The 2001 R.M. Hardy Lecture: The limits of limit equilibrium analyses

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Abstract: Limit equilibrium types of analysis have been in use in geotechnical engineering for a long time and are now used routinely in geotechnical engineering practice. Modern graphical software tools have made it possible to gain a much better understanding of the inner numerical details of the method. A closer look at the details reveals that the limit equilibrium method of slices has some serious limitations. The fundamental shortcoming of limit equilibrium methods, which only satisfy equations of statics, is that they do not consider strain and displacement compatibility. This limitation can be overcome by using finite element computed stresses inside a conventional limit equilibrium framework. From the finite element stresses both the total shear resistance and the total mobilized shear stress on a slip surface can be computed and used to determine the factor of safety. Software tools that make this feasible and practical are now available, and they hold great promise for advancing the technology of analyzing the stability of earth structures.

Key words: limit equilibrium, stability, factor of safety, finite element, ground stresses, slip surface.

Résumé : Les analyses du type équilibre limite ont été utilisées depuis longtemps en géotechnique et sont maintenant utilisées de façon routinière dans la pratique de l'ingénierie géotechnique. Les outils informatiques graphiques modernes ont permis d'obtenir une bien meilleure compréhension des détails numériques internes de la méthode. Un examen plus poussé des détails révèle que la méthode d'équilibre limite des tranches comporte de sérieuses limitations. Le défaut fondamental des méthodes d'équilibre limite, qui ne satisfont que les équations de la statique, est qu'elles ne prennent pas en compte la compatibilité entre la déformation et le déplacement. Cette limitation peut être surmontée en utilisant des contraintes calculées par éléments finis à l'intérieur du cadre conventionnel d'équilibre limite. En partant des contraintes d'éléments finis, on peut calculer le long de la surface de glissement la résistance totale au cisaillement de même que la contrainte totale de cisaillement mobilisée et les utiliser pour déterminer le coefficient de sécurité. Les outils informatiques qui rendent faisables et pratiques ces analyses sont maintenant disponibles et offrent de grandes possibilités pour faire avancer la technologie d'analyse de la stabilité des structures en terre.

Mots clés : équilibre limite, stabilité, coefficient de sécurité, éléments finis, contraintes du terrain, surface de glissement.

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Introduction

Limit equilibrium types of analysis to assess stability have been used in geotechnical engineering for decades. The concepts have been widely applied to the stability analysis of earth slopes. The idea of discretizing a potential sliding mass into vertical slices was introduced early in the 20th century. In 1916, Petterson (1955) presented the stability analysis of the Stigberg Quay in Gothenberg, Sweden where the slip surface was taken to be circular and the sliding mass was divided into slices. During the next couple of decades or so, Fellenius (1936) introduced the Ordinary or Swedish method of slices. In the mid-1950s, Janbu (1954) and Bishop (1955) developed advances in the method. The advent of electronic computers in the 1960s made it possible to more readily handle the iterative procedures inherent in the method, which led to mathematically more rigorous formulations such as those developed by Morgenstern and Price (1965) and by Spencer (1967). The introduction of powerful desktop personal computers in the early 1980s made it economically viable to develop commercial software products based on these techniques, and the ready availability today of such software products has led to the routine use of limit equilibrium stability analysis in geotechnical engineering practice.

Modern limit equilibrium software is making it possible to handle ever-increasing complexity in the analysis. It is now possible to deal with complex stratigraphy, highly irregular pore-water pressure conditions, various linear and nonlinear shear strength models, almost any kind of slip surface shape, concentrated loads, and structural reinforcement. Limit equilibrium formulations based on the method of slices are also being applied more and more to the stability analysis of structures such as tie-back walls, nail or fabric reinforced slopes, and even the sliding stability of structures subjected to high horizontal loading arising, for example, from ice flows.

While modern software is making it possible to analyze

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ever-increasingly complex problems, the same tools are also making it possible to better understand the limit equilibrium method. Computer-assisted graphical viewing of data used in the calculations makes it possible to look beyond the factor of safety. For example, graphically viewing all the detailed forces on each slice in the potential sliding mass, or viewing the distribution of a variety of parameters along the slip surface, helps greatly to understand the details of the technique. From this detailed information, it is now becoming evident that the method has its limits and that it is perhaps being pushed too far beyond its initial intended purpose. Initially, the method of slices was conceived for the situation where the normal stress along the slip surface was primarily influenced by gravity (weight of the slice). Including reinforcement in the analysis goes far beyond the initial intention.

Based on the author's many years of experience with supporting a commercial slope stability software package (SLOPE/W 2001), it seems that the fundamentals of the limit equilibrium method of slices are not well understood, despite the routine use of the method in practice. The fact that the limit equilibrium method of slices is based on nothing more than statics often seems to be forgotten, and the significance of one factor of safety for all slices is not appreciated.

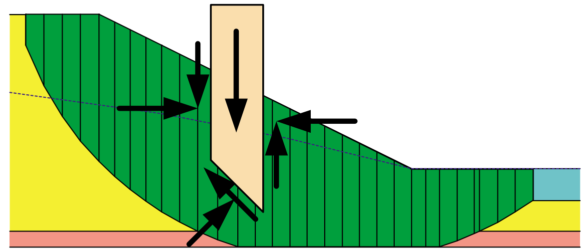
The objective here is to firstly take a fresh look at the fundamentals of the limit equilibrium method of slices with the aid of graphically presented results using modern software tools. The second objective is to use these graphical presentations to highlight the consequences of a stability analysis method that satisfies statics but ignores strain and displacement. The third main objective is to show how the results from a finite element stress–strain analysis can be married with a limit equilibrium framework to overcome the method's shortcomings.

General limit equilibrium method

Various solution techniques for the method of slices have been developed and are in common use. The primary difference among all these methods lies in which equations of statics are considered and satisfied, which interslice normal and shear forces are included, and the assumed relationship between the interslice forces. Figure 1 illustrates a typical slice in a potential sliding mass with the forces acting on the slice. Normal and shear forces act on the slice base and on the left and right sides of the slice. Table 1 summarizes the conditions for some of the common methods. This table lists which equations of equilibrium are satisfied, whether the interslice normal is included, whether the interslice shear is considered, and what the assumed relationship between the interslice normal and shear forces is.

A general limit equilibrium (GLE) formulation was developed by Fredlund at the University of Saskatchewan in the 1970s (Fredlund and Krahn 1977; Fredlund et al. 1981). This method encompasses the key elements of all of the methods listed in Table 1. The GLE formulation is based on two factor of safety equations and allows for a range of interslice shear-normal force conditions. One equation gives the factor of safety with respect to moment equilibrium (F_m), while the other equation gives the factor of safety with respect to horizontal force equilibrium (F_f). The idea of using two factor of safety equations was actually first published by Spencer (1967).

Fig. 1. Slices and forces in a sliding mass.



The interslice shear forces in the GLE method are handled with an equation proposed by Morgenstern and Price (1965). The equation is

$$[1] \qquad X = E\lambda f(x)$$

where $f(x)$ is a function, λ is the percentage (in decimal form) of the function used, E is the interslice normal force, and X is the interslice shear force. Figure 2 shows a typical half-sine function. The upper curve in this figure is the actual specified function. The lower curve is the function used. The ratio between the two curves is λ. Lambda (λ) in Fig. 2 is 0.43. At slice 10, $f(x) = 0.83$. If, for example, $E = 100$ kN, then $X = E\, f(x)\, \lambda = 100 \times 0.43 \times 0.83 = 35.7$ kN. The arctan of 35.7/100 is equal to 19.6°. This means that the interslice resultant force is inclined at 19.6° from the horizontal at slice 10. One of the key issues in the limit equilibrium formulation, as will be illustrated later, is knowing how to define this interslice function.

The GLE factor of safety equation with respect to moment equilibrium is

$$[2] \qquad F_m = \frac{\sum[c'\beta R + (N - u\beta)R\tan\phi']}{\sum Wx - \sum Nf \pm Dd}$$

The factor of safety equation with respect to horizontal force equilibrium is

$$[3] \qquad F_f = \frac{\sum[c'\beta\cos\alpha + (N - u\beta)\tan\phi'\cos\alpha]}{\sum N\sin\alpha - D\cos\omega}$$

where

c'	is the effective cohesion
ϕ'	is the effective angle of friction
u	is the pore-water pressure
N	is the slice base normal force
W	is the slice weight
D	is the line load
β, R, x, f, d, ω	are geometric parameters
α	is the inclination of slice base

(There are additional terms in eqs. [2] and [3], but they are not required here for this discussion.)

One of the key variables in both equations is N, the normal at the base of each slice. This equation is obtained by the summation of vertical forces. Vertical force equilibrium is consequently satisfied. In equation form, the base normal is defined as

Table 1. Statics satisfied and interslice forces in various methods.

Method	Moment equilibrium	Horizontal force equilibrium	Interslice normal (*E*)	Interslice shear (*X*)	Inclination of *X*/*E* resultant
Ordinary or Fellenius	Yes	No	No	No	No force
Bishop's simplified	Yes	No	Yes	No	Horizontal
Janbu's simplified	No	Yes	Yes	No	Horizontal
Spencer	Yes	Yes	Yes	Yes	Constant
Morgenstern–Price	Yes	Yes	Yes	Yes	Variable
Corps of Engineers – 1	No	Yes	Yes	Yes	Inclination of a line from crest to toe
Corps of Engineers – 2	No	Yes	Yes	Yes	Slice top ground surface inclination
Lowe–Karafiath	No	Yes	Yes	Yes	Average of ground surface slope and slice base inclination

Fig. 2. Half-sine interslice force function.

