), etc. Notice that \( s \) is here a dummy variable; we could just as well have used any other symbol but \( s \) is conventional. \( X \) is the position of a point on a line, where just above \( x \) meant the position of a parcel. This reuse of symbols is certainly a risky practice, but it’s also almost unavoidable. Given the velocity components Eq. (67), these equations are also readily integrated to yield a family of streamlines:

\[
X = X_0 e^{s - s_0}; \quad Y = Y_0 e^{\frac{s - s_0}{1 + 2t_f}}. \tag{71}
\]

and recall that \( t_f \) is the fixed time that we draw the streamlines. We are free to choose the integration constants so that a given streamline will pass through a position that we specify. There is no rule for choosing these positions; in Fig. 7 we arbitrarily picked five positions and then let \( s \) vary over sufficient range to sweep through the domain. Other streamlines could be added if needed to help fill out the picture. No particular value is attached to a given streamline. In the future we will consider the streamline’s sophisticated cousin, the streamfunction, which has isolines that are also parallel to velocity, but which assigns values that are related to the speed of the flow.

5 Eulerian to Lagrangian Transformation by Approximate Methods.

The previous sections emphasized the purely formal steps required to transform from one reference frame to the other. An understanding of the formal steps is important, of course, but the ease with which we could make the transformation in those cases could be positively misleading. In actual practice, an explicit and invertible specification of trajectories over an entire domain is highly unlikely, and even in the case that a complete field specification is available, it probably can not be integrated by elementary methods. In this section we will consider an approximate method based upon an expansion of the velocity field in Taylor

\[\text{In fact, } s \text{ could be regarded as time, provided we make certain not to confuse this use of time with the time-dependence of the velocity field (which is suppressed while we draw a given map of streamlines). This helps make clear that streamlines are parallel to parcel trajectories in steady flows. If we marked off equal increments of time along a streamline we could depict the speed of the flow.}\]
5 EULERIAN TO LAGRANGIAN TRANSFORMATION BY APPROXIMATE METHODS.

Figure 9: (a) Trajectories of five parcels that were released from a common source, \((x,y) = (2,3)\), and tracked until \(t = 1\). The parcels were released at different initial times, \(t_0 = 0, 1/4, 2/4, 3/4, \) and 1. The latter trajectory has zero length. The end points of the trajectories are the open circles, the locus of which forms a streakline. (b) Streaklines from several different sources. These streaklines start at \(t_0 = 0\) and the ‘photograph’ was taken at \(t_p = 1\). Notice that these streaklines end at the endpoint of the trajectories of Fig. 8 (they have that one point in common) but that streaklines generally have a different shape (different curvature) from the trajectories made over the same time range. In this figure and in previous ones (Fig. 2b) the two quantities being compared were only slightly different and one might well wonder if, for example, streaklines are some kind of approximation to trajectories. The answer is ‘no’, in general, they are qualitatively different.

series. This yields results that are interesting and important of themselves, and introduces some new tools, e.g., the velocity gradient tensor, that are widely useful.

5.1 Tracking parcels around a steady vortex

The power and the limitations of the Taylor series method can be appreciated by analysis of parcel motion in a steady, irrotational vortex in \(R^2\). The radial and azimuthal velocity components are given by

\[
V = (u_{rad}, u_{azi}) = (0, \frac{C}{2\pi r}),
\]

where \(r\) is the distance from the vortex center. The \(1/r\) dependence of azimuthal speed is the distinguishing feature of an irrotational vortex. \(C\) is a constant, termed the circulation,

\[
C = \oint V \cdot ds,
\]

where \(ds\) is the vector line segment along a path that encloses the vortex center and that is traversed in an anti-clockwise direction. \(C\) measures the vortex strength, and without loss of generality we can set
$C = -2\pi$ to define a vortex that rotates clockwise, Fig. (10). An irrotational vortex is an idealization of the vortex flow produced by the convergent flow into a drain, for example, and has several interesting properties that we will consider in later sections (including why it is said to be irrotational). For now it makes a convenient flow into which we can insert floats and current meters to investigate kinematics. It is apparent that parcel trajectories in this steady vortex will be circular, and that a parcel will make a complete orbit in time $T = (2\pi r)^2/C$. The Cartesian velocity components are

$$\begin{pmatrix} u \\ v \end{pmatrix} = \frac{C}{2\pi r} \begin{pmatrix} \sin \theta \\ \cos \theta \end{pmatrix}$$

(73)

where $r = (x^2 + y^2)^{1/2}$ and the angle $\theta = \text{atan}(y/x)$ is measured counter-clockwise from the x-axis.

For the sake of this development let’s suppose that the only thing we know is the velocity observed at one fixed site, say $(x_s, y_s) = (0, 1)$.\(^{31}\) The velocity observed at this fixed site, $V_0 = V(x_s, y_s)$, an Eulerian velocity, would then be a steady, uni-directional flow having Cartesian components $(U_0, V_0) = -(C/2\pi r, 0) = (1, 0)$. What, if anything, can be inferred about parcel trajectories from this limited data? If no other information was available, then we might make a first attempt at estimating the displacement of the parcel by integrating the Eulerian velocity in time as if it were the Lagrangian velocity, i.e.,

$$\delta X_0 = \int V_0 dt = \begin{pmatrix} U_0 t \\ 0 \end{pmatrix}.$$  

(74)

It is essential to understand that such a procedure is wrong, formally. But it is also fruitful to see the result, termed a progressive vector diagram, or PVD, as a lowest (zero) order approximation of the trajectory.\(^{32}\) The PVD in this case indicates a linear (pseudo-)trajectory, consistent with the uni-directional velocity observed at the fixed site, Fig. 6. A PVD is a useful way to visualize a current meter record or a wind record insofar as it gives a direct measure of how much fluid has gone past the observation site. But the question here is to what extent does a PVD show where fluid parcels will go after they pass through the observation site? That’s a hard question to answer generally, but a related question - is there any flow condition under which we could interpret a PVD as if it were a parcel trajectory? - leads to a useful analysis. A PVD would represent a true trajectory if the Eulerian velocity field was spatially uniform. Observations made at any position would then be equal to observations made anywhere else, including at the moving position of a parcel. A spatially uniform flow is a degenerate case of little interest, but this helps us to see that the issue insofar as this Eulerian to Lagrangian transformation is concerned is the spatial variation of the flow.

If we consider the example of a steady vortex flow, then it would appear that the PVD is an acceptable trajectory estimate only for (pseudo-)displacements that are much less than the horizontal distance over which the flow changes significantly. By inspection, the horizontal scale of this vortex is estimated to be the

\(^{31}\)When we write the Cartesian components of a vector within the body of the text as here, they will be written as if they were in a row matrix, i.e., $(x, y)$; when they are written as a separate equation they will be written as a column matrix, Eq. (73), which is the form in which they actually appear in tensor equations.

\(^{32}\)The notation $\delta X$ would usually mean a displacement vector that is small in some sense, e.g., compared to the radius of convergence of a power series. Here we are going to integrate long enough for the displacement to be substantial, and then we will call $\delta X$ the trajectory. This abuse of $\delta$ is intentional, because we want to see the consequences of violating the small displacement restriction.
radius (at a given point), and so this condition could be written \( \delta X_0 \ll r \). When the displacement is greater than this, the velocity at the position of the parcel (the Lagrangian velocity that we should be integrating) will begin to differ significantly from the velocity observed back at the fixed site (the Eulerian that we are integrating in this PVD-approximation). Once this discrepancy is evident, the PVD will soon fail to make a good approximation to the actual trajectory.

We could improve on this first attempt at computing a trajectory from Eulerian data if we could take some account of the spatial variation of the velocity. To do this we can represent the velocity field in the vicinity of the observation point by expanding in a Taylor series, here for each component separately,

\[
\begin{align*}
  u(x, y) &= u_0 + \frac{\partial u}{\partial x} \delta X + \frac{\partial u}{\partial y} \delta Y + HOT, \\
  v(x, y) &= v_0 + \frac{\partial v}{\partial x} \delta X + \frac{\partial v}{\partial y} \delta Y + HOT,
\end{align*}
\]

where \((u_0, v_0)\) is the velocity observed at the observation site, the partial derivatives are evaluated at the observation site, and \(HOT\) is the sum of all the higher order terms that are proportional to \(\delta x^2, \delta x^3\), etc. In effect, we are now allowing that we know not only the velocity but also the four partial derivatives, though at one position only. It is very convenient to use a vector and tensor notation to write equations like these in a format

\[
V(x, y) = V_0 + \mathcal{G} \cdot \delta X + HOT,
\]

where \(\mathcal{G}\) is the velocity gradient tensor encountered already in Section 2.3, and repeated here for the
two-dimensional case at hand,

\[
G = \begin{pmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}
\end{pmatrix}
\]  

(77)

\(G\) recurs in the study of kinematics and we will encounter it again in Section 8.3. For now \(G\) is a device to streamline notation; when we (matrix) multiply \(G\) into a displacement vector (written as a column vector), we get the velocity difference that corresponds to that displacement vector. It is easy to see that if we doubled or halved the length of the displacement vector we would get twice or half the velocity difference. Thus, multiplication by the velocity gradient tensor serves to make a linear transformation on a displacement vector. In general the result will be a velocity difference vector having a different direction from that of the displacement vector, and of course it has a different amplitude and different dimensions as well. The velocity gradient tensor evaluated at the observation site \((x_s, y_s) = (1, 0)\) has a simple form

\[
G = \frac{C}{2\pi} \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} = \begin{pmatrix}
0 & -1 \\
-1 & 0
\end{pmatrix}.
\]

The PVD approximation amounts to an integration of the first term only of the Taylor series, while ignoring the spatial variation of the velocity altogether. Since this approximation omits all terms that are first and higher power in the displacement, the PVD is termed the zeroth order displacement or trajectory. The first order trajectory, by which we mean the first correction to the trajectory, can then be computed by dropping the \(HOT\) from Eq. (76), by approximating \(\delta X = \delta X_0 = \int V_0 dt\) and integrating in time,

\[
\delta X_1 = \delta X_0 + G \cdot V_0 t^2 / 2.
\]  

(78)

This first order trajectory is a considerable improvement upon the PVD (Fig. 10). Nevertheless, after sufficient time has passed and the displacement becomes comparable to the length scale of the flow, the radius, then this first order trajectory also accumulates a noticeable error. Adding an evaluation of the next term of the \(HOT\) would delay the failure, in general, but in any steady vortex flow the displacement will eventually carry a parcel long distances from its origin. Approximation methods built around a Taylor series expansion are not uniformly valid in time when applied to a steady vortex flow.