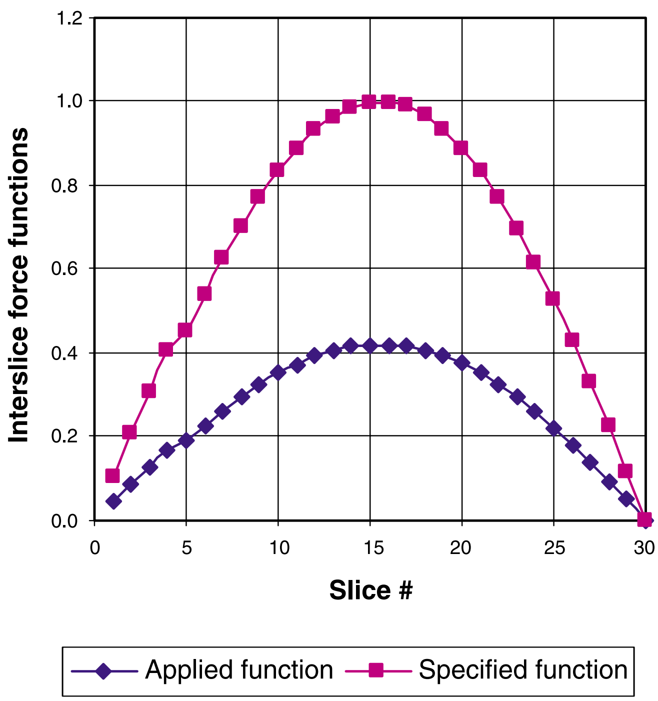
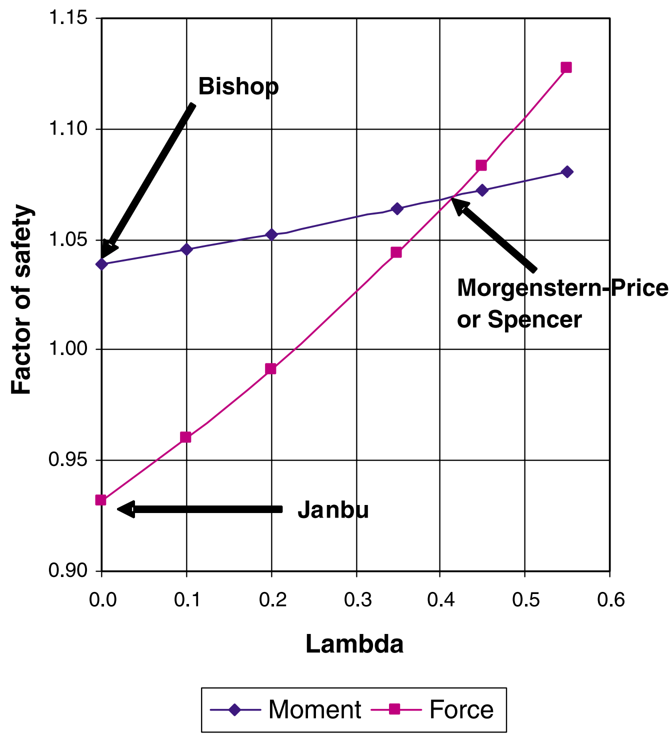


Fig. 3. A factor of safety versus λ plot.



$$[4] \quad N = \frac{W + (X_R - X_L) - \dfrac{c'\beta \sin\alpha + u\beta \sin\alpha \tan\phi'}{F}}{\cos\alpha + \dfrac{\sin\alpha \tan\phi'}{F}}$$

where F is F_m when N is substituted into the moment factor of safety equation, and F is F_f when N is substituted into the force factor of safety equation. The literature on slope stability analysis often refers to the denominator as m_a.

A very important point here is that the slice base normal is dependant on the interslice shear forces X_R and X_L on either side of a slice. The slice base normal is consequently different for the various methods depending on how each method deals with the interslice shear forces.

The GLE method computes F_m and F_f for a range of λ values. With these computed values, a plot such as that shown in Fig. 3 can be drawn that shows how F_m and F_f vary with λ.

As listed in Table 1, Bishop's simplified method ignores interslice shear forces and satisfies only moment equilibrium. In the GLE terminology, no shear forces means λ is zero. As a result, the Bishop factor of safety falls on the moment curve in Fig. 3 where λ is zero. Janbu's simplified method also ignores interslice shear forces and only satisfies force equilibrium. The Janbu factor of safety consequently falls on the force curve in Fig. 3 where λ is zero. The Spencer and Morgenstern–Price (M-P) factors of safety are determined at the point where the two curves cross in Fig. 3. At this point, the factor of safety satisfies both moment and force equilibrium. Whether the crossover point is the Spencer or M-P factor of safety depends on the interslice force function. Spencer only considered a constant *X*/*E* ratio for all slices, which in the GLE formulation corresponds to a constant (horizontal) interslice force function. The M-P method can utilize any general appropriate function. The Corps of Engineers and Lowe–Karafiath factors of safety fall on the force curve in Fig. 3. The position on the force curve depends on the procedure used to establish the inclinations of the interslice resultant. The inclination of the interslice resultant is arctan (λ) when $f(x)$ is a constant 1.0 as in the Spencer method.

In the GLE formulation, the methods are not restricted by the shape of the slip surface. It is true that the Bishop method was initially developed for circular slip surfaces, but the same assumptions inherent in the Bishop method can also be applied to noncircular slip surfaces. All the methods listed in Table 1 can be used to analyze any kinematically admissible slip surface shape with the GLE formulation.

Sliding mass contortion

The importance of the interslice force function depends to a large extent on the amount of contortion the potential sliding mass must undergo to move. The function is not important for some kinds of movement while the function may significantly influence the factor of safety for other kinds of movement. The following examples illustrate this sensitivity.

Circular slip surfaces

Figure 4 presents a simple circular slip surface together with the associated factor of safety (FS) versus λ plot. In this case the moment equilibrium is completely independent of the interslice shear forces as indicated by the horizontal moment equilibrium curve. The force equilibrium, however, is dependant on the interslice shear forces.

The moment equilibrium is not influenced by the shear forces because the sliding mass as a free body can rotate without any slippage between the slices. Substantial interslice slippage is, however, necessary for the sliding mass to move laterally; as a consequence the horizontal force equilibrium is sensitive to interslice shear.

Since the moment equilibrium is completely independent of interslice shear, any assumption regarding an interslice force function is irrelevant. The interslice shear can be assumed to be zero, as in the Bishop simplified method, and yet obtain an acceptable factor of safety, provided the method satisfies moment equilibrium. This is, of course, not true for a method based on satisfying only horizontal force equilibrium, such as the Janbu simplified method. Ignoring the interslice shear when only horizontal force equilibrium is satisfied for a curved slip surface results in a factor of safety significantly different than when both force and moment equilibrium are satisfied.

The moment equilibrium curve is not always perfectly horizontal for circular slip surfaces. The moment curve in Fig. 3 was obtained from a circular slip surface analysis, and it is slightly inclined. Usually, however, the slope of the moment curve is nearly horizontal. This is why the Bishop and Morgenstern–Price factors of safety are often similar for circular slip surfaces.

Planar slip surface

Figure 5 illustrates a planar slip surface. The moment and force equilibrium curves now have reverse positions from those for a circular slip surface. Now force equilibrium is completely independent of interslice shear, while moment equilibrium is fairly sensitive to the interslice shear. The soil wedge on the planar slip surface can move without any slippage between the slices. Considerable slippage is, however, required for the wedge to rotate.

Composite slip surface

A composite slip surface is one where the slip surface is partly on the arc of a circle and partly on a planar surface as illustrated in Fig. 6. The planar portion in this example follows a weak layer, a common situation in many stratigraphic settings. In this case, both moment and force equilibrium are influenced by the interslice shear forces. Force equilibrium factors of safety increase while moment equilibrium factors of safety decrease as the interslice shear forces increase (higher λ values).

This illustrates that a Bishop simplified type of analysis does not always err on the safe side. A more rigorous formulation such as the Morgenstern–Price or Spencer method will give a lower factor of safety than a Bishop simplified factor of safety. This is not necessarily true for all composite slips surfaces. For some composite slips surfaces a mathematically more rigorous factor of safety may be higher than the Bishop simplified. It is not possible to generalize as to when a more simplified factor of safety will or will not err on the safe side.

Slippage between the slices needs to occur for both moment and force equilibrium for a slip surface of this shape and, consequently, the interslice shear is important for both types of equilibrium.

Block slip surface

Figure 7 shows a block-type slip surface. As with the previous composite slip surface, the moment and force equilibrium are both influenced by the interslice shear. The force equilibrium is more sensitive to the shear forces than the moment equilibrium, as indicated by the curve gradients in Fig. 7. Once again it is easy to visualize that significant slippage is required between the slices for both horizontal translation and rotation, giving rise to the importance of the shear forces.

Shoring wall

Figure 8 provides an example that examines the deep seated stability of a shoring wall. The slip surface is beneath the lower tip of the sheet piling. This example comes from the analysis of a deep excavation in downtown Calgary, Alberta. The FS versus λ plot shows that the moment and force equilibrium curves are similar in this case. They are both very sensitive to the interslice shear forces. Ignoring the interslice shear forces for this case results in a significant underestimation of the factor of safety. Without including the interslice shear forces, the factor of safety is less than 1.0, indicating an unstable situation. Including the shear forces increases the factor of safety to 1.22. The difference again is due to the contortion the potential failing mass would have to undergo to rotate or move laterally.

Contrasting material

Switzerland uses berms to deflect the path of snow avalanches. The photograph in Fig. 9 shows one of these berms constructed to protect a church high in the mountains. These

Fig. 4. Conditions for a simple circular slip surface.

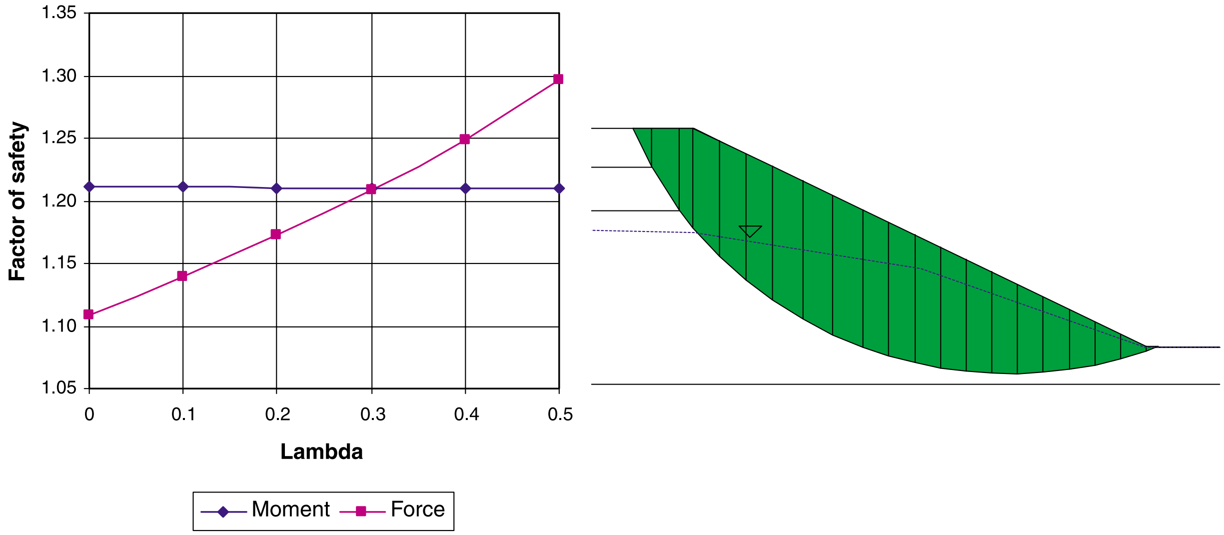


Fig. 5. Situation for a planar slip surface.

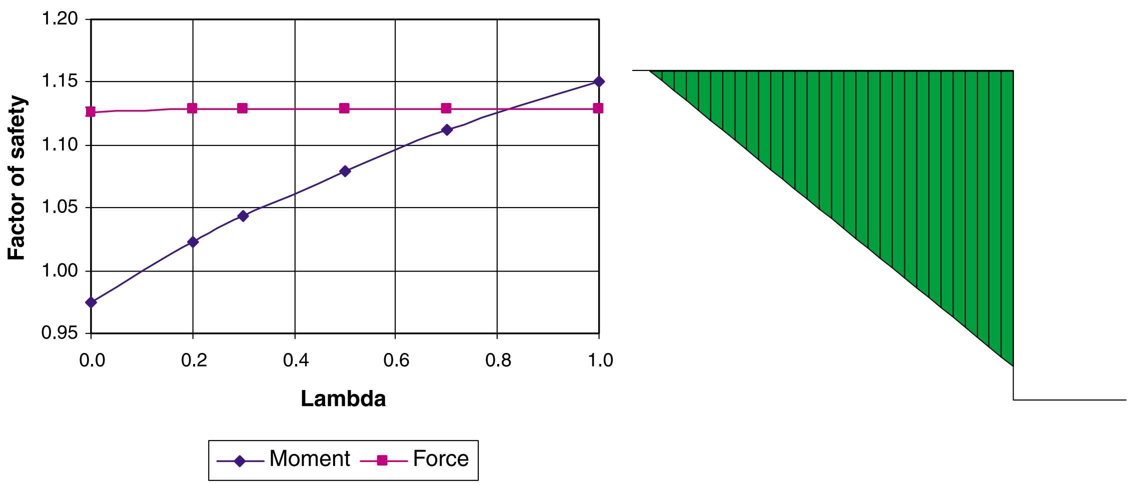


Fig. 6. Situation for a typical composite slip surface.

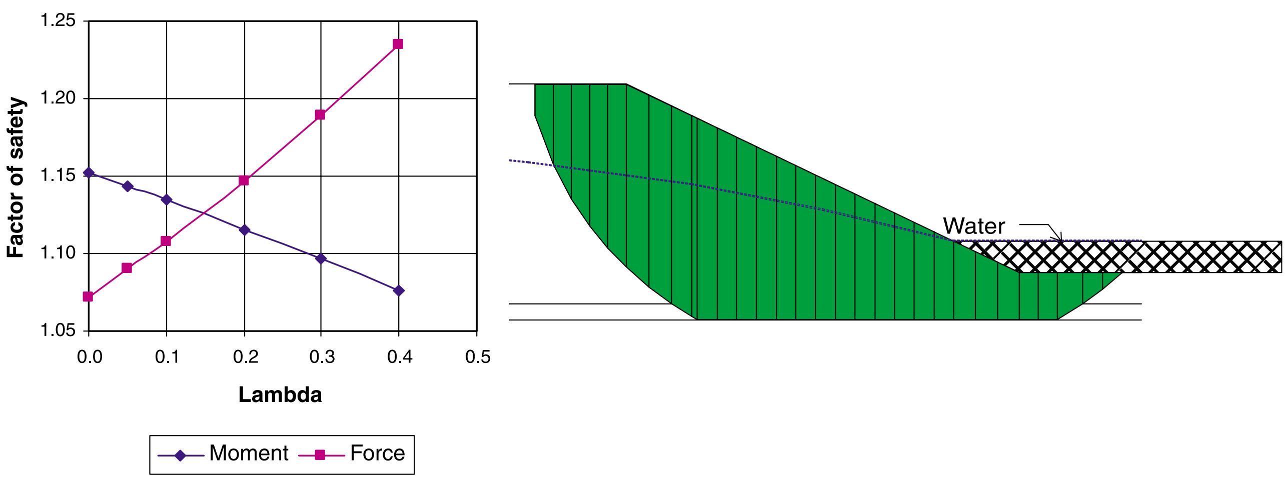


Fig. 7. Typical situation for a block slip surface.

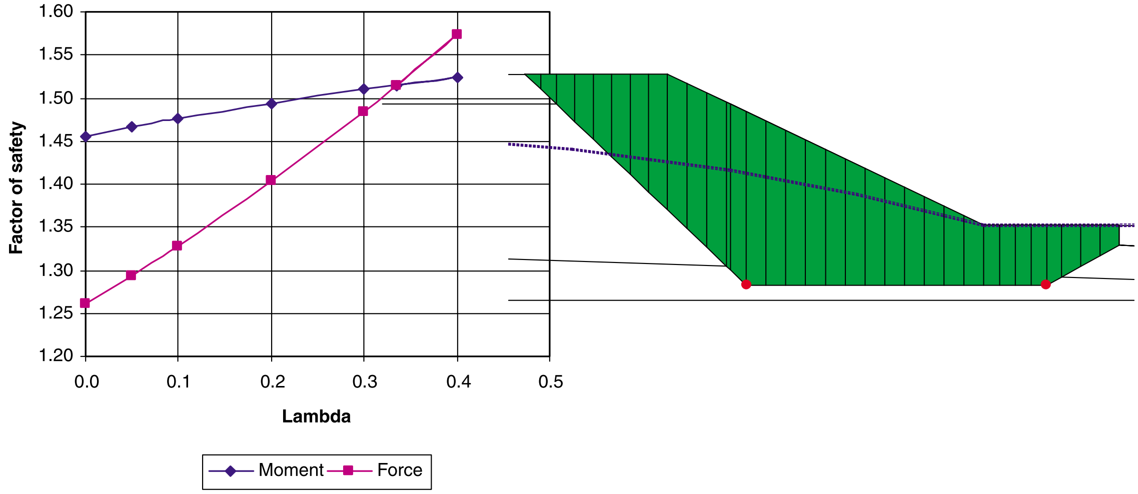


Fig. 8. A deep stability analysis of a shoring wall.

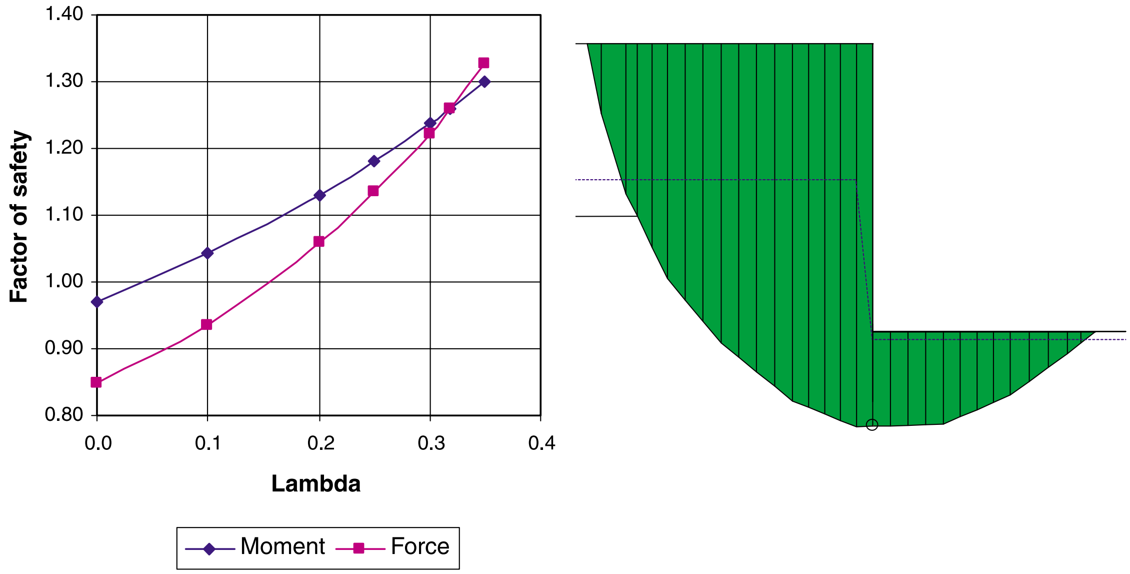


Fig. 9. Snow avalanche berm in Switzerland (courtesy, Dr. Steiner, Bern, Switzerland).

berms must have a steep face to deflect the snow flow. If the face is too flat, there is danger of the avalanche running up and over the face. A typical inclination is 68° (5v:2h). At this angle the embankment slope is unstable, and consequently they are protected and stabilized with a stone face. Another important element in the stability is that the stone face be keyed into the ground at the toe of the berm.

Steiner (W. Steiner[1], personal communication, 1995) was faced with the design task of increasing the height of one of these berms from 12 to 16 m. As part of his work, he back-analyzed some stable berms that had been in existence for a long time. The existing performance of the berms inferred a high factor of safety. A routine limit equilibrium analysis however, gave very low safety factors. The Bishop factor of safety was around 0.9, and a Spencer factor of safety was around 1.1. Steiner was of the opinion that the analysis did not properly represent the high shear strength of the stone face. In his view, the interlocking nature of the stone provided a much higher resistance than the analysis indicated.

Figure 10 shows an analysis cross-section of the berm. The stone face is 1.5 m thick at the base and narrows to 1.0 m at the top. The material properties are as follows:

Property	Soil	Stone
γ (kN/m^3)	21	22
C (kPa)	0.0	0.0
ϕ (°)	36	42

Note: γ, unit weight; C, cohesion; ϕ, friction angle.

Only one slip surface is presented here to illustrate the effect of the interslice function.

Analyzing the situation with a constant interslice function results in the FS versus λ plot shown in Fig. 11. The force and moment equilibrium curves are nearly parallel, which confirms the importance of the interslice shear in this case. The crossover point is at a factor of safety of about 1.11, which Steiner believed was not representative of the actual stability conditions.

Steiner was of the view that the interslice forces could be better represented by a step function like the one in Fig. 12. When using this step function the resulting FS versus λ plot is as in Fig. 13. The moment and force equilibrium curves are now even closer but the crossover point is now at a factor of safety of 1.42.

The purpose here is not to comment on the merits or applicability of the step function but to show an example where the solution is very sensitive to the assumed interslice force function. The difficulty and uncertainty of defining a suitable function for a case like this limits what can be done with a limit equilibrium type of analysis.

Concentrated loads

As mentioned in the introduction, limit equilibrium techniques are ever more being extended to analyzing walls stabilized with tie-back anchors, nails, or geofabrics. Since limit equilibrium is all about statics, any of these structural elements in essence become concentrated line loads. Locally, these loads can create some unrealistic conditions.

Fig. 10. Avalanche berm analysis section.

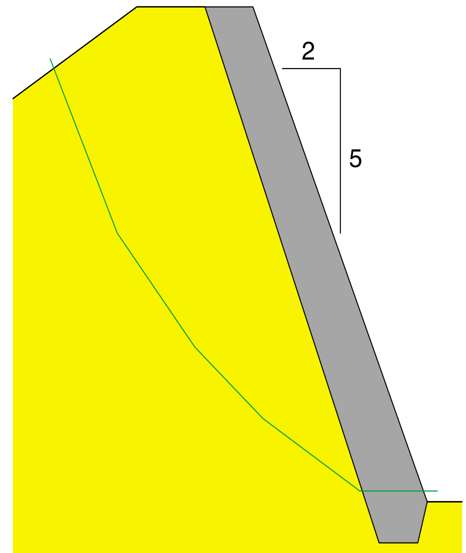
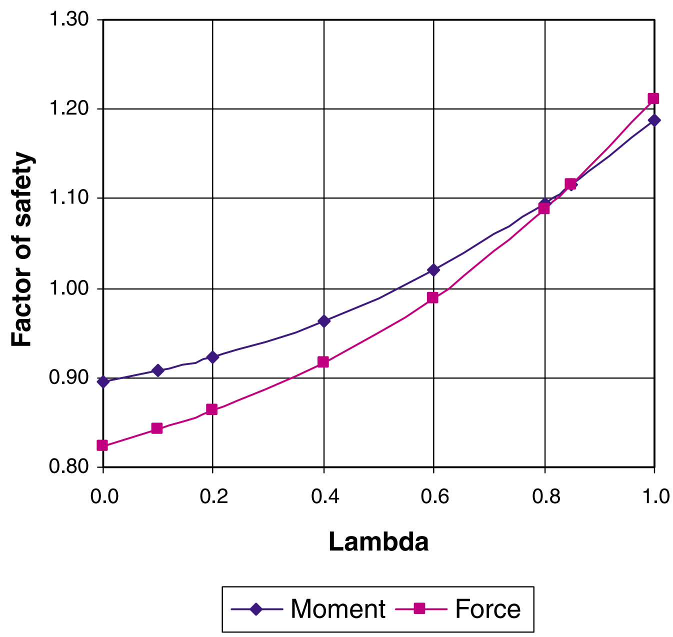


Fig. 11. Berm analysis results with a constant (horizontal) interslice force function.