5.2 Tracking parcels in gravity waves; Stokes drift

This method has somewhat better success when applied to a wavelike motion in which the parcel displacements over a single wave passage are small compared to the wavelength. In this flow there are then two length scales (where in the vortex flow above there was only one, the radius). As an example, we will analyze the parcel motion associated with a surface gravity wave having a surface displacement

\[
\eta(x, t) = a \cos(k x - \omega t),
\]  

(79)

\(33\)\(G\) is a Cartesian tensor that can be manipulated as if it were a matrix. \(G\) is a tensor insofar as its elements will transform with a rotation of coordinate axes in a way that leaves tensor equations invariant to any (time-independent) coordinate rotation. No such transformation properties are implied for the elements of a matrix.
where $a$ is the amplitude of the surface displacement, $k = 2\pi/\lambda$ is the wavenumber given the wavelength $\lambda$ and $\omega$ is the wave angular frequency (it is assumed that $\omega$ and $k > 0$). The argument of the trigonometric function shows that this surface displacement moves rightward as a progressive wave having a phase speed $c = \omega/k$. The two-dimensional and time-dependent velocity field associated with this wave

$$V(x, z, t) = U e^{kz} \begin{pmatrix} \cos(kx - \omega t) \\ \sin(kx - \omega t) \end{pmatrix}, \quad (80)$$

where $z$ is the depth, positive upwards from the surface. The amplitude or speed at the surface is $U = a\omega$ and decays with depth on an $e$-folding scale $1/k$. This exponential decay with depth is appropriate for a wave whose wavelength is less than the water depth, a so-called deep water wave. If the wavelength is much greater than the water depth, a shallow water wave, the $x$ component of the velocity is independent of depth and the $z$ component is linear with depth and vanishes at the (flat) bottom.

The (Eulerian) velocity observed at a fixed point is a rotary current, often called the orbital velocity, of amplitude $a\omega e^{kz}$ that turns clockwise with time at the angular frequency $\omega$. Given the known velocity we can readily calculate the PVD-like parcel displacements by integrating $V(x, z, t)$ with respect to time while holding $x$ and $z$ constant,

$$\delta X_0 = a e^{kz} \begin{pmatrix} -\sin(kx - \omega t) \\ \cos(kx - \omega t) \end{pmatrix}. \quad (81)$$

The PVD indicates that parcels move in a closed rotary motion with each wave passage and that the net motion is zero, consistent with the wave-average of the Eulerian velocity.

From the analysis of motion around a vortex we might have developed the insight that this PVD for a gravity wave would probably give a fairly accurate prediction for the actual parcel displacements provided that the parcel displacements were very much less than the scale over which the wave orbital velocity varies. In this case the scale is $k^{-1}$ in either direction, so that this condition is equivalent to requiring that the wave steepness, $ak = 2\pi a/\lambda$, must be much less than 1. This is also the condition under which the linear solution gives an accurate waveform of the surface displacement, a pure sinusoid, which we have assumed with Eq. (79).

If we are dubious of this zeroth order approximation of trajectories then we may want to calculate the first order velocity via Eq. (76), and neglecting the HOT. The velocity gradient tensor for this wave is just

$$\mathcal{G} = a\omega e^{kz} \begin{pmatrix} -\sin(kx - \omega t) & \cos(kx - \omega t) \\ \cos(kx - \omega t) & \sin(kx - \omega t) \end{pmatrix}.$$
Figure 11: (a) Parcel trajectories underneath a deep-water, surface gravity wave that is presumed to be propagating from left to right. The amplitude of the surface displacement was taken to be $a = 1$ m and the wavelength $\lambda = 50$ m. The Eulerian trajectories (or PVDs) are closed circular orbits around which the parcels move clockwise. The Lagrangian trajectories were computed by integrating numerically for four periods ($= 22.6$ sec) and are open loops indicating there is a net drift of fluid parcels from left to right (shown as an arrow). (b) Stokes drift for this wave computed from Eq. (82). An animation of a gravity wave and parcel trajectories is in the next figure.

and matrix-multiplying into the zeroth order displacement given by Eq. (81) gives the first order velocity,

$$V_1(z) = a^2 \omega k e^{2kz} \begin{pmatrix} \sin^2(kx - \omega t) + \cos^2(kx - \omega t) \\ 0 \end{pmatrix}$$

$$= Uak e^{2kz} \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

(82)

where recall $U = a\omega$ is the amplitude of the orbital velocity at the surface. The coefficient $Uak = a^2 k^2 c$ is the wave steepness squared times the phase speed. This velocity is independent of time and $x$, and so the first order displacement is easily computed,

$$\delta X_1(z) = V_1(z)t,$$

(83)

where there is no particular initial position. Notice that this formula relates the displacement of a parcel to a depth, so that it gives a field representation of a Lagrangian property.

Quite unlike the PVD, Eq. (83) indicates that fluid parcels have a substantial net motion in the direction of the wave propagation, often called Stokes drift or mass transport velocity, that is a fraction $2\pi k e^{kz}$ of the
orbital motion amplitude. For example, for a wave having an amplitude of \( a = 1 \) m and wavelength of \( \lambda = 50 \) m, the orbital motion at the surface is about 1.11 m \( s^{-1} \) where the Stokes drift is about 0.14 m \( s^{-1} \). The Stokes drift decreases very rapidly with depth; twice as rapidly as does the wave orbital velocity. The vertically integrated Stokes drift is the mass transport per unit length,

\[
M = \rho \int_{-\infty}^{0} V(z)dz = \left( \frac{\rho Ua/2}{0} \right),
\]

and also the momentum per unit area associated with the wave. The kinetic energy per unit area of the wave motion is

\[
K = \rho \int_{-\infty}^{0} V(z)^2dz = \rho Uca/4.
\]

Thus the momentum in the direction of gravity wave propagation and the kinetic energy of gravity waves are related by the particle-like relation

\[
M = \frac{2K}{c},
\]

which holds for other kinds of waves, e.g., electromagnetic waves, that have genuine momentum.\(^{35}\) Thus, the Stokes drift turns out to be much more than a residual effect of switching from an Eulerian to a Lagrangian

\(^{35}\)Waves that exist on a physical medium need not have a genuine momentum of this sort, though they will have a momentum
coordinate system and indeed it is one of the most important means by which surface gravity waves interact with other scales of motion. It is notable that all of the information needed to calculate the Stokes drift was present in the Eulerian velocity field, Eq. (80). However, to reveal this important phenomenon, we had to carry out an analysis that was explicitly Lagrangian, i.e., that tracked parcels over a significant duration.

6 Aspects of Advection.

As we noted at the close of Section 3, the most distinctive feature of the Eulerian balance equations is the occurrence of advection terms. These terms represent the process (or at least the effect of) fluid flow, and contribute most of the physical and mathematical complexity of fluid mechanics. In this section we will begin to consider ways to set some bounds upon what advection alone can do in a fluid flow, and just as important, to understand what advection can not do. There are four topics in this section. Of these, the first, second and fourth are essential elements of fluid kinematics: Modes of an advection equation (6.1), Fluxes in flux, a point made famously in this context by M. E. McIntyre, ‘On the wave momentum myth’, *J. Fluid Mech.*, 106, 331-347, 1981. An excellent recent reference on the topic of wave momentum is by D. Rowland, ‘Comments on “What happens to energy and momentum when two oppositely-moving wave pulses overlap?”’, *Am. J. Phys.*, 72(11), 1425-1429, 2004. Surface gravity waves have momentum by virtue of the displaced free surface. Here’s a small problem for you: show that the Eulerian mean momentum over the water column is equal to the mass transport associated with Stokes drift by making the same kind of (bilinear) approximation for transport that we made for Stokes drift, i.e., assume that the wave velocity \( V \) under the displaced surface is given by Eq. (26) evaluated at \( z = 0 \), and compute the wave mean of \( \eta V \).


Stokes drift is a very robust phenomenon that can be produced and observed with simple means: fill a flat container with water to a depth of about 2-4 cm. A bath tub works well, but even a large cake pan will suffice. To make gravity waves use a cylinder having a diameter of roughly the water depth and a length that is about half the width of the tank. Oscillate the wave maker up and down with a frequency that makes gravity waves and observe the motion of more or less neutrally buoyant particles; some that float and others that sink to the bottom. You can easily vary three things, the amplitude of the waves, the depth of the water, and the width of the wave maker. Are the waves in your tank shallow water or deep water waves? Describe the mean flow (if any) set up by the oscillating wave maker, and how or whether it varies with the configuration of the tank and wave maker.

Suppose that the one-dimensional velocity in a progressive wave is given as \( U \cos(kx - \omega t) \). Calculate the Stokes drift approximation of the mean parcel motion in this wave, and compare the result to the numerical integration of the full trajectory equation, i.e., \( \frac{dx}{dt} = U \cos(kx(t) - \omega t) \). This requires a very small program or script and can be accomplished with a rather crude numerical method. For what range of wave steepness does the Stokes drift estimate give an accurate estimate of the mean flow? Why do parcels have a Stokes drift in this wave?, i.e., explain why parcels in this wave velocity field have a net motion. What happens at very large steepness? How does this compare with your observations from the bath tub? How does this compare with the Stokes drift of a deep water gravity wave?

We will not go through a comparable, lengthy discussion of the complementary transformation from Lagrangian velocity measurements to an Eulerian velocity field. But here are some things for you to think about. Suppose that we can make perfect Lagrangian measurements at arbitrarily fine temporal resolution. How would you use these data to construct the corresponding Eulerian velocity field? Now suppose, more realistically, that our Lagrangian measurements are averages over some finite time interval, say many wave periods in a case where surface gravity waves are present. How would you (or could you) construct the corresponding Eulerian velocity field from these data?
space (6.2) and The Cauchy-Stokes Theorem (6.4). The third topic, The method of characteristics (6.3), is perhaps a little less so, but is (almost) irresistible given the Lagrangian/Eulerian theme of this essay.

6.1 Modes of an advection equation

While the thermal energy balance equation is still fresh we will take the opportunity to note the possible balances among the terms in a two-dimensional (horizontal) version of the equation, (61),

$$\frac{\partial T}{\partial t} = \frac{\partial T}{\partial x} + V \cdot \nabla T = -\nabla \cdot \mathbf{Q},$$

(84)

where $V \cdot \nabla T = u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y}$ is the two-dimensional advection of internal energy (proportional to temperature), and $\mathbf{Q}$ is the heat flux in kinematic units, $\mathbf{Q} = \mathbf{Q} / \rho_0 C_p$ where $\rho_0$ is the nominal density and $C_p$ is the nominal heat capacity at constant pressure. The instantaneous thermal energy balance will generally involve all three terms (taking advection as one term). But it may also happen that at some times, or averaged over some time scales, the balance will be dominated by just two terms, which we will dub a ‘mode’ of the thermal energy balance equation. There are three modes inherent in a three term equation like this one, and as part of building a vocabulary for describing fluid dynamics it is useful to note each of the possibilities:

**Local balance:** The advective term $V \cdot \nabla T$ will vanish if the fluid is at rest, if the temperature is spatially uniform so that $\nabla T$ vanished, or if the fluid velocity is parallel to isolines of temperature. In that case the local time rate of change of temperature would reflect the source term, $\nabla \cdot \mathbf{Q}$, in what is called a local balance,

$$\frac{\partial T}{\partial t} = -\nabla \cdot \mathbf{Q},$$

where ‘local’ means at one place, ‘nonlocal’ meaning that spatial gradients are important (the next two modes). For example, on a summer day that includes strong solar radiation, the air at ground level will likely warm rapidly during the late morning, indicating a local, source-driven thermal energy balance. Much the same thing happens on a seasonal cycle, especially at middle and higher latitudes, when data are averaged over a period of a few months. A local balance of this kind will occur in a solid, e.g., the Earth’s land surface, and in many cases the dynamics of such a local balance are essentially linear, i.e., not inherently fluid mechanical since fluid motion need not be involved.