Consider the simple case of a near vertical slope with two anchors as shown in Fig. 14. The applied force for each anchor is 150 kN.

One of the issues in a case like this is deciding where to apply the line load on the sliding mass free body. Should the line load be applied where the line of action intersects the slip surface, or at the wall face, or somewhere in between? Additionally, should the line load be included in the force

[1] Use of a stepped-interslice function for the back-analysis and design of a rock-faced avalanche protection dam. Report prepared for the Department of Public Work, Canton Grisons, Switzerland (1985, unpublished).

Fig. 12. Berm interslice step function.

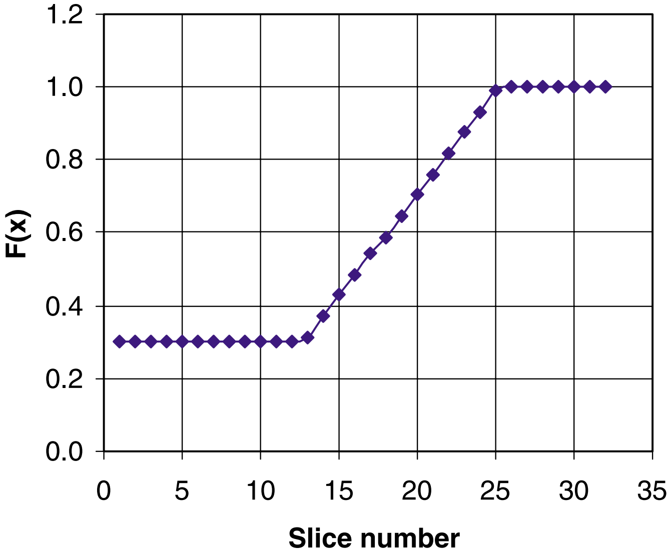


Fig. 13. Berm analysis results with interslice step function.

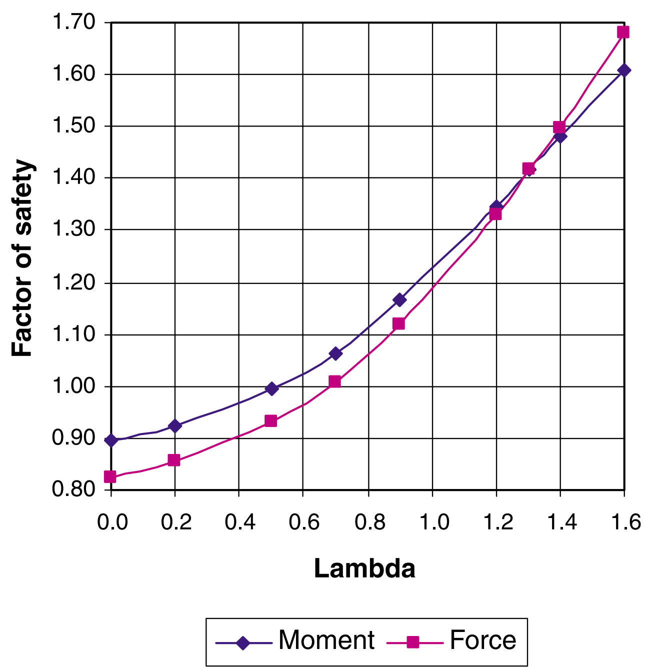


Fig. 14. A tie-back wall example.

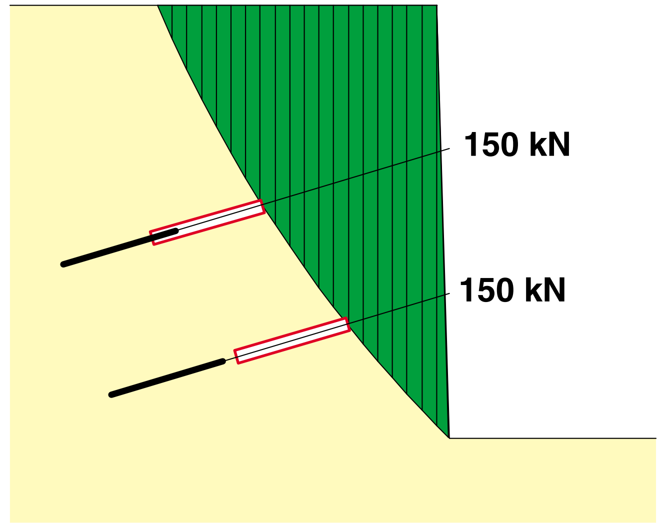
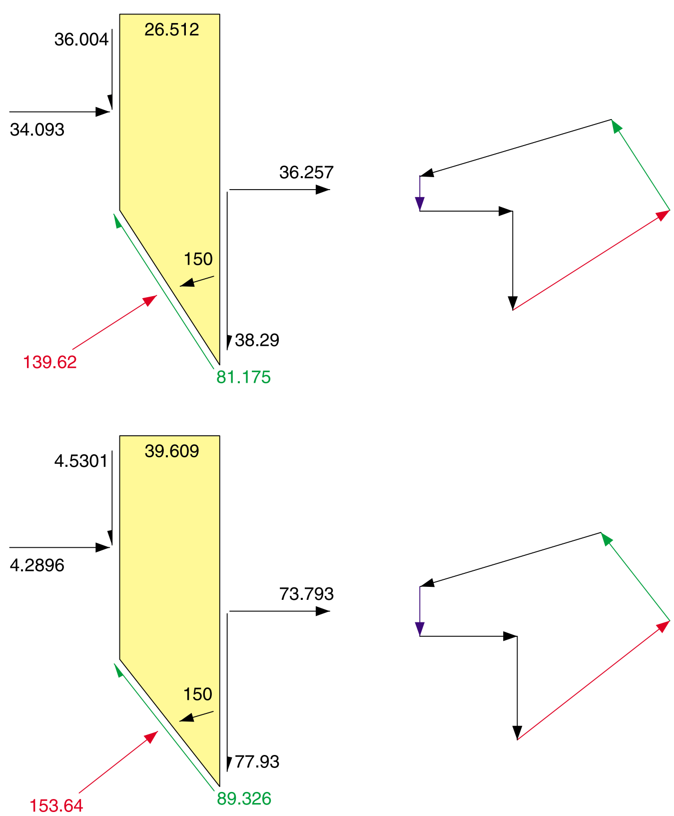


Fig. 15. Free bodies and force polygons for slices 8 and 13.

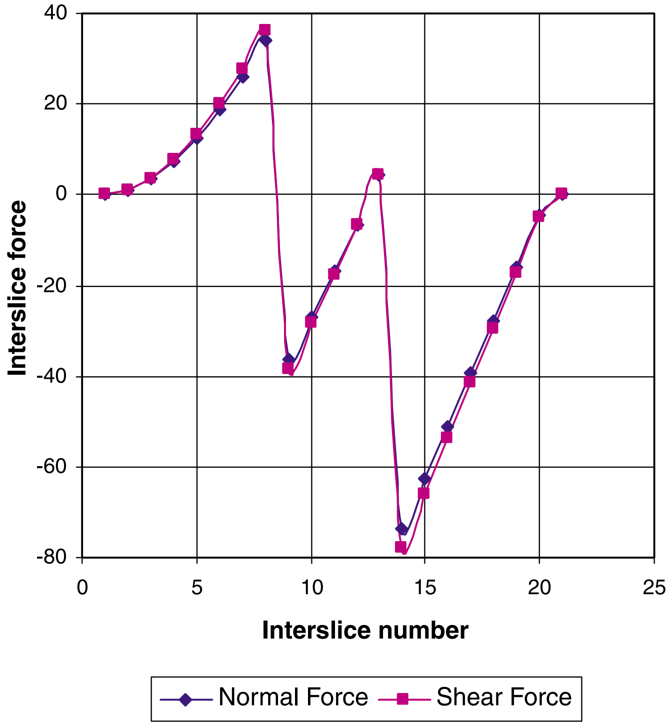


equilibrium of each slice or just in the equilibrium of the sliding mass as a whole? If the line load is included in the equilibrium of each slice, then the line load will affect the base normal of the slice which, in turn, will affect the shear strength at the base of the slice. Fundamentally, the question is, how does the structural line load influence the normal stress along the slip surface?

First, consider the case with the line (anchor) load applied on the slip surface where the line of action intersects the slip surface. The anchor force is included in the equilibrium of the slice that has the line of action extending through the slice base. For this particular example, the upper anchor passes through the base of slice 8 and the lower anchor passes through the base of slice 13. Free body diagrams and force polygons for the two slices with the anchor load are presented in Fig. 15. The force polygons close indicating the slices are in force equilibrium.

The interslice normals on the right side of the slices point away from the slices, indicating tension. This is actually true for many of the slices as can be seen by the interslice force plot in Fig. 16. The interslice normal is nearly equal to the shear since λ is nearly 1.0 for this case. The upper anchor makes the interslice normal become negative at slice 8, then the normal becomes less negative until it is positive for one slice. The lower anchor, however, makes the interslice normal become negative again. Finally, the normal becomes less

Fig. 16. Interslice shear and normal forces with anchor loads applied at the slip surface.



negative and ends up close to zero. These interslice normal forces may be totally unreasonable but, despite this, all the slices are in complete force equilibrium as indicated by the closure of the force polygons for each slice.

When looking at the exact same situation but with the anchor loads applied at the wall, the interslice forces are now completely different. Figure 17 again shows the interslice shear and normal forces. The normal force increases evenly and gradually except for the last two slices. Of interest is the interslice shear force. The direction is now the reverse of that which usually occurs when only the self weight of the slices is included (simple gravity loading). The shear stress reversal is a reflection of a negative λ.

Applying the anchor line loads on the wall results in perhaps a somewhat better interslice force distribution, but now the base normal force for the first slice or two behind the wall is affected by the anchor load. This also does not seem proper.

The large differences in the interslice forces also lead to significantly different normal stress distributions along the slip surface as shown in Fig. 18. It was noted earlier that the equation for the normal at the base of the slices includes terms for the interslice shear forces. This example vividly illustrates this effect.

Interestingly, in spite of the different stresses between the slices and along the slip surface, the factors of safety are nearly identical for these two treatments of the concentrated line loads, i.e.,

Anchor force location	Factor of safety
On slip surface	1.075
On wall	1.076

Fig. 17. Interslice shear and normal forces with anchor loads applied at the wall.