**Steady balance:** The term $\partial T/\partial t$ is the rate of change of temperature observed at a fixed position and a steady flow is one that has $\partial/\partial t = 0$ for all relevant properties throughout the domain. A steady flow may nevertheless be subject to significant external forcing in which the advection term balances the source,

$$V \cdot \nabla T = -\nabla \cdot \mathbf{Q}.$$

For example, if the source was positive or heating and yet we observed that there was little or no local time rate of change (i.e., that the temperature was steady) then we would infer that the advection term was acting to cool the observation site.
Such a steady balance may occur instantaneously, or more often when the thermal energy balance equation is time-averaged over a time scale that spans a full cycle of heating, say diurnal or annual. The temperature at a given site in the lower atmosphere is likely to go through a nearly closed annual cycle, so that the annual average of the local rate of change will nearly vanish. Nevertheless the local source term \( \nabla \cdot \mathbf{Q}_* \) may be significant when averaged over the same period. In that case we can infer that the advection term averaged over the annual cycle must also have been significant. For example, the ocean’s overturning circulation transports comparatively warm waters from the tropics to middle and higher latitudes where thermal energy is given up from the ocean to the atmosphere. The ocean at higher latitudes is thus cooled by heat loss to the atmosphere, \(-\nabla \cdot \mathbf{Q}_* < 0\), and warmed by advection of thermal energy from lower latitudes, \(\mathbf{V} \cdot \nabla T > 0\) so that a more or less closed annual cycle holds (aside from climate drift, of course!). The signs of the local source and the advective term are reversed at lower latitudes; the source term is positive due to an excess of solar radiation and the ocean’s overturning circulation acts to bring in cooler waters from higher latitudes. At a given location, the advection term may well be three-dimensional, including an important contribution from vertical motion.\(^{39}\) This advection or transport of thermal energy from lower to higher latitudes, which occurs in both the atmosphere and ocean, makes a very significant contribution to the moderation of Earth’s meridional temperature gradient.

Frozen field: Finally, we note that a third important mode inherent to Eq. (84) is that

\[
\frac{\partial T}{\partial t} + \mathbf{V} \cdot \nabla T = 0,
\]

while both terms are considerably larger than the source, \(\nabla \cdot \mathbf{Q}_*\). Said a little differently, the local rate of change of temperature may be due mainly to advection rather than to a local (heat) source term. This kind of balance is sometimes referred to as a ‘frozen field’, though the thing imagined to be frozen is the spatial structure of temperature that is embedded within the moving fluid (and not frozen in space). When this spatial structure is carried past fixed points by the fluid flow there is then a local rate of change of the property. For example, the passage of an atmospheric frontal boundary will often cause a rapid and significant change in the locally observed air temperature or humidity.

A comment on field measurements: The object of many (geophysical) field experiments is to observe the source term, here \(\mathbf{Q}_*\), and its relationship to the local flow environment, the topography, etc. When only Lagrangian measurements are available to define the heating rate following fluid parcels, then the estimation of the source term will probably be straightforward (Section 2.1). Observing the connection with the environment may be challenging, however, since the measurement array will be uncontrolled. When only Eulerian measurements are available, then the inference of a source term from measurements of the local heating rate must account for what are usually important effects of advection. The scope of the observations needed to do this is usually quite significant, but of the kind needed to define the environment, generally.\(^{40}\)

\(^{39}\)The vertical advection term is just \(w \partial T/\partial z\). In natural flows of the atmosphere and ocean the vertical velocity \(w\) is usually much, much smaller than the horizontal velocity, but then the vertical gradient of most properties is also much, much larger than the horizontal gradient. The vertical advection term is sufficiently different from the horizontal advection terms that it would be reasonable to treat it as a separate, fourth term in a three-dimensional thermal energy budget. We won’t do that, however, as three modes are quite sufficient.

\(^{40}\)Two questions for you: 1) This description of modes has been for the Eulerian balance. Go back and describe the Lagrangian
6 ASPECTS OF ADVECTION.

6.2 Fluxes in space

It is often useful to write the Eulerian budget equations in 'flux' form or 'conservation' form

\[ \frac{\partial b}{\partial t} + \nabla \cdot \mathbf{b} = 0 \]  

(85)

where the variable \( b \) is any intensive property of the fluid, e.g., mass per unit volume, \( \rho \), or momentum per unit volume, \( \rho \mathbf{V} \). Thus \( b \times Volume \) is said to be an extensive property, e.g., mass or momentum and for \( b \) we will say that the extensive quantity is \( B \). The flux \( \mathbf{b} \) could be due to transport by the moving fluid, e.g., \( \mathbf{b} = \mathbf{bV} \) obtains for any fluid property, \( b \), which is the reason that fluid flow is of first importance in so many applications, or molecular diffusion, \( \mathbf{b} = -K\nabla b \), if a Fourier diffusion law is appropriate. In either event, \( \mathbf{b} \) is a vector with dimensions \( b \text{ length time}^{-1} \) so that \( b \times area \text{ time}^{-1} \). For the moment it will be assumed that \( \mathbf{b} \) depends upon \( b \), so that \( b = 0 \) over some neighborhood implies that \( \mathbf{b} = 0 \) as well.

6.2.1 Global conservation laws

A conservation equation of the form Eq. (85) has the following property. We can assume without loss of generality that the variable \( b \) and the flux \( \mathbf{b} \) are vanishingly small as \( x, y \) go to infinity, i.e., that the flow is bounded in space. Denote the total amount of \( b \) by the volume integral

\[ B = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} b \, dx \, dy \, dz. \]

By integrating both sides of the conservation law from \( -\infty < x, y,z < +\infty \), and use of the (Gauss') divergence theorem on the right hand side we find that \( B \) is constant in time,

\[ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\partial b}{\partial t} \, dx \, dy \, dz = - \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \nabla \cdot \mathbf{b} \, dx \, dy \, dz \]

\[ \frac{dB}{dt} = \oint \mathbf{b} \cdot \mathbf{n} \, da = 0. \]  

(86)

In going from the second to the third line we have used that \( \mathbf{b} \) vanishes at infinity along with \( b \), and thus vanishes along the surface of the areal integral. \( B \) must be constant in time, even though the flux \( \mathbf{b} \) may act balance for each of these modes. Consider whether the temperature can or must vary with \( \xi_0 \). 2) The temperature \( T \) of our little thermal energy budget Eq. (84) could be the temperature of the atmosphere observed near ground level, i.e., your local climate. In that case \( Q \) would be due mainly to solar insolation and radiative cooling, and the advection term would be associated with the advection of differing air masses to your observation site. Over the next week or two, take notice of your local climate and how it varies on a diurnal to weekly basis. What causes the local temperature to change, advection or the local source? Assuming that the latter is mainly radiative, then it can be inferred, very roughly, from the diurnal variation due to radiative fluxes. To evaluate advection requires that you monitor the mesoscale temperature in your region and the local wind; the necessary data are available from good weather maps in the newspaper or better, from a weather forecasting center such as FNMOC. The angle between the wind and the gradient of temperature may be rather small even when horizontal advection is quite important, and so the inference of advection may be semi-quantitative, at best. We are not expecting precise quantitative estimates so much as a qualitative discussion of the two-dimensional thermal energy balance sorted upon time scale.
to redistribute $b$ within the domain so that at a given point $b$ could be highly time-dependent, as in the frozen field balance of Section 6.1. Nevertheless, the conservation law assures us that the total amount of $b$ will remain constant. The same thing would result if the flux $b$ vanished on the boundary of a domain, e.g., if the flux was due to fluid motion only and the domain boundary was a solid, impermeable surface.

The mass conservation equation Eq. (48) can be readily written in flux form by expanding the material derivative and collecting terms under the divergence operator,

$$\frac{D\rho}{Dt} + \rho \nabla \cdot V = \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho V) = 0, \tag{87}$$

or in Cartesian components,

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0,$$

so that the total mass is constant. We would expect exactly this kind of conservation property on purely physical grounds when $b$ is fluid density and $B$ is the net mass within the domain, there being no sources or sinks for mass in the classical physics that we have presumed. But now we see that the same integral conservation relation holds just as well for a few other important physical quantities, notably momentum and total energy, provided that the appropriate momentum and energy budgets can be written in the form of Eq. (85), with no other terms.

If we regard the density as variable, then we can not write the velocity of the Euler fluid in flux form directly, we have to use instead the momentum density, i.e., $\rho V$. By adding the mass conservation equation to the momentum equation and minor rearrangement (that you should be sure to verify) we can write that

$$\begin{align*}
\frac{\partial}{\partial t} \begin{pmatrix} \rho u \\ \rho v \\ \rho w \end{pmatrix} + \frac{\partial}{\partial x} \begin{pmatrix} \rho u^2 + P - \nu \frac{\partial U}{\partial x} \\ \rho uv - \nu \frac{\partial V}{\partial x} \\ \rho uw - \nu \frac{\partial W}{\partial x} \end{pmatrix} + \frac{\partial}{\partial y} \begin{pmatrix} \rho vu - \nu \frac{\partial U}{\partial y} \\ \rho v^2 + P - \nu \frac{\partial V}{\partial y} \\ \rho vw - \nu \frac{\partial W}{\partial y} \end{pmatrix} + \frac{\partial}{\partial z} \begin{pmatrix} \rho wu - \nu \frac{\partial U}{\partial z} \\ \rho vw - \nu \frac{\partial V}{\partial z} \\ \rho w^2 + P - \nu \frac{\partial W}{\partial z} + \Phi \end{pmatrix} &= 0 \tag{88}
\end{align*}$$

where the gravitational potential $\Phi = \rho gz$. \footnote{Most arbitrary quantities are not subject to a strict conservation law. For example, in a two particle collision, the kinetic energy will be conserved only in the special case that the collision is "elastic", so that no energy is lost to deformation, acoustic waves, etc. On the other hand, if the particles stick together after the collision, then the kinetic energy will decrease by an amount that depends upon the particular conditions of mass and initial velocity. Thus the kinetic energy and higher moments of the velocity are not conserved in most collisions or during mixing events in a fluid. Very often a conservation law will not obtain because of the presence some external source, e.g., gravity, that does not vanish with the fluid velocity and hence the global integral need not be conserved. In that case we should probably call the governing equation the momentum "balance" or "budget" rather than "conservation", though this distinction is often ignored. Two questions for you: Can you show the relationship between the RTT, and the conservation form of the differential balance? Discuss the case Eq. (88) in which a gravitational potential is present, and specifically, does the conservation property hold in that case?}
pressure stress tensors, $T$ and $P$ (Section 2.3). The momentum flux tensor

$$\mathbb{A} = \rho \begin{pmatrix} uu & uv & uw \\ vu & vv & vw \\ wu & wv & ww \end{pmatrix},$$

can be written as the direct vector product

$$\mathbb{A} = V V^T$$

(89)

where $V$ is a 3x1 column vector and the transpose $V^T$ is a 1x3 row vector; hence $\mathbb{A}$ is 3x3. The gravitational tensor

$$\mathbb{Y} = \rho g z \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$  

The tensor form of the Eulerian momentum balance is then

$$\frac{\partial (\rho V)}{\partial t} + \nabla \cdot (\mathbb{A} + \mathbb{P} - T + \mathbb{Y}) = 0,$$

(90)

where $\nabla$ is a three element row vector, $\nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$. Admittedly, there is little gain in this for the pressure and gravitational terms. But having written the momentum balance equation in tensor form we now can assert two very important properties: 1) The tensor equation (90) is valid in any (Cartesian) coordinate system, assuming that it is correct in one coordinate system, and 2) The components of the tensors transform under a rotation of the coordinate axes as

$$\mathbb{A}' = V'(V')^T = R \mathbb{A} R^T,$$

(91)

where $R$ is the rotation tensor.\(^{42}\)

### 6.2.2 Control volume budgets

If the volume and areal integrals of Eq. (87) are taken over a (fixed) portion of a domain, often called a control volume (Fig. 13), then another important interpretation of the RTT is that

$$\int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} \frac{\partial b}{\partial t} \ dx dy dz = - \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} \nabla \cdot b \ dx dy dz + \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} Q \ dx dy dz$$

$$\frac{\partial B}{\partial t} = - \oint b \cdot n \ da + \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} Q \ dx dy dz$$

(92)

\(^{42}\)We can readily infer this last result from the form of the momentum flux tensor, Eq. (89). If the velocity vector (components) transform under rotation of coordinate axes as $V' = RV$ (rotation matrices (tensors) are discussed in Price (2004)\(^{14}\) and references therein), and given that $(V')^T = (RV)^T = V^T R^T$, then by substitution into the middle term of Eq. (91) and using the associative property of matrix multiplication we verify the transformation rule for tensors. (A good review of tensor algebra is Ch. 2 of Kundu and Cohen (2001) noted in footnote 13.)
where the time derivative can be moved inside or outside the spatial integral over what are now presumed to be the fixed limits, \(x_1, x_2,\) etc., of a control volume. The total amount of \(b\) in this control volume, \(B,\) can thus change because of a flux across the surface of the control volume or due to a body source integrated over the control volume. This result is often utilized as the means for finding the differential budget equations for instances of unusual geometry and/or flow conditions. We will consider a simple example; suppose that we aim to keep account of the \(x\)—component of momentum within a control volume of \(x\)—length \(l,\) and width and height \(w\) and \(h,\) and suppose too that \(y\) and \(z\) variations of the fluxes can be ignored. The \(x\)—component of the flux of \(\rho u\) is then from Eq. (88),

\[
b = \rho u^2 + P - v(\partial^2 u / \partial x^2), \tag{93}
\]

and the integral momentum budget for this fixed control volume is

\[
lwh \frac{\partial \rho u}{\partial t} = wh((b(x = l) - b(x = 0)) \]

\[
= \frac{wh((-\rho u^2 - P + v \frac{\partial u}{\partial x}) |_{x=l} - (-\rho u^2 - P + v \frac{\partial u}{\partial x}) |_{x=0})}{lwh}. \tag{94}
\]

Dividing through by the volume, \(lwh,\) taking the limit that \(l\) becomes very small, application of mass conservation, and \textit{voila,} out pops the differential form of the momentum balance,

\[
\frac{\partial u}{\partial t} = -u \frac{\partial u}{\partial x} - \frac{1}{\rho} \frac{\partial P}{\partial x} + \frac{v \partial^2 u}{\rho \partial x^2}. \tag{95}
\]

The only slightly disagreeable part of this procedure is that if there had been an external forcing term, \(Q,\) then it would necessarily be applied at points in space, i.e., the control volume, rather than to specific fluid parcels, but we have seen that the RTT comes to the same thing.
The value of this kind of derivation is that it may be most natural to make assertions about the physical properties of a flow at the first stage of this procedure, Eq. (93), where the flux is prescribed. Here we have kept all three of the terms that contribute to momentum flux, in general, but very often this is not necessary or appropriate. In most flows the pressure gradient is of leading importance, and the question is how the advective and diffusive terms compare to one another. To make a rough estimate of these terms we have to make an estimate of the length scale, \( L \), over which the current component varies by about 100%. That is, we seek to write

\[
\frac{\partial u}{\partial x} = O\left(\frac{U}{L}\right) \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = O\left(\frac{U}{L^2}\right)
\]

where by the big \( O \) notation we mean the order of magnitude without regard to the sign. For example, if \( u = \sin(kx) \), then we would know quite precisely that \( \partial u/\partial x = Ak\cos(kx) \) is \( O(Ak) \) and so \( U = A \) and \( L = k^{-1} = \lambda/2\pi \). If our aim was a rough estimate, then we might ignore the factor \( 2\pi \), and say that \( L \) is \( O(\lambda) \). Of course, when factors of \( 2\pi \) or 2 cascade, as they do for the second derivative, then this could eventually lead to trouble. But with that in mind, let’s proceed to estimate the ratio of the advective term to the diffusion term as

\[
\frac{u}{v \frac{\partial^2 u}{\partial x^2}} = O\left(\frac{u^2/L}{\nu u/L^2}\right) = O\left(\frac{UL}{\nu}\right) = O(Re).
\]

where \( Re \) is the Reynolds number. The Reynolds number is nondimensional, and serves here as a measure of the ratio of the advection term to the diffusion term. For the wave-like motion shown in the cover graphic, we can estimate that \( U \approx 0.1 \text{ m s}^{-1} \), and that \( L \approx 100 \text{ km} \), very roughly. Using the known viscosity of water, \( \nu = 0.001 \text{ m}^2 \text{ s}^{-1} \), then \( Re \approx 10^7 \). Almost no matter how crude the estimate of \( L \), it is unmistakable that the viscous diffusion term is much, much smaller than the advection term. After making a few such estimates, you will be entirely justified in dropping (without even mentioning) the viscous, diffusive contribution to the momentum flux in the case that the momentum budget is going to be applied to such large-scale motions of the atmosphere or ocean.\(^{43}\)

### 6.3 The method of characteristics

Advection transports, and in the simplest case, translates, fluid fields. To begin to examine this, consider the case of a constant (spatially and temporally uniform) velocity, \( u \geq 0 \), and some scalar property of the fluid, say \( T \), in \( \mathbb{R}^1 \). Assuming that there is no body force for \( T \), then the Eulerian (field) equation for the evolution of \( T \) reads

\[
\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} = 0.
\]

\(^{43}\)You might well wonder if the viscosity term is ever important. Emphatically yes, but it is of leading importance only for scales of motion that are very, very small compared to most oceanic or atmospheric motions. Recall the three-dimensional flow in a tea cup; the smallest scales of motion are vortices having \( L = 0.01 \text{ m} \) and a typical speed \( U = 0.05 \text{ m s}^{-1} \), and hence \( Re \approx 1/2 \). You may have noticed that these small scales of motion are damped rather quickly. Can this be attributed to viscosity? The time scale required for diffusion to propagate a signal over a distance \( L \) is roughly \( t_{diff} = L^2/\nu \), deduced from dimensional analysis or from a solution of the diffusion equation (Price, 2003), while the rotation time for a vortical motion is \( t_{adv} = L/U \). The ratio is once again the Reynolds number, and diffusion of momentum, i.e., viscosity, evidently is sufficient to damp the smallest vortical motions in a time that is the order of several rotations.
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In Lagrangian (material) terms this states that $T$ is constant in time on each parcel, though $T$ could very well be a different constant on each parcel. If we could specify $T$ for each parcel at some point along its trajectory, and if in addition we could solve for the trajectories of all parcels, we would have a complete solution. Of course, solving for the trajectories is, in general, a formidable task, but there are idealized flows that are both easily solved and illuminating.

For the problem at hand we will assume that $T(x)$ is known along the line $(x, t = 0)$,

$$T(x, t = 0) = H(x) = B \exp(-x^2/2W^2),$$  

where $H(x)$ is a Gaussian hump of amplitude $B = 1$ and width $W = 1$. Advection by the uniform velocity $U$ will act to shift each parcel to the right at the rate $U$ and according to our governing equation (97), each parcel retains its initial $T$. Since all parcels move with the same speed, the initial profile $T(x)$ is unchanged in width or amplitude. Advection by a uniform velocity amounts to a simple translation of the $T$ field, Fig. (14).

The field $T(x, t)$ is constant in the $x, t$ plane along lines given by $x - Ut = constant$ that are said to be the characteristics of the PDE, Eq. (97). If we set

$$T(x, t) = f(x - Ut),$$  

where $f(x, t)$ is any differentiable function, this $T(x, t)$ satisfies the governing PDE, Eq. (97). If we further set $f = H$ we can satisfy the initial condition, and thus $T(x, t) = H(x - Ut)$ is the solution to the problem posed (as you should verify by substitution into the governing equation).

This is a nearly trivial example, of course, but it illustrates the basis of a very powerful solution technique for first order PDEs, including some nonlinear forms, called the method of characteristics.\textsuperscript{\textit{44}}

\textsuperscript{\textit{44}}The method of characteristics is described well in many textbooks on partial differential equation. An excellent text that emphasizes numerical solution methods built upon the idea of characteristics is by R. J. LeVeque, \textit{Numerical Methods for Conservation Laws} (Birkhauser Verlag, Basel, 1992). A clear and concise online source is at http://www.scottscarr.org/shock/shick.html
appeal of this method for our present purpose is that it solves the advection equation by recognizing that
advection alone does not change the scalar $T$ of a given parcel (though, of course, advection can change $T$ at
a fixed point in space). The solution strategy is to convert the governing PDE into a set of (usually coupled)
ODEs; one of the ODEs will describe the rate of change of the property $T$ along the trajectory of a parcel
(zero if there is no forcing, as in Eq. (97)), and the other ODEs will serve to define the trajectory of each
parcel. When combined, these will give the full solution.