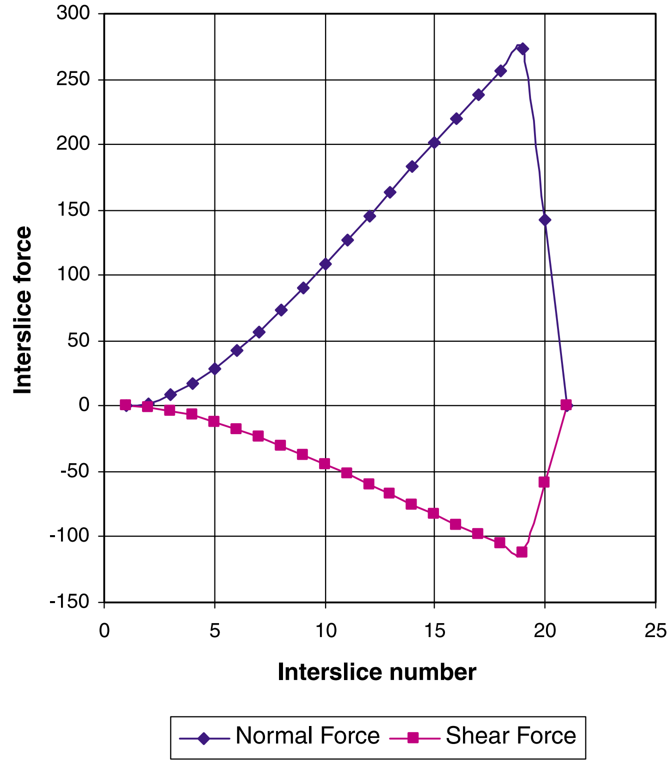
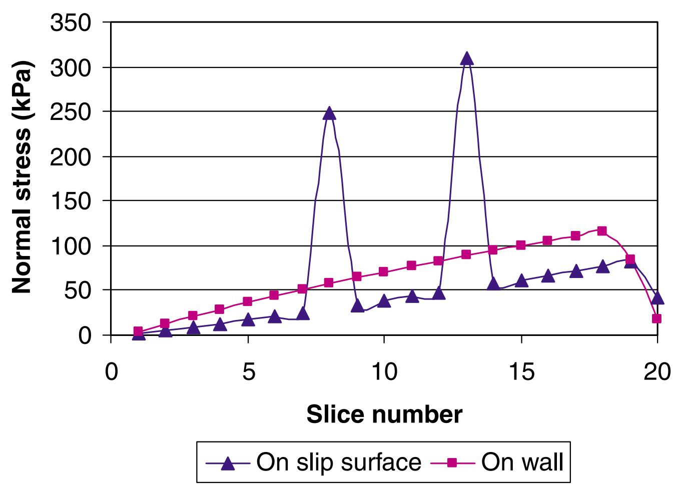


Fig. 18. Normal stress distributions along the slip surface.



With the anchors applied at the slip surface location, the factor of safety is 1.075; while when they are applied at the wall, the factor of safety is 1.076. For all practical purposes they are the same. The reason for this is discussed later.

Convergence issues

Another important issue with using a limit equilibrium approach to analyze the stability of very steep slopes (walls) is the difficulty of obtaining convergence. The denominator in the slice base normal equation is commonly known as m_a, as noted earlier. As the slice base inclination increases, the m_a term increases and eventually reaches a point where it is not possible to compute a normal that will satisfy the force equilibrium of the slice. In numerical terms this manifests itself in nonconvergence of the factor of safety equations that must be solved by iterative techniques because of the nonlinearity of the equations. The nonlinearity arises from the fact that the term for the factor of safety (F) occurs on both sides of the equation.

Not being able to compute a factor of safety is bad enough, but the minimum factor of safety is often located immediately next to the area where nonconvergence becomes a problem. Sometimes the minimum factor of safety is among nonconverged slip surfaces. The critical slip surface is often in the proximity of the active wedge line inclined at $45 + \phi/2$, which is also the inclination at which the convergence difficulties frequently start. The difficulty is that the critical factor of safety can be somewhat suspect when its neighbors represent a nonconverged case.

A typical situation is portrayed in Fig. 19. In this example a converged solution could not be obtained for most of the grid rotation centres in the lower right hand triangular area. The grid rotation centres without a factor of safety beside them are the ones where it was not possible to get a converged solution. These nonconverged grid centres mostly represent steep slip surfaces. The critical grid rotation centre is represented by the point with the larger dot and is surrounded by points for which it was not possible to obtain a converged solution.

Forces outside the sliding mass

When structural components are included in a limit equilibrium analysis, the influence of the structure can extend outside the potential sliding mass. A typical case is the embedment of steel sheet piling beneath the base of an excavation as illustrated in Fig. 20. The issue is how to include the lateral resistance provided by the buried portion of the sheet piling when looking at a potential mode of failure where the slip surface exits at the excavation base. The passive resistance in front of the pile is an integral part of the stability of the system, but it is outside the free body of the sliding mass. It is not possible to include the shear strength of the steel piling. The analysis of the snow avalanche berm showed the difficulty with trying to incorporate the strength of a structure. This is even worse for a high strength material such as steel.

One way of including the passive resistance in front of the wall is to do an independent analysis. This could be done with closed-form solutions or even with a limit equilibrium analysis. The computed available force can then be included in the wall stability analysis as a line load just above the slip surface exit point as shown in Fig. 20.

The issue is further complicated by the fact that the passive resistance is sensitive to the friction between the wall and the soil. The passive earth pressure coefficient can vary greatly depending on the assumed friction between the soil and the steel. Also, large movement is sometimes required to develop the complete passive resistance. To account for this, only a portion of the passive resistance can be relied upon in the wall stability analysis. In other words, the passive resistance needs its own factor of safety, which is likely much higher than the factor of safety for the wall stability.

Another possible approach is to include a line load in the summation of moments and forces that acts outside the free body diagram representing the potential sliding mass. The passive toe resistance could be represented by such a line load even though the load itself does not act on the soil wedge. Not all slices will then, however, be in force equilibrium.

Slip surface normal forces

As was pointed out earlier, one of the key variables in a limit equilibrium method of slices analysis is the normal force at the base of the slice. Further, the normal force is dependent on the assumptions made regarding the interslice shear forces. The examples discussed here indicate that the base normals can vary greatly depending on the point where the concentrated loads are applied. This section looks at the meaning of the normal stress distribution along a slip surface as obtained from a limit equilibrium analysis.

To begin, take the simple 45° slope in Figs. 21 and 22 with a slip surface through the toe and another deeper slip surface below the toe. The normal stress distribution along the slip surface from a limit equilibrium Morgenstern–Price analysis with a constant interslice force function is compared with the normal stress distribution from a linear–elastic finite element stress analysis. For the toe slip surface, the normal stresses are quite different, especially in the toe area. The normal stress distributions for the deeper slip surface are closer but still different for a good portion of the slip surface.

Figure 23 presents a case with reinforcement. The line loads are applied at the point where the slip surface intersects the line of action. Again there are significant differences between the limit equilibrium normal stresses and the finite element stresses, particularly for the slices that include the line loads. The finite element stresses show some increase in normal stresses due to the nails but not as dramatic as the limit equilibrium stresses.

These examples show that the stress conditions as computed from a limit equilibrium analysis may be vastly different from finite element computed stresses. The finite element stresses are more realistic and are much closer to the actual conditions in the ground. The implication is that the limit equilibrium computed stresses are not representative of actual field conditions, yet the limit equilibrium seems to give reasonable factors of safety.

Limit equilibrium forces and stresses

The question is, why can such unrealistic stresses give a seemingly reasonable factor of safety? The answer lies in the fundamental assumption that the factor of safety is the same for each slice. The limit equilibrium method of slices requires iterative techniques to solve the nonlinear factor of safety equations. In the Morgenstern–Price or Spencer methods, a second level of iterations is required to find the slice forces that result in the same F_m and F_f. Fundamentally, the iterations are required to meet two conditions, namely

(1) **to find the forces acting on each slice so the slice is in force equilibrium, and**
(2) **to find the forces on each slice that will make the factor of safety the same for each slice.**

Fig. 19. Convergence difficulties at grid location points without safety factors.

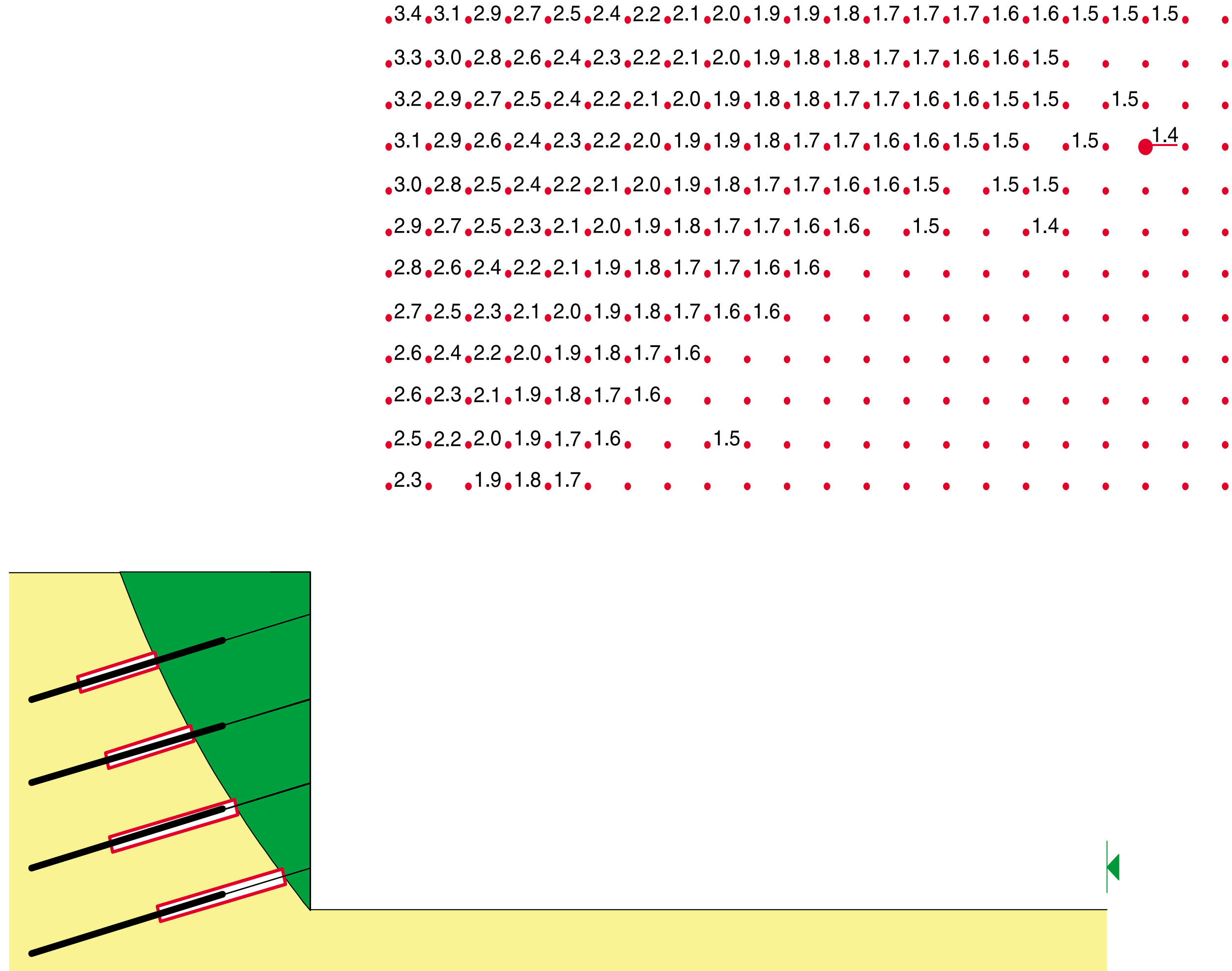
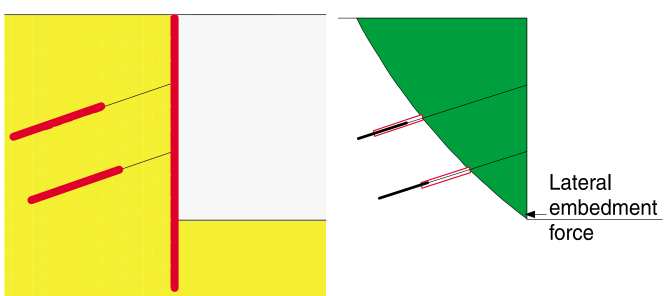


Fig. 20. Inclusion of pile embedment resistance in a limit equilibrium analysis.



This means that interslice and slip surface forces are not necessarily representative of the actual in situ conditions, but they are the forces that satisfy the above two conditions for each slice.

If the slice forces are not representative of actual in situ ground conditions, then it is also not possible to determine a realistic line of thrust for the interslice shear-normal resultant. The forces on each slice that meet the above two conditions can result in a line of thrust outside the slice, a further indication that the slice forces are not always realistic.

Fortunately, even though the limit equilibrium statics formulation locally does not give realistic slice forces, the factor of safety globally is nonetheless realistic. Once all the mobilized driving forces and base resisting shear forces are integrated, the local irregularities are smoothed out making the overall factor of safety for the entire sliding mass quite acceptable.

As a footnote, it is interesting that the early developers of the method of slices recognized the limitations of computing realistic stresses on the slip surface. Lambe and Whitman (1969) in their text book *Soil Mechanics* point out that the normal stress at a point acting on the slip surface should be mainly influenced by the weight of the soil lying above that point. This, they state, forms the basis of the method of slices. Morgenstern and Sangrey (1978) state that one of the uses "… of the factor of safety is to provide a measure of the average shear stress mobilized in the slope." They go on to state that, "This should not be confused with the actual stresses." Unfortunately, these fundamental issues are sometimes forgotten as use of a method is gradually adopted in routine practice.

While the early developers of the method of slices intuitively recognized that the slice stress may not be real, they did not have finite element tools to demonstrate the way in which they differ from the actual ground stresses. Now, with the help of finite element analyses, it is possible to show that the difference is quite dramatic.

Fig. 21. Normal stress distributions along a toe slip surface.

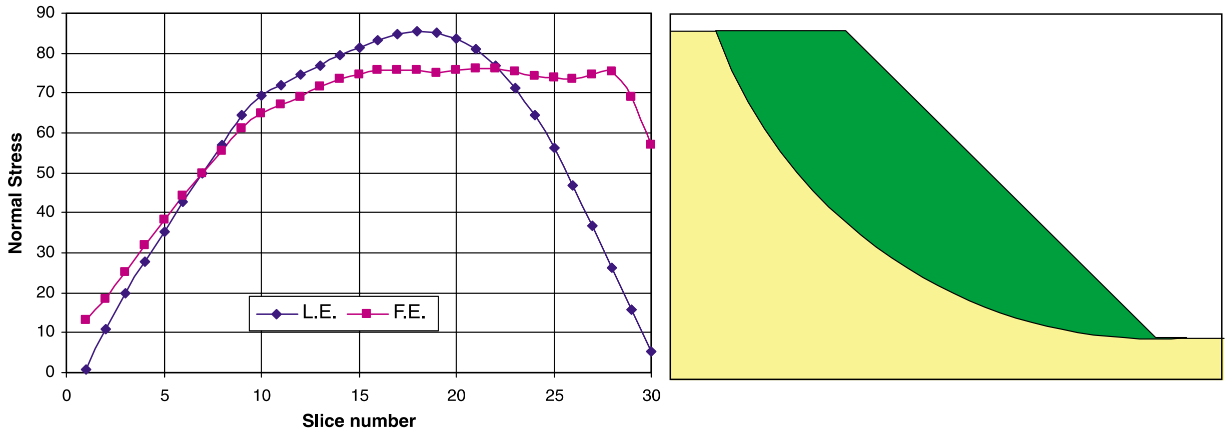


Fig. 22. Normal stress distributions along a deep slip surface.

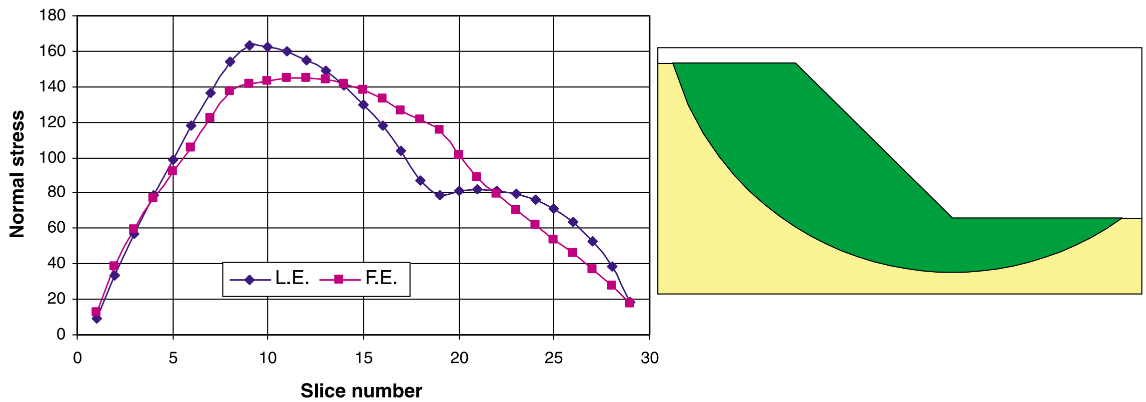
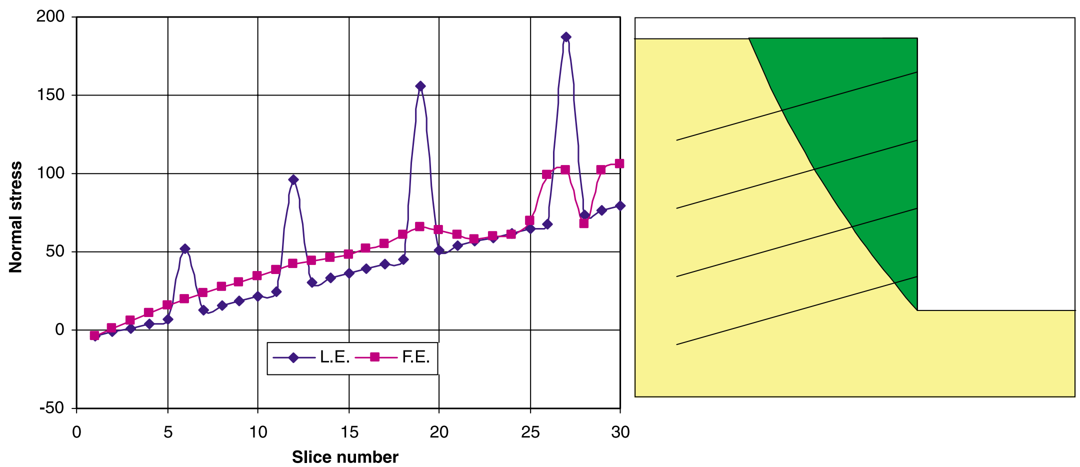


Fig. 23. Normal stress distributions along a slip surface with wall reinforcement.



In the context of stress distributions it is of interest to examine the Janbu generalized formulation (Janbu 1954, 1957). The Janbu generalized method imposes a stress distribution on each slice. The interslice stress distribution is often assumed to be hydrostatic, and the resultant is assumed to act on the lower third point along the side of the slice. A line passing through the resultants of the slice vertical sides is known as the line of thrust. Assuming a line of thrust and taking moments about the base of each slice makes it possible to determine the magnitudes of the interslice force.

This approach works reasonably well provided the actual stress distribution in the ground is close to the imposed stress distribution. Like, for example, when the slip surface has no sharp corners and the sliding mass is long relative to the slide depth. More generally, the approach works well when the potential sliding mass has no significant stress concentrations. If stress concentrations exist that deviate significantly from the Janbu generalized imposed stress distribution, the problem is overconstrained. This leads to convergence problems and lack of force equilibrium for some slices. This is particularly true when features like anchors or nails are included in the analysis. As Abramson et al. (2002) point out, the calculations for the Janbu generalized method are very sensitive to the line of trust location.

Earlier it was mentioned that the line thrust could potentially fall outside the slice. With the GLE method the slices are always in force equilibrium, but it is possible that the interslice forces would have to act outside the slice for the slice itself to be in moment equilibrium. The Janbu generalized approach, on the other hand, forces the line of thrust to be at a particular point on the side of the slice, but this may lead to the slice not being in force equilibrium. So it is not always possible to achieve both conditions. Sometimes the line of thrust needs to be outside the slice to have slice force equilibrium, or the slice cannot be in force equilibrium if the line of thrust is fixed at a particular point on the slice.

The behavior of the Janbu generalized method reinforces the earlier observation that limit equilibrium methods based purely on statics can in some circumstances overconstrain the problem, which results in unrealistic stress conditions. In this sense the Janbu generalized approach is no different from any other limit equilibrium method. The inherent interslice force assumptions are different, but in the end the limitations are similar.

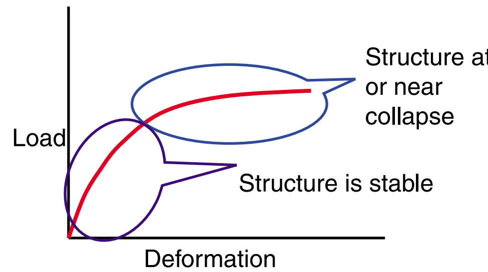
Missing physics

The limit equilibrium method of slices is based purely on the principle of statics; that is, the summation of moments, vertical forces, and horizontal forces. The method says nothing about strains and displacements, and as a result it does not satisfy displacement compatibility. It is this key piece of missing physics that creates many of the difficulties with the limit equilibrium method.

One alternative to dealing with the difficulties inherent in limit equilibrium types of analysis is to set aside the whole concept of using statics and move totally to a stress–strain based approach. Ideally, the objective would be to model the complete load–deformation behavior of a structure right up to the point of failure and in many cases well past the point of failure. The question being asked more and more these days is, "What will the structure or ground look like after failure?" Stated another way, the interest is in post-failure deformation or in the consequence of failure.