To start we will seek the path in $(x, t)$ along which the governing equation reduces to an ordinary
differential. To do this we will seek the parametric form of a curve $(x(s), t(s))$ where $s$ is distance along the
curve. Assuming that there is such a curve, then we can write $T(x(s), t(s))$, and the (directional) derivative
along $s$ is just

$$\frac{dT}{ds} = \frac{\partial T}{\partial t} \frac{dt}{ds} + \frac{\partial T}{\partial x} \frac{dx}{ds}.$$ 

(100)

Notice that we can write the differentials of $x$ and $t$ with respect to $s$ as ordinary differentials. Comparing
this with Eq. (97), it appears that we can set

$$\frac{dT}{ds} = 0, \text{ and thus } T = \text{constant}$$ 

(101)

along this path, provided that the parametric representation of the path satisfies

$$\frac{dt}{ds} = 1$$ 

(102)

and

$$\frac{dx}{ds} = U.$$ 

(103)

These latter two ODEs define the family of lines along which Eq. (101) holds. The first of these can be
immediately integrated to yield

$$t = s,$$ 

(104)

where the integration constant can be set to zero, and the second condition integrates to

$$x = Us + b,$$ 

(105)

where $b$ is the value of $x$ when $t = 0$. Using Eq. (104) this last can be written

$$x = U t + b.$$ 

(106)

Thus the family of lines along which $T =$constant are given parametrically by Eqs. (104) and (105) or by
Eq. (106). These lines are called the characteristics of the governing PDE, Eq. (97). In this extremely
simple problem all of the characteristics have the same slope, $dt/dx = 1/U$, Fig. (14), since we have
presumed that $U$ is constant.

Along each characteristic line $T$ remains constant, according to Eq. (101), and to find what constant
value holds on a given characteristic we need initial data on each characteristic; here we have Eq. (98), or

$$T(x, t) = H(b).$$
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We can then eliminate \( b \) using Eqs. (104) and (105), and find the explicit solution,

\[
T(x, t) = U_0 e^{-(x - Ut)^2 / 2W^2}.
\]

(107)

Thus the field \( T(x, t) \) is translated to the right at the constant rate \( U \), as we had surmised already from very basic considerations.

The method of characteristics can be applied to real advantage to many first order PDEs, including some that are not linear. For example, consider a problem in which the field being advected is the current \( U \) itself,

\[
\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0,
\]

(108)

so that \( u \) is now the dependent variable. The initial condition is again presumed to be a Gaussian hump, Fig. (15),

\[
u(x, t = 0) = U_0 e^{-(x^2 / 2W^2)}.
\]

(109)

This governing equation, often called the inviscid Burgers’ equation, is not linear because the advection term is the product of two unknowns and so the wide array of methods that are available to solve linear PDEs will not be applicable. It is linear in the partial derivatives, however, and the method of characteristics is well-suited for such quasi-linear problems.\(^{45}\) We again seek solutions of

\[
\frac{du}{ds} = 0
\]

and as above find that the characteristics \( x(s) \) and \( t(s) \) are (straight) lines, Figs. (15) and (16). A significant difference is that the slope of the characteristic lines now varies from characteristics to characteristic since \( u \) varies with \( x \). Far from the origin the characteristic lines are nearly vertical in the \( x, t \) plane (large slope corresponds to small \( u \)) while close to the origin the characteristics have a minimum slope and thus a maximum \( dx/dt \) and \( 1/u \). The solution for this problem can be written,

\[
u(x, t) = U_0 e^{-(x - u(x, t)t)^2 / 2W^2}.
\]

(110)

which is not separable into an explicit solution (in which \( u \) is on the left side and the right side is a function of \( x \) and \( t \)). It can be readily graphed, however, Fig. (15), and interpretation of the solution is made clear by the characteristics, Fig. (16). The most rapidly moving part of the hump that begins near the origin starts to overtake the slower moving part that starts at larger \( x \); after some time \( t_c \), some characteristics will intersect, and \( u \) at that point will there have an infinite derivative; if the method of characteristics is continued, the solution for \( u \) will then appear to be triple-valued. In the case that \( u \) is the current speed, a multi-valued solution makes no physical sense and has to be rejected.\(^{46}\) It isn’t that the method of characteristics has failed, but rather that the governing equation (108) has omitted some physical process(es), e.g., diffusion or}

\(^{45}\)It should be noted that this is not a complete model of any fluid flow in that we have not considered the conservation of volume (or mass) nor the possibility of a pressure gradient. These will be considered in Part II in the context of acoustic and shallow water waves.

\(^{46}\)To calculate when characteristics will first cross, consider the following problem, the governing equation is the inviscid Burgers’ equation, and the initial data is piece-wise linear: \( u = U_0 \) for \(-\infty < x \leq 0\); \( u = U_0 - x/L \) for \( 0 < x \leq L \) and \( u = 0 \) for \( x > L \). Assume that \( U_0 > 0 \). Sketch the characteristics and the solution at several times, and show that the characteristics starting
viscosity, that will become important when the derivative $\partial / \partial x$ becomes very large. Even with diffusion present, the derivative may, nevertheless, become very large, and the flow said to form a shock wave, across which the current speed is nearly discontinuous. The conservation of momentum holds regardless of the details of the field, and the subsequent motion of a shock wave can be determined using fundamental principles.

The method of characteristics can be readily extended to higher dimensions and to problems in which there is some forcing. Suppose that we have a model

$$\frac{\partial T}{\partial t} + u(x, y) \frac{\partial T}{\partial x} + v(x, y) \frac{\partial T}{\partial y} = Q(x, y).$$

(111)

The governing equation can be reduced to an ODE:

$$\frac{dT}{ds} = Q$$

(112)

provided that

$$\frac{dt}{ds} = 1,$$

from $x = 0$ and $x = L$ will cross at $t_c = L/U_0$. In the limit that $L$ is small this can be written $t_c = -1/(\partial U_0/\partial x)$, and for a continuous initial $U_0(x)$ it is plausible that the first crossing will be due to the largest (negative) value of $(\partial U_0/\partial x)$. For the Gaussian of Eq. (109) the crossing is expected at $t_c = (W/U_0) \exp(-1/2)$. Thus, even a small amplitude current pulse will steepen into a shock given enough time, and the practical question is whether other effects — dispersion, spreading and dissipation — will reduce the amplitude sufficiently to avoid a shock. Where in space does the first crossing occur? What would happen in the piece-wise linear case if $U_0 < 0$?
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Figure 16: The characteristics of the flow shown in Fig. (15). The short dashed lines denote $x = 2W$ and $t = \frac{W}{U_0} exp(1/2)$ where characteristics are expected to first cross in time. Since only a few characteristic lines are shown here, the first crossing that actually occurs in this figure is at a slightly later time, and hence is slightly displaced as well.

and

\[
\frac{dx}{ds} = u \quad \text{and} \quad \frac{dy}{ds} = v. \tag{113}
\]

Notice that Eq. (113) is exactly Eq. (19) for streamlines that we considered in Section 4.3. Thus streamlines and characteristics are one and the same when the velocity is steady and the governing PDE is the advection equation. In more realistic fluid models that include acoustic or gravity waves, the characteristic lines are lines along which certain properties may be constant, but again are not, in general, particle paths (more on this in Part II).\(^{47}\)

6.4 Advection of a finite parcel; the Cauchy-Stokes Theorem

You have probably noticed that advection does much more than simply translate parcels from place to place; the flow in a teacup will usually also act to draw out parcels into long, thin streaks (allowing that they have finite size) and will also change their orientation. This second aspect of advection, that it may change the shape and orientation of fluid parcels, is an important part of kinematics that we first considered in Section 1.1, and take up here in some detail.

To pose a definite problem, we will exploit the last result of Section 6.2 to calculate the motion of a parcel (identified with a colored tracer) that is embedded in a steady, clockwise rotating, vortical (circular) flow; either an irrotational vortex, Fig. (17, left), discussed in Section 5.1 and defined by Eq. (72), or a solid

\(^{47}\)Model PDE systems in which the fields are propagated at a definite, finite rate are said to be hyperbolic; in the advective equation (111) the speed is simply the fluid velocity. The elementary wave equation is also hyperbolic, since fields are propagated at the wave phase speed. A system that includes diffusion is said to be parabolic; at a given point the field will be influenced by the entire domain at all previous times. The identification of a model system as hyperbolic or parabolic is a key step in the design and implementation of efficient numerical schemes (e.g., LeVeque\(^{44}\)).
body rotation, Fig. (17, right) in which the azimuthal speed increases with radius as

\[ V = (U_r, U_\theta) = (0, \Omega r), \]  

(114)

where \( \Omega \) is the uniform rotation rate. It is assumed that \( C < 0 \) and \( \Omega < 0 \) so that both vortical flows are clockwise. These are idealizations, of course, and yet with a little effort (imagination?) something akin to both kinds of vortices can be observed in the flow in a teacup: more or less irrotational vortices are observed to spill off the edges of a spoon that is pushed through the fluid, and at longer times the azimuthal motion that fills the teacup will often approximate a solid body rotation (except very near the edges).  

The first thing to note is that the parcels are transported clockwise with the clockwise flow in either vortex; in these flows there is nothing quite as exciting as Stokes drift (Section 5.2) that can make the Lagrangian mean flow (i.e., what we see as the displacement of the parcel) qualitatively different from the Eulerian mean flow (what you would expect given the field of vector velocity). Aside from that, the effects of advection are remarkably different — the irrotational vortex produces a very strong deformation of the parcel, while the solid body rotation leaves the shape of the parcel unchanged. The irrotational vortex leaves the (average) orientation of the parcel unchanged, though this is impossible to verify in Fig. (17) given the very large deformation, while the solid body rotation changes the orientation at the rate \( \Omega \) that characterizes the rotation rate of the vortex. The area of the parcel is unchanged in either case.

These changes in the orientation and shape of a fluid parcel are caused solely by advection and are thus a consequence of the velocity field. Our goal in this section is to find out what specific properties of the velocity field are relevant. Once again we are asking for what amounts to Lagrangian properties — the size, shape, etc. of a fluid parcel — in terms of the Eulerian velocity field, \( V(x, y, z, t) \). To make the analysis tractable we are going to consider flows that are two-dimensional and we will follow the parcel only for short times (unlike the examples of Fig. 17) so that the velocity field can be assumed steady. Given these restrictions, the Eulerian velocity field around a given point, \( x_0, y_0 \) can be calculated with sufficient accuracy by the first terms of a Taylor expansion (as in Section 5.1 and repeated here),

\[ V(x, y) = V(x_0, y_0) + \mathbb{G} \cdot \delta X + \text{HOT}, \]  

(115)

where \( \delta X \) is a small displacement and

\[ \mathbb{G} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \]

is the velocity gradient tensor, which represents all that we presume to know regarding the velocity field. These restrictions to small displacement and steady flow may seem severe, but in the end we come to results that can be applied to the differential (Eulerian) conservation equations, which is just what we need.

\[ \]  

\[ \]  

\[ 48 \text{The irrotational vortex has a singularity in } U_\theta \text{ at } r = 0, \text{ while in a solid body rotation } U_\theta \text{ grows linearly and without bound with } r. \text{ A hybrid made from these two idealizations — solid body rotation near the center of a vortex, Eq. (114), matched to an irrotational profile } U_\theta(r) \text{ from Eq. (72), that continues on for larger } r — \text{ avoids both problems and can make a convenient, useful approximation to a real vortex, e.g., a hurricane. This kind of hybrid is called a Rankine vortex.} \]
Figure 17: (a) A small patch of colored tracer, or parcel, has been set into a steady, clockwise rotating irrotational vortex and advected through most of one revolution. The largest velocity vectors near the center of the vortex were not plotted. The parcel was square, initially (at 12 o’clock) and then rather severely deformed by this flow. Nevertheless, the area of the parcel was conserved, as was its average orientation (though the orientation is obscured by the very large deformation). (b) In this experiment the vortex flow was a solid body rotation. As the name implies, this motion could just as well be that of a solid, rotating object. The orientation of the parcel changes in time, but the area and the shape are conserved, i.e., there is no deformation.