Stress–strain finite element software based on a displacement formulation has been available for some time, and such software is readily available for use in practice. A displacement formulation requires that all the elements in a mesh remain attached at the nodes. This limits the amount of deformation that can be modelled. Furthermore, displacement formulations that can handle nonlinear constitutive relationships have great difficulty with convergence as the system approaches failure (Fig. 24). The finite element equations are in essence equations of equilibrium, and when loads are applied that push the system past the point of limiting equilibrium, it is not possible to obtain a solution to the finite element equations. This manifests itself in nonconvergence.

Fig. 24. An illustrative load deformation curve.



Discrete element formulations are better suited to looking at failure and post-failure behavior. In these formulations (Cundall and Strack 1979), individual particles, blocks, or elements that were initially connected may separate, translate, and rotate individually, and they can potentially form new contacts with other blocks. These techniques are particularly useful for studying the mechanism of failure and large post-failure displacements. Discrete element software products are not yet routinely used in practice, particularly not in geotechnical engineering, and the method is therefore not an alternative to replacing limit equilibrium methods — at least not yet.

Another drawback to finite element analyses by themselves is that they do not give a direct indication of the margin of safety; that is, a quantitative measure of how close the system may be to collapse. A statement frequently seen in publications is, "it was not possible to obtain convergence past a certain loading, indicating the structure had reached failure." The end point where the factor of safety is 1.0 is indirectly known, but the margin of safety along the path is not known.

The exact point of failure is difficult to determine. One of the criteria used, as already mentioned, is the nonconvergence of the solution (Zienkiewicz 1971). With this approach the shear strength parameters in a nonlinear analysis are reduced until numerical instability occurs. The factor of safety is then taken to be a ratio of the actual available shear strength and the lowest strength that resulted in a converged solution. Recent studies on slope stability analysis using the strength reduction approach have been published by Dawson et al. (1999) and Griffiths and Lane (1999). The difficulty with using nonconvergence as a criteria is that there are many factors that can cause numerical instability. Some of the more common factors are incremental load-step size, gravity loading procedures, low confining stresses near the ground surface, purely frictional materials with no cohesion, and initial in situ stress conditions. So deciding on what is causing the numerical instability is not trivial and consequently nonconvergence is a somewhat uncertain criteria for determining the point of failure and defining the margin of safety.

A further downside of leap-frogging to an entirely new approach for analyzing the stability of geotechnical structures is that it does not provide a reference point. One of the attractions of the limit equilibrium method is that it has been in use for many years and has been calibrated with experience and observations. A totally new approach provides no such anchor.

The remainder of this paper presents an approach that combines the benefits from a finite element stress–strain analysis with the familiarity of a traditional method of slices limit equilibrium analyses. It is an approach that advances the technology and yet provides an anchor to the past.

Finite element computed stresses in a limit equilibrium framework

As already noted above, one of the key factors in the limit equilibrium factor of safety equation is the expression that defines the normal force at the base of each slice. If determining the normal base force is so problematic, then why not determine this important parameter from the results of a finite element stress analysis? The integration of limit equilibrium software such as SLOPE/W (2001) with finite element stress–deformation software such as SIGMA/W (2001) makes this possible.

Figure 25 shows a simple 45° slope discretized into finite elements. Using a simple gravity turn-on technique, the stresses in the ground can be computed. Using a linear–elastic constitutive relationship, the vertical stresses are as presented in Fig. 26. This is typical of the information available from a finite element analysis. The basic information obtained from a finite element stress analysis is σ_x, σ_y, and τ_{xy} within each element.

Worth noting at this stage is the 50 kPa contour, which is not a constant distance from the ground surface. The contour is closer to the surface under the toe. This means that the vertical stress is not just influenced by the overburden weight. It is also affected by the shear stress.

The finite element computed stresses can be imported into a conventional limit equilibrium analysis. The stresses σ_x, σ_y, and τ_{xy} are known within each element, and from this information the normal and mobilized shear stresses can be computed at the base midpoint of each slice. The procedure is as follows:

(1) The known σ_x, σ_y, and τ_{xy} at the Gauss numerical integration point in each element are projected to the nodes and then averaged at each node. With the σ_x, σ_y, and τ_{xy} known at the nodes, the same stresses can be computed at any other point within the element.
(2) For slice 1, find the element that encompasses the *x*–*y* coordinate at the base mid-point of the slice.
(3) Compute σ_x, σ_y, and τ_{xy} at the midpoint of the slice base.
(4) The inclination (α) of the base of the slice is known from the limit equilibrium discretization.
(5) Compute the slice base normal and shear stress using ordinary Mohr circle techniques.
(6) Compute the available shear strength for the computed normal stress.
(7) Multiply the mobilized shear and available strength by the length of the slice base to convert stress into forces.
(8) Repeat the process for each slice in succession up to slice number *n*.

Once the mobilized and resisting shear forces are available for each slice, the forces can be integrated over the

Fig. 25. A finite element mesh for establishing in situ stresses.

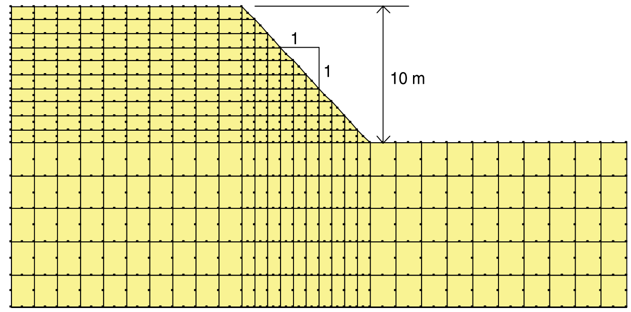
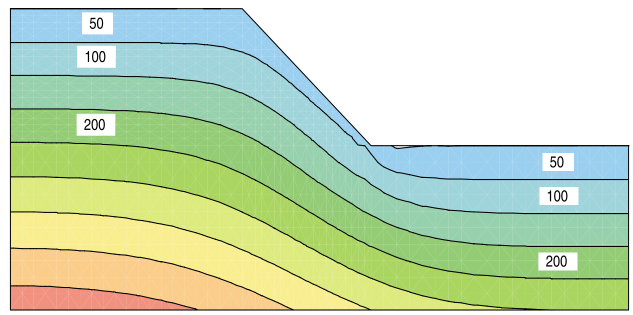


Fig. 26. Vertical stress contours.



length of the slip surface to determine a stability factor. The stability factor is defined as

$$[5] \qquad \mathrm{F.S.} = \frac{\sum S_r}{\sum S_m}$$

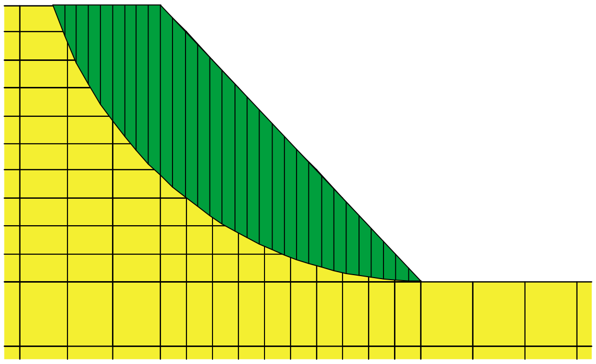
where S_r is the total available shear resistance and S_m is the total mobilized shear along the entire length of the slip surface. Similar stability factor expressions have been presented by others (Kulhawy 1969; Naylor 1982).

Figure 27 shows a potential sliding mass discretized into slices superimposed on the finite element mesh. Following the procedure listed above, the stability factor for this slip surface is 1.318. This compares with a Morgenstern–Price factor of safety of 1.145 (constant interslice function). This is about a 15% difference.

The reason for the difference in the margin of safety is primarily related to the normal stress distribution along the slip surface. The finite element and limit equilibrium normal stress distributions for this particular slip surface were presented earlier in Fig. 21. The significantly different normal stresses in the toe area result from the shear stress concentration in this part of the section. Localized shear stress concentrations are, of course, not captured in a limit equilibrium formation where the slice base normal is derived primarily from the slice weight. This is one of the limitations of the limit equilibrium method.

The situation is somewhat different for a deeper slip as shown in Fig. 22. The finite element and limit equilibrium normal stress distributions along the slip surface are much closer for this case. Consequently the stability factor based on finite element stresses is almost the same as the Morgenstern–Price factor of safety. The stress-based stability factor is 1.849 while the Morgenstern–Price factor of safety is 1.804. This shows that, when the normal stress dis-

Fig. 27. A toe slip surface on a finite element mesh.



tribution along the slip surface is fairly representative of the actual ground stresses, the limit equilibrium factor of safety is as good as a stress-based factor of safety.

An added advantage of the finite element stress based approach is that it creates the possibility of looking at local safety factors for each slice. Figure 28 shows the variation of the local safety factor for the toe and deep slip surfaces. Included in the figure is the limit equilibrium factor of safety, which is the same for each slice.

For the deep slip surface, the two global factors of safety are almost identical. Locally, however, the safety factors are both smaller and greater than the global value represented in this figure by the constant limit equilibrium factor of safety. Integrating the total available shear resistance and total mobilized shear along the slip surface averages the variation making the two factors of safety the same.

The difficulties with convergence in limit equilibrium analyses were discussed earlier. The stress-based approach completely eliminates this problem. The factor of safety is obtained directly without any iterations. Figure 29 shows an example of a grid of rotation centres with factor of safety contours. The contours now cover the complete grid confirming that a valid value was calculated for each grid point. Moreover, any question as to which is a valid minimum factor of safety is no longer an issue.

As an aside, the large number of grid points in Fig. 29 inside the 1.5 contour is of interest. This shows there are a large number of potential slip surfaces with a factor of safety between 1.409 and 1.500. In other words, there is a zone where many slip surfaces have similar safety factors. This is typical for a near vertical wall with the slip surface exiting at the base of the wall.

One of the very attractive features of doing a stability analysis based on finite element computed stresses is that soil–structure interaction can be handled in a direct manner. The difficulty of dealing with forces outside the sliding mass was earlier noted in the discussion on dealing with sheet piling embedment below the slip surface. Another similar situation is the use of a shear key wall placed across a slip surface to stabilize a slope as illustrated in Fig. 30. The factors of safety, both based on finite element stress, with and without the structure are 1.52 and 1.17, respectively. In this case there is no need to try to represent the wall resistance with a line load as in a limit equilibrium analysis, and there is no need to independently determine the line load magnitude. The stiffness of the structure is included in the finite element analysis, which alters the stress state and which, in turn, increases the margin of safety.

This approach also opens the door to looking at stability variations due to ground shaking during an earthquake. The stresses can come from a dynamic finite element analysis (QUAKE/W 2001) the same as from a static stress analysis. The stresses computed during a dynamic earthquake analysis can be saved at regular intervals during the shaking. A factor of safety then can be computed for each moment in time that the stresses are available, and in the end a plot of factor of safety versus time, such as in Fig. 31, can be created. This type of a plot can be readily created for each and every trial slip surface. This is a great improvement over the historic pseudostatic approach still used so routinely in practice.