The velocity gradient tensor is the center of attention for now, and we’d like to know what it looks like. We can not make a diagram of a tensor per se, but we can show what \( \mathcal{G} \) does when it operates on a displacement vector, and that is what counts. We will illustrate this with a very simple shear flow

\[
V(x, y) = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} U_0 + \Delta y \\ 0 \end{pmatrix},
\]

where the spatial mean flow is \( U_0 = 0.5 \) and the shear is \( \partial u/\partial y = \Delta = 0.7 \) and constant. A fluid parcel embedded in this flow evolves as shown in Fig. (18). The velocity gradient tensor evaluated at any point in this flow is just

\[
\mathcal{G} = \begin{pmatrix} 0 & \Delta \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0.7 \\ 0 & 0 \end{pmatrix}.
\]

To see what \( \mathcal{G} \) does, we can multiply \( \mathcal{G} \) into a set of unit vectors, \( e \), that span 360 degrees in direction, and plot the resulting velocity difference, \( \delta V = \mathcal{G}e \), at the end of each unit vector, Fig. (19a). The velocity difference plotted in this way looks a lot like the velocity field itself since the shear is presumed spatially uniform. However, these diagrams show properties of the velocity field at the specific point where \( \mathcal{G} \) has been evaluated, and are not a map of the velocity field per se.
Figure 18: (a) A parcel has been set into the shear flow defined by Eq. (116) and advected for a time interval, $\delta t$. The sides on the lower and left edges of the parcel were initially orthogonal, and of length $L_x$ and $L_y$. After a time interval $\delta t$, the upper left corner has been displaced a distance $d = \delta t L_y \partial u / \partial y$ with respect to the lower left corner and so the left edge of the parcel has rotated clockwise through an angle $\phi \approx \tan \phi = -d / L_y$. The angle $\phi$ thus changes at the rate $d\phi / dt \approx \phi / \delta t = -\partial u / \partial y$, as long as $\phi$ is small. (b) The same flow and the same parcel, but compared to the example at left, the initial orientation of the parcel was rotated by 45 deg. The original lower left and lower right sides are shown at $t + \delta t$ as the dotted lines. In this case the angle defined by the lower left and lower right edges remains 90 deg, while the length of these sides is compressed or stretched. Evidently this particular shear flow has both a shear strain rate, emphasized at left, and a linear strain rate, emphasized at right (strain rate will be defined in the text below). An orthogonal axes pair, e.g., the lower and left sides of this parcel, will thus sample one or the other (or a little of both) of these strain rates depending upon their orientation with respect to the flow.
Figure 19: (a) The velocity gradient tensor $G$, Eq. (117), has been multiplied into a sequence of unit (displacement) vectors with varying directions (the dotted lines) and the resulting velocity plotted at the end of the unit vectors. This is a useful way to show what the velocity gradient tensor does, but keep in mind that all of this should be envisioned to hold at a single point. (b) The eigenvectors of the strain rate tensor $E$ are shown as the vectors, and the linear strain rate + 2 is the peanut-shaped ellipse plotted in a radial coordinate system. The maximum and minimum values of the linear strain rate are aligned with the eigenvectors and the value of the maximum and minimum linear strain rate is equal to the eigenvalues. (c) The rotation rate tensor $R$ has been multiplied into a sequence of unit vectors. (d) The strain rate tensor $E$ multiplied into the unit vectors. Notice how the sum of the velocity vectors in c) and d) compares with the velocity vectors of a), and notice too how $e \cdot \delta V$ compares with the linear strain rate shown in the upper right panel.
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6.4.1 The rotation rate tensor

Consider a spatial variation of the velocity that could cause the sides of the parcel to change orientation: in the example of Eq. (116) and Fig. (18a), the $u$ component of velocity increases with increasing $y$, and this causes the left and right sides of the parcel to rotate clockwise. If we denote the angle of the left side of the parcel with respect to the $y$-axis by $\phi$, then by simple geometry we can see that

$$\frac{d\phi}{dt} = -\frac{\partial u}{\partial y}.$$  

Similarly, if we denote the angle between the lower side of the parcel and the $x$-axis by $\nu$, then the lower side of the parcel would rotate counter-clockwise if the $y$-component of velocity increased with $x$, i.e.,

$$\frac{d\nu}{dt} = \frac{\partial v}{\partial x}$$

(the angle $\nu = 0$ in this figure, and so you should make a sketch to verify this, keeping in mind that the angle $\nu$ can be assumed to be very small). The angles $\nu$ and $\phi$ may change independently. A sensible measure of the average rotation rate of the parcel, $\bar{\omega}$, also called the physical rotation rate, is the average of these angular rates,

$$\bar{\omega} = \frac{1}{2} \left( \frac{d\nu}{dt} + \frac{d\phi}{dt} \right) = \frac{1}{2} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right).$$  \hspace{1cm} (118)

A small, rigid orthogonal vane embedded in the flow would rotate at this rate (assuming that the force on the vanes is linear in the velocity). For many purposes it is convenient to use a measure of the rotation rate called the vorticity, $\chi = 2\bar{\omega}$, which is just twice the physical rotation rate,

$$\chi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \nabla \times V$$

The vorticity is invariant to the orientation of the parcel with respect to the flow, and hence is invariant to a rotation of the coordinate system. In $R^2$ vorticity is effectively a scalar, i.e., a single number; in $R^3$ the vorticity $\nabla \times V$ is a vector with three components. The vorticity of a fluid is analogous in many respects to the angular momentum of a rotating solid and because it follows a particularly simple conservation law (no pressure effects!) it often makes an invaluable diagnostic quantity.

6.4.2 The strain rate tensor

In the case that the two angles $\nu$ and $\phi$ change at different rates, then our parcel will necessarily change shape or deform. One of several plausible measures of the shape of the parcel is the angle made by the lower and left sides, $\Gamma$, and evidently $\Gamma = \pi/2 + \phi - \nu$, Fig. (18a). If $\phi$ and $\nu$ change by the same amount, then $\Gamma = \text{constant}$, and the parcel will simply rotate without deforming; this is what we could call a solid body rotation, considered just above. But if the angles change at a different rate, then the shape of the parcel will necessarily change, and

$$\frac{d\Gamma}{dt} = \frac{d\nu}{dt} - \frac{d\phi}{dt} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}.$$  \hspace{1cm} (120)
If the angle $\phi$ alone changes while the lengths of the parcel sides remain constant, then this kind of time-varying deformation is called a shear strain (or deformation) rate. This is the case in Fig. (18a) when the angle $\phi$ is very small, i.e., for very small times after the parcel is released in the flow. This particular shear strain rate is in the plane parallel to the x-axis (and recall the shear deformation noted in Section 1.1; in that case the deformation was due partly to the no-slip lower boundary condition).

But suppose that we rotate the parcel by 45 degrees before we release it into the same shear flow. The result of advection then appears to be quite different if we continue to emphasize an orthogonal axes pair (Fig. 18b); the sides of the parcel remain orthogonal, but now the lengths of the sides are compressed (lower left side, $d(L_{x'})/dt < 0$) or stretched (lower right side, $d(L_{y'})/dt > 0$). A strain rate that causes a change in the length of a material line is termed a linear strain rate, e.g., in the x'-direction or y'-direction, rewriting Eq. (41),

$$\frac{dL_{x'}}{dt} = L_{x'} \frac{\partial u}{\partial x'} \quad \text{and} \quad \frac{dL_{y'}}{dt} = L_{y'} \frac{\partial v}{\partial y'}.$$  

(121)

Divergence, another very important quantity that we have encountered already in Section 6.2, is the sum of the linear strain rates measured in any two orthogonal directions and gives the normalized rate of change of the area of the parcel, $A = L_x L_y$, or rewriting Eq. (42),

$$\frac{1}{dA/dt} \left(\frac{dA}{dt}\right) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \nabla \cdot V.$$

Unlike the vorticity and the divergence, the strain rate can not be written as a vector operator on the velocity, and from Fig. (18) it seems that the strain rate that we diagnose with a given orthogonal axes pair (the sides of the parcel), is entirely dependent upon the orientation of these axes with respect to the flow. It is highly unlikely that a quantity that depends entirely upon the orientation of the coordinate system can have any fundamental role, and this in turn implies there is more to say about the strain rate than Eqs. (120) and (121) taken separately.\textsuperscript{49}

The strain rate may change the shape and the size (area) of a parcel, while rotation alone can not. This suggests that there is something fundamentally different in these quantities, and that it may be useful to separate the rotational part of the velocity gradient tensor from all the rest. This turns out be straightforward because the rotation is associated with the anti-symmetric part of the velocity gradient tensor, and any tensor can be factored into symmetric and anti-symmetric component tensors by the following simple procedure. Let $G'$ be the transpose of $G$,

$$G' = \begin{pmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}
\end{pmatrix}.$$

We can always subtract and add $G'$ from $G$

$$G = \frac{1}{2} (G - G') + \frac{1}{2} (G + G').$$  

(122)

\textsuperscript{49}It was noted above that the rotation rate defined by Eqs. (118) or (119) is independent of the orientation of the axes with respect to the flow. Can you verify this (semi-quantitatively) from Fig. (18)?
and thereby decompose $G$ into two new tensors

$$
\mathbb{R} = \frac{1}{2} \begin{pmatrix}
0 & \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \\
\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} & 0
\end{pmatrix}
$$

and

$$
\mathbb{E} = \begin{pmatrix}
\frac{\partial u}{\partial x} & \frac{1}{2}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \\
\frac{1}{2}(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}) & \frac{\partial v}{\partial y}
\end{pmatrix}
$$

$\mathbb{R}$ is called the rotation rate tensor and is anti-symmetric, $R_{12} = -R_{21}$ (this is a new and distinct use of the symbol $R$ compared with that of Section 6.2.1); $\mathbb{E}$ is called the strain rate tensor and is symmetric, $E_{12} = E_{21}$. In $R^2$, $\mathbb{R}$ has only one unique component that is proportional to the rotation rate $\omega$. When $\mathbb{R}$ is multiplied onto a set of unit vectors the result is a velocity difference $\delta V = \mathbb{R}e$ that is normal to the unit vector and has the same amplitude (same speed) for all directions of the unit vectors. The rotation associated with $\mathbb{R}e$ is apparent, Fig. (19, lower left), and the magnitude is just $\omega = -\frac{1}{2} \frac{\partial u}{\partial y} = -0.35$; the vorticity of this shear flow is then $2\omega = -0.7$.

**Eigenvectors of the strain rate tensor.** The strain rate tensor $\mathbb{E}$ has three independent components (in general) and is a little more involved. The resulting velocity difference, $\delta V = \mathbb{E}e$, varies in direction and amplitude depending upon the direction of the unit vector (though in the specific case shown in Fig. (19d) the amplitude happens to be constant). The linear strain rate in a given direction is given by the component of $\delta V$ that is parallel or anti-parallel to the unit vector in that direction. There are two special directions in which the linear strain rate is either a minimum or a maximum. Given the specific $G$ of Eq. (117), the minimum linear strain rate is -0.35 and is found when the unit vector makes an angle of 135 (or 315) degrees with respect to the $x$ axis (Fig. 19b); the maximum linear strain rate is 0.35 and is at 45 (or 225) degrees.

Thus a parcel will be compressed along a line that is oriented 135 degrees (with respect to the $x$ axis) and will be stretched along a line normal to this, 45 degrees. Notice too that when the unit vector is pointing in these special directions the velocity difference and the unit vector are either anti-parallel or parallel, and the relationship among $\mathbb{E}$, $e$, and $\delta V$ may be written

$$
\mathbb{E}e = \delta V = \lambda e,
$$

where $\lambda$ is a real number. These directions are thus the directions of the eigenvectors of the symmetric tensor $\mathbb{E}$, and the amplitude of the linear strain rate in those directions is given by the corresponding eigenvalues, $\lambda = -0.35$ and $+0.35$ for this particular $\mathbb{E}$.\(^{50}\) The divergence, $\nabla \cdot V = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$, is the sum of the eigenvalues, and also the trace of $\mathbb{E}$; in the specific case of the shear flow Eq.(116), the divergence happens to be zero though the linear strain rate in most directions is not. Like the rotation rate, the divergence is invariant to rotation of the coordinate system.

---

\(^{50}\)Eigenvectors and eigenvalues are a central theme of linear algebra and will not be reviewed here. Excellent references are by J. Pettofrezzo *Matrices and Transformations* (Dover Pub., New York, 1966) and for undergraduate-level applied mathematics generally, M. L. Boas, *Mathematical Methods in the Physical Sciences, 2nd edition* (John Wiley and Sons, 1983). If you have access to Matlab, a search on 'eigenvector' will return several useful, concise tutorials.
Stirring and mixing. There can be a significant strain rate even when the divergence vanishes, as it does in the case of the irrotational vortex Fig. (17.a), the plane shear of Fig. (18.a) or to an excellent approximation, the flow in a tea cup. From the ‘conservation form’ of the Eulerian budget equations (Section 6.2) we can be sure that the total amount of a tracer is thus unaffected by advection and strain rate, and the same holds also for the mean square of the tracer (when divergence vanishes). What has changed, however, is that the mean square of the gradient of the tracer field, $\langle \nabla T^2 \rangle d\text{Area}$, with $T$ the tracer density, is likely to be increased dramatically by a sustained or a random strain rate. Molecular diffusion acting upon the greatly increased surface area of the long, thin streaks produced by a sustained or random strain rate can then act much more effectively to mix the tracer with the surrounding fluid, and so produce a homogeneous equilibrium more rapidly than would occur in the absence of straining. At this level of detail, there is a very significant difference between molecular diffusion, which acts to reduce tracer gradients, and a strain rate, often called ‘stirring’ when it is random, which acts to increase mean square gradients.

6.4.3 The Cauchy-Stokes Theorem

In the discussion above we have sketched out The Cauchy-Stokes Decomposition Theorem:

An instantaneous fluid motion may, at each point, be resolved into three components:

1) a translation — the velocity $V(x_0, y_0)$ of Eq. (115),

2) a rigid body rotation — the amplitude of which is given by the off- diagonal elements of the rotation rate tensor, $\mathbf{R}$, and,

3) a linear strain rate along two mutually perpendicular directions — the directions and the amplitudes of the maximum and minimum linear strain rate are given by the eigenvectors and eigenvalues of the strain rate tensor, $\mathbf{E}$.

In most real fluid flows the rotation rate and the strain rate will vary spatially and in time, right along with the velocity itself, and each of them have important roles in Eulerian theories that seek to predict the evolution of fluid flows: the strain rate is proportional to the rate at which adjacent fluid parcels are slide past one another. Since the solid body-rotational part of the velocity gradient tensor does not contribute to linear or shear strain rate, it may as well be subtracted from the velocity gradient tensor before computing the viscous stress (Section 2.2.3). Thus the viscous stress tensor is often written $\mathbf{T} = \nu\mathbf{E}$, which shows that the stress tensor is symmetric. We will see these tensors and these concepts again.51

51So soon? 1) Go back and take another look at the (stirred) fluid flow in a teacup. Do you see evidence of divergence, rotation or strain rate? 2) Compute the velocity gradient tensor, and the associated divergence, rotation, and strain rates for the case of an irrotational vortex, and for the case of a solid body rotation. You may compute the derivatives either analytically or numerically, and use Matlab to calculate the eigenvalues and vectors. Choose several points at which to do the calculation and interpret your results in conjunction with Fig. (17). 3) The Cauchy-Stokes Theorem is useful as a systematic characterization and interpretation of the velocity expansion formula, Eq. (115), and especially of the velocity gradient tensor. A characterization of the strain rate tensor by its eigenvectors/values is surely the most natural, but suppose that we have a special interest in the shear strain rate, or, perhaps we just want to be perverse — can you state an equivalent theorem in terms of the shear strain rate? 4) Familiarity with tensors sometimes breeds a certain affection for them, and we decide to make yet another one: move the trace (the divergence) of the strain rate tensor
Figure 20: Maps of SOFAR float trajectories at weekly intervals. The region is the central Sagasso Sea, about 700 km west-southwest of Bermuda, and these floats were at a depth of 1300 m. A speed scale is in the upper left panel. The dashed contour lines in the background are lines of constant $f/H$ in units of percent of $f$. The center of this figure just happened to be the location where the relative vorticity of the motion was approximately zero, and the zero of $f/H$ is assigned there. There are two obvious kinds of motion evident here, the dominant motion is a northeast to southwest oscillation, the potential vorticity aspects of which are described in the main text and also a small, anticyclonic eddy centered on 31 N 69 W (on 16 May) that is not dealt with here. The wave-like aspects of this oscillation are described by Price and Rossby (1982).

6.4.4 Rotation and divergence of a geophysical flow; the potential vorticity

The ocean current observed by the cluster of SOFAR floats shown on the cover image and in Fig. (20) may be analyzed fruitfully by a macro-scale application of the Cauchy-Stokes Theorem. The east and north gradients of the east and north velocity components were estimated by fitting a 2-dimensional plane to each velocity component at daily intervals.\footnote{At a given time we have measurements of say the east or $u$-component of velocity at $n = 12$ float locations, $(x_i, y_i)$, where $i$ denotes a specific float. Define the average of $u$, $x$, and $y$ over this cluster to be $\bar{u} = \frac{1}{n} \sum u_i$ and similarly for $\bar{x}$ and $\bar{y}$. Now fit a linear function of $x$ and $y$ to the cluster of float-measured $u(x_i, y_i)$ by minimizing the mean square difference $\sum (u(x_i, y_i) - (u_0 + \alpha(x_i - \bar{x}) + \beta(y_i - \bar{y}))^2$ to find the best fit values $\alpha = \partial u/\partial x$ and $\beta = \partial u/\partial y$. The same is done for the $v$ component of the velocity. Thus the float cluster is used to make a single estimate of the velocity gradient tensor on each day.}

The vorticity (twice the rotation rate) and the divergence may then into what we might call the divergence tensor, $D$, i.e., $D(1, 1) = D(2, 2) = \frac{1}{2}(V \cdot V)$, and $D(1, 2) = D(2, 1) = 0$. What properties does this new tensor $D$ have? What simplification results so far as the eigenvectors/values of the strain rate tensor are concerned? 5) In a similar vein, the velocity gradient tensor in $R^2$ has four independent elements, $\partial u/\partial x, \partial u/\partial y, \partial v/\partial x$ and $\partial v/\partial y$. We have seen that three of the four possible combinations of these terms have real importance, the divergence, $\partial u/\partial x + \partial v/\partial y$, the rotation rate or vorticity, $\xi = \partial v/\partial x - \partial u/\partial y$, and the shear strain rate, $\partial v/\partial x + \partial u/\partial y$ of Eq. (120). Can you interpret the fourth possible combination, $\partial u/\partial x - \partial v/\partial y$? (Don’t expect something of cosmic significance.)
be estimated by sums and differences of the horizontal shears, exactly as noted above in Sections 6.4.1 and 6.4.2. These estimates are Lagrangian in the sense that the floats act as a tag on a specific material volume that we can follow for an extended period of time. As we will describe below, the motion of this float cluster appears to have been representative of the full water column.

The vorticity of a fluid column is analogous to the rotation rate of a spinning, solid column, and the conservation of vorticity is closely analogous to the conservation of angular momentum. Many fluid phenomenon are illuminated by analysis of vorticity, and nowhere is this more true than in geophysical fluid dynamics. Indeed, most understanding of geophysical flows comes by understanding the distribution and the balance of the potential vorticity, $\Psi$, which for a layer of thickness $H$ is

$$\Psi = \frac{f + \chi}{H},$$

(125)

where $f$ is the Coriolis parameter, described in the context of momentum balance by Price (2004). In the present context of vorticity balance, $f$ is dubbed appropriately the 'planetary vorticity' because even when a fluid column is at rest with respect to Earth-bound observers, i.e., there is no horizontal wind or ocean current, it will nevertheless be rotating with respect to an inertial observer by virtue of Earth’s rotation. The magnitude of this Earth-induced vorticity is given by the projection of Earth’s rotation vector onto the unit normal of the horizontal face of the column,

$$\Omega \cdot n = 2\Omega \sin(\text{latitude}) = f,$$

where $\Omega = 2.729 \times 10^{-5} \text{rad s}^{-1}$ is Earth’s rotation rate (Fig. 21). Thus, for a fluid column sitting directly over the north pole, the planetary vorticity is twice the Earth’s rotation rate, $f = 2\Omega$; for a column at 30 N, the planetary vorticity is $f = \Omega$ and for a column on the equator, the planetary vorticity vanishes and $f = 0$ and column sitting over the south pole would have a planetary vorticity $f = -2\Omega$. If $f$ is the planetary vorticity, then it is appropriate to call $\chi = \nabla \times \mathbf{V}$ the ‘relative vorticity’, since $\mathbf{V}$ is the fluid velocity relative to the Earth. An inertial observer will observe that the vorticity of a column is the sum of planetary and relative vorticity, called the 'absolute vorticity' $\Upsilon = f + \chi$, while an Earth-bound observer will see only the relative vorticity, $\chi$.

The potential vorticity $\Psi$ is an intensive, scalar property of a fluid, and aside from what are usually small effects of bottom drag or diffusion, the $\Psi$ of a given fluid column is conserved,

$$\frac{d\Psi}{dt} = 0.$$  

(126)

Often our interest is in the balance between changes in relative vorticity and planetary or stretching vorticity (described below) and so it is useful to consider the difference form;

$$\delta \chi = -\delta f + (f + \chi) \frac{\delta H}{H},$$

(127)

The goodness of fit may be diagnosed by the fraction of the velocity variance accounted for by the fit (termed FIT in Fig 22).
6 ASPECTS OF ADVECTION.

Figure 21: A rotating Earth and schematic fluid columns. The darker blue columns at right show that the horizontal rotation of a column due to Earth’s rotation is \( f = 2\Omega \sin(\text{latitude}) \); the angle \( \phi \) between the unit normal and Earth’s rotation vector \( \Omega \) is the co-latitude. The three lighter blue columns show the sense of relative vorticity acquired by a column that is displaced north and south while maintaining constant thickness. It is natural to show a column spinning about its center when it acquires relative vorticity, as here. In fact, the float cluster of Fig. (20) indicates that the relative vorticity was due mainly to shear rather than curvature.

where

\[
\delta() = \int_{t_0}^{t} \frac{d}{dt}(\cdot) dt
\]

is the change in time following a given float cluster and presumably a material volume. Note that no one part of the expanded potential vorticity is conserved, but only the sum of the three. To make a complete Lagrangian model (as in the method of characteristics, Sec. 6.3) we would next have to write down an equation for the column trajectory. In this case, we observed the float trajectories and so to understand why the relative vorticity changed we need consider only Eq. (127) or equivalent (ah, the joys of Lagrangian measurement). Two properties of \( \Psi \) make it extraordinarily useful for the analysis of many geophysical flows: (1) The potential vorticity is transported with the fluid velocity (rather than with gravity wave phase speeds, as applies for momentum), so that \( \Psi \) serves to tag the fluid, much like temperature or salinity. (2) \( \Psi \) is nevertheless very closely related to the velocity, and in many cases knowledge of the distribution of \( \Psi \) is sufficient to recover the horizontal velocity (though not when gravity waves are present).

Assuming that the moment of inertia (volume times radius\(^2\)) of a fluid column is constant (it generally will not be) then the absolute vorticity of a column will be conserved, \( d\Gamma / dt = 0 \), or \( \Gamma = f + \chi = \text{constant} \), including when a column changes latitude. For example, suppose that a fluid
column is at rest at a mid-latitude so that it has a planetary vorticity $f_0$ but no relative vorticity, and thus its absolute vorticity is just $\Psi = f_0$. If this column is displaced toward the pole, as more or less happened to the float cluster starting on ca. 16 May (Figure 20), then the planetary vorticity of the column will increase. To maintain a constant absolute vorticity, $\Upsilon = f + \chi = f_0$, the column will thus acquire negative or anticyclonic relative vorticity, $\delta \chi = -\delta f$, the amount given by the change in the planetary vorticity (the Coriolis parameter). Notice that as the float cluster moved northeastward, it seemed to acquire clockwise or anticyclonic relative vorticity, and after a couple of weeks the entire cluster turned and moved back toward the southwest. As the float cluster moved equatorward it acquired positive or cyclonic relative vorticity, which in this case served to erase the negative relative vorticity acquired during the poleward displacement.

An Earth-bound observer (i.e., what we see in Fig. 20) will note that the cluster lost and gained relative vorticity as it oscillated north to south. An observer who witnessed this motion from an inertial reference frame would note that the absolute vorticity of the cluster did not appear to change (aside from thickness changes, which we will discuss next), though she would be able to tell that the absolute vorticity of this cluster was different from the surrounding water (assuming that it is at rest).

Another and often very important aspect of vorticity balance is that column thickness will change if there is a divergence of the horizontal velocity within the fluid column, as would occur if the column moved over variable water depth. For a given column, the moment of inertia is $\propto 1/H$, and the absolute angular momentum is thus $\propto \Upsilon/H$, which is just the potential vorticity noted above. If the thickness changes, then so too does the moment of inertia and the absolute vorticity, exactly as would be expected from angular momentum conservation. Thus when a column is squashed it will slow it’s rotation rate (aside from changes in latitude and thus a change in planetary vorticity), and the reverse must happen if a column is stretched. In the example of the float cluster described here, the water depth shoaled gently toward the east and thus when the column moved northeast, it was apparently squashed by about 100 m out of a nominal thickness of 5000 m, or by roughly 2 percent, while at the same time it also went to larger $f$, by about 3 percent. Thus the changes in thickness and planetary vorticity were roughly comparable and in phase for this particular motion (Fig. 22).

It is noteworthy that a change in $f/H$ of only about 5 percent was sufficient to account for the rather impressive change in the relative vorticity, about 5 percent of $f_0$, and by extension, of the current, which had an amplitude of roughly 0.15 m s$^{-1}$. Thus what would seem to be comparatively small changes in latitude or column thickness result in very significant changes in the relative vorticity and the current itself. The reason, hinted at above, is that the planetary vorticity $f$ is generally much larger then the relative vorticity of most ocean currents (or winds) and hence the planetary vorticity represents a very large ‘potential’ of the relative vorticity, which can be released by changing either the latitude, and thus the Coriolis parameter, $f$, or column thickness.$^{56}$

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$^{54}$Relative vorticity or rotation that is in the same direction as Earth’s rotation is said to cyclonic, which comes from the Greek kyklon for circular motion. Cyclonic rotation is thus counterclockwise in the northern hemisphere and clockwise in the southern hemisphere. Anticyclonic is the reverse.

$^{55}$The conservation of potential vorticity makes such a useful and vivid diagnostic tool that it seems to govern these motions. However, potential vorticity conservation is probably better thought of as our guide to geophysical fluid dynamics rather than the governing dynamics per se, since it is once removed from the momentum equations, which more nearly govern the motion.

$^{56}$This analysis was carried out in a Lagrangian coordinate or reference frame, in that the floats followed a material volume of
7 Concluding Remarks.

The broad goal of this essay has been to introduce some of the central concepts of kinematics applied to fluid flows, and especially to develop an understanding of Eulerian and Lagrangian representations of fluid flow. The starting point is the so-called Fundamental Principle of Kinematics, or FPK, (Section 1.2), which asserts that there is one unique fluid velocity. The fluid velocity can be sampled either by tracking fluid parcels or by placing current meters at fixed locations. We have seen by way of a simple example (Sections 2 and 3) that it is possible to shift back and forth from a Lagrangian to an Eulerian representation provided that we have either (1) a complete knowledge of all parcel trajectories, or, (2) the complete velocity field at all relevant times. In Section 2 we first presumed (1), which happens only in the special world of homework problems. However, in the usual course of a numerical model calculation we probably will satisfy (2), and thus can compute parcel trajectories on demand (Sections 4 and 5). Numerical issues and diffusion (numerical and physical) will complicate the process and to some degree the result. Nevertheless, the procedure is straightforward in principle and is often an important step in the diagnostic study of complex flows computed in an Eulerian frame. Approximate methods may be usefully employed in the analysis of some flows, an important example being the time-mean drift of parcels in a field of surface gravity waves (Section 5.2). The Eulerian mean motion below the wave trough is zero on linear theory, while the Lagrangian mean flow may be substantial. The Eulerian statement of potential vorticity conservation (that we would be much more likely to use for a complete, predictive model) is just $\frac{D\mathcal{P}}{Dt} = \frac{\partial}{\partial t} \mathcal{V} + \nabla \cdot \mathbf{V} \mathcal{V} = 0$. How would the northeast to southwest oscillation appear if it was observed at a fixed site, i.e., from an Eulerian reference frame? Consider three cases: (1) the idealized case that only variations of planetary vorticity are present, and assume that $f$ can be expanded in a Taylor series, $f = f_0 + \beta y$. (2) that only variations of water depth are important, and finally, (3) consider the actual, observed current field of Fig (20) and whether 'frozen field' advection of relative vorticity would have been observed (Sec. 6.1). Finally, when the cluster changed directions from northeast to southwest the turn appeared to propagate across the cluster (see Fig. 20 or the animation on the cover page). Can you estimate a phase speed and direction? How about a period and a wavelength?
depending upon wave steepness.

The statement of conservation laws for mass, momentum, etc., applies to specific parcels or volumes of fluid, and yet to apply these laws to a continuum it is usually preferable to transform these laws from an essentially Lagrangian perspective into Eulerian or field form (Section 3). There are three key pieces in this transformation; the first is the FPK (Section 1.2). The second is the material derivative (Section 3.1); an ordinary time derivative transformed into the Eulerian system is 
\[ \frac{D}{Dt} = \frac{\partial}{\partial t} + V \cdot \nabla (\cdot) \]
the sum of a local time rate of change and an advective rate of change (Eq. (37)). The third is that integrals and their time derivatives can be transformed from material to field coordinates by way of the Reynolds Transport Theorem (Eq. (44) and Section 3.2). Important applications of the RTT yield the mass conservation relation and the momentum balance (Sections 3.2.1 and 3.2.2), which are the starting point for classical fluid dynamics.

The process of advection contributes much of the interesting and most of the challenging dynamics and kinematics of fluid flows. The advection term is semi-linear in that it involves the product of an unknown (generally) velocity component and the first partial derivative of a field variable. There are some important bounds on the consequences of advection. For variables that can be written in a conservation form (e.g., mass and momentum), advection alone can not be a net (globally integrated) source or sink, though it may cause variations at any given point in the domain (Section 6.1). Advection alone transports fluid properties at a definite rate and direction, that of the fluid velocity. The method of characteristics (Section 6.3) exploits this hyperbolic property of the advection equation to compute solutions of nonlinear PDEs; along a characteristic line (which are streamlines in steady flow) the governing equation is exactly as seen by a parcel. In the instances where we can solve for the characteristics, this leads to insightful solutions. The idea of characteristics is the basis of efficient numerical advection schemes and for the interpretation of many fluid flows. Besides merely transporting fluid properties, advection by a nonuniform velocity field (which is to say nearly all velocity fields) will also cause a rotation of fluid parcels that is akin to angular momentum (Section 6.4). Advection may also cause straining or deformation of fluid parcels that may lead to greatly increased mixing rates in a stirred fluid compared to diffusion alone.

The next steps for us are to consider the appropriate boundary and initial conditions for the ideal or Euler fluid model (boundary conditions being the defining element in many problems), and to determine several very useful first integrals of the motion, the Bernoulli functions. The Euler fluid model has built in limitations in that it ignores diffusion and dissipation; the inclusion of these physical processes leads to a more general fluid model, the Newtonian fluid, and a new set of boundary conditions, physics and phenomena. The one thread that runs through all of the vast subject of fluid dynamics is the foundation of continuum kinematics that we have begun to lay down here.

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