Commentary on finite element stress-based stability analysis

The use of finite element computed stresses inside a limit equilibrium framework to assess stability has many advantages. Some of them are as follows:

- There is no need to make assumptions about interslice forces.
- The stability factor is deterministic once the stresses have been computed, and consequently there are no iterative convergence problems.
- The issue of displacement compatibility is satisfied.
- The computed ground stresses are much closer to reality.
- Stress concentrations are indirectly considered in the stability analysis.
- Soil–structure interaction effects are readily handled in the stability analysis.
- Dynamic stresses arising from earthquake shaking can be directly considered in a stability analysis.

The finite element based approach presented here overcomes many of the limitations inherent in a limit equilibrium analysis. At the same time, it does raise some new issues.

It is necessary to first carry out a finite element stress analysis with the proposed approach. Fortunately, the necessary software tools are now readily available and relatively easy to use. However, it does mean that the analyst must become familiar with finite element analysis techniques.

Fortunately, a finite element stress analysis is fairly straightforward if the material properties are restricted to simple linear–elastic behavior. Besides being relatively simple, using only linear–elastic soil models always ensures a solution since there are no convergence difficulties as with nonlinear constitutive models. A linear–elastic analysis is adequate in many cases to obtain a reasonable picture of the stress conditions. It certainly gives a much better stress distribution picture than that obtained from a limit equilibrium analysis. Nonlinear constitutive relationships are often essential if the main interest is deformation but not if the main interest is a stress distribution. Furthermore, even approximate linear–elastic properties are adequate to get a reasonable stress distribution and consequently not a great deal of effort is required to define the linear–elastic parameters.

The results from a simple linear–elastic analysis may mean that the computed stresses in some zones are higher than the available soil strength. This manifests itself as a local factor of safety of less than 1.0 for some slices, which

Fig. 28. Local factors of safety for the toe (left) and the deep (right) slip surfaces.

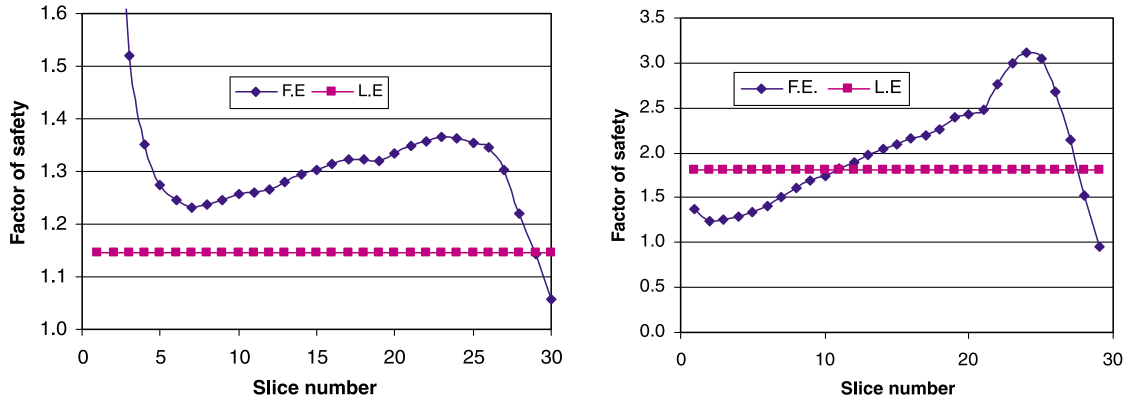


Fig. 29. Grid of safety factors obtained from a finite element stress-based analysis.

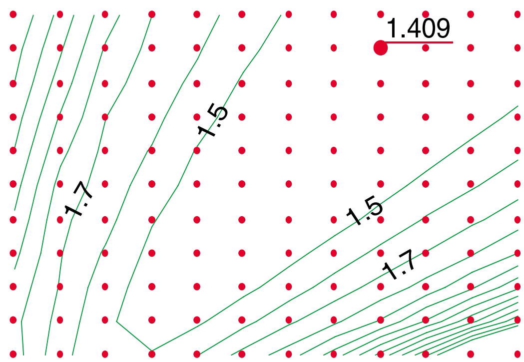
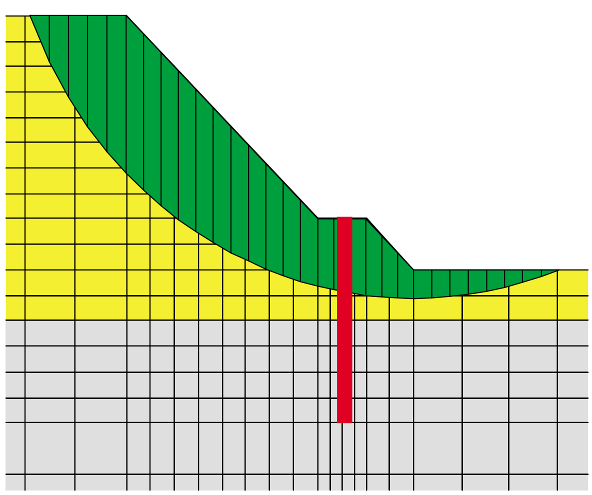
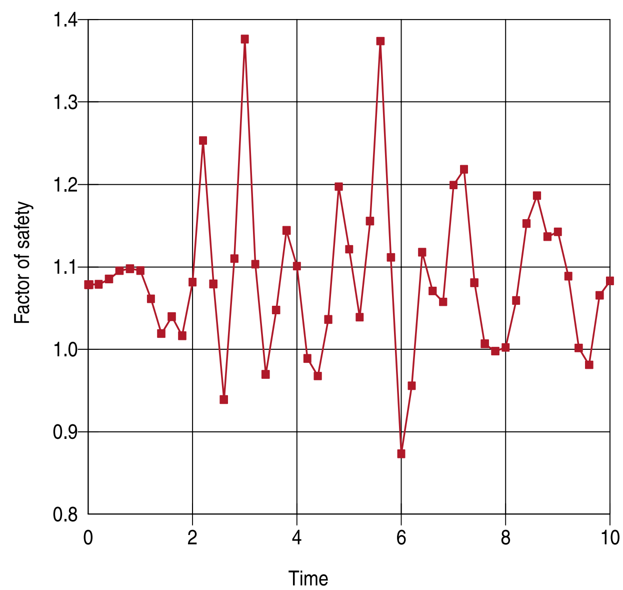


Fig. 30. A shear-key wall for slope stabilization.



is not physically possible. Ideally, nonlinear constitutive models should be used to redistribute the stresses such that the applied stresses do not exceed the strength. However, using nonlinear constitutive relationships greatly complicates the analysis, primarily because of the associated numerical convergence issues. Ignoring local safety factors that are less than unity is not all that serious. Physically, it means that neighboring slices have a local safety factor that is too high. Since all the mobilized and resisting shear forces are tallied

Fig. 31. Factor of safety variations during earthquake shaking.

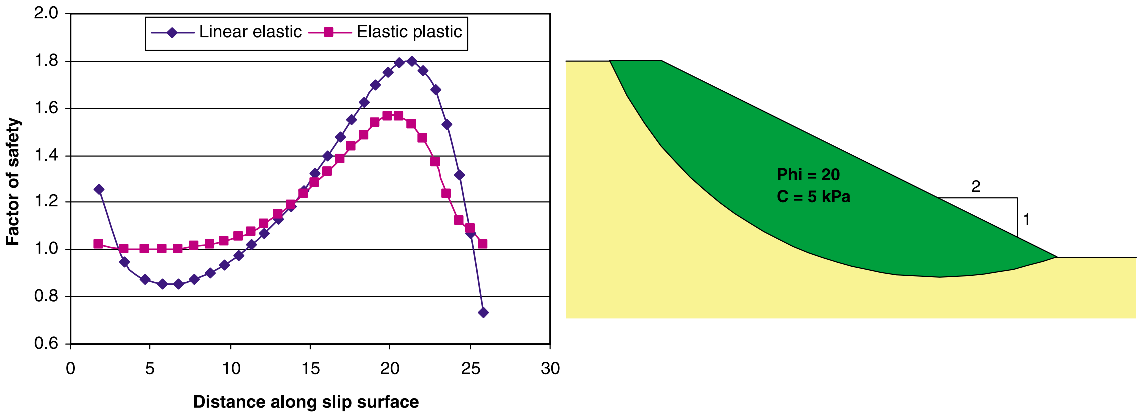


along the entire slip surface, local irregularities are smoothed out and therefore have little effect on the total forces that are used in computing the global factor of safety for the entire sliding mass. This is an indirect form of averaging but not nearly to the extent that is inherent in the limit equilibrium formulation where the factor of safety is the same for all slices.

Figure 32 shows the local factor of safety distribution for a simple 2h:1v slope when the stresses are determined using a linear–elastic analysis and an elastic–plastic analysis. The linear–elastic stresses result in local safety factors less than 1.0. The elastic–plastic analysis redistributes the stresses and then none of the local safety factors are less than 1.0. The global factors of safety are however, nearly identical. For the linear–elastic case the global factor of safety is 1.206 and for the elastic–plastic case the global factor of safety is 1.212, less than half a percent difference.

Using the finite element computed stresses means that the stability calculations now involve the horizontal stresses as well as the vertical stresses. This is good and bad. The good part is that various K_o (σ_x/σ_y ratio) conditions can be considered in a stability analysis. The bad part is that the K_o must

Fig. 32. Local safety factor distributions for linear–elastic and elastic–plastic stress conditions.



be defined. In a linear–elastic gravity turn-on analysis, the ratio of σ_x/σ_y is reflected through Poisson's ratio (ν). For level ground, $K_o = \nu/(l - \nu)$. Different K_o conditions will give different safety factors. Fredlund et al. (1999) studied the effect of varying Poisson's ratio on the factor of safety. Fortunately, defining appropriate K_o conditions is not an impossibility. It is certainly not so difficult as to prevent the use of finite element stresses in assessing stability.

The biggest disadvantage of using a finite element stress-based approach to analyzing stability at this time is not so much technical as it is the lack of experience with the method in geotechnical engineering practice. This will likely change now that the necessary software tools are readily available.

Concluding remarks

Geotechnical limit equilibrium stability analysis techniques, as has been demonstrated in this paper, have limitations. The limitations arise chiefly because the method does not consider strain and displacement compatibility. This has two serious consequences. One is that local variations in safety factors cannot be considered, and the second is that the compute stress distributions are often unrealistic. To allow for variations in local safety factors along the slips surface and to deal with somewhat realistic stresses, the formulation and analysis technique needs to include a stress–strain constitutive relationship. It is the absence of a stress–strain relationship in conventional limit equilibrium analysis methods that is the fundamental piece of missing physics.

The limit equilibrium method for analyzing stability of earth structures remains a useful tool for use in practice in spite of the limitations inherent in the method. Care is required, however, not to abuse the method and apply it to cases beyond its limits. To effectively use limit equilibrium types of analyses, it is vitally important to understand the method, its capabilities, and its limits, and not to expect results that the method is not able to provide. Since the method is based purely on the principles of statics and says nothing about displacement, it is not always possible to obtain realistic stress distributions. This is something the method cannot provide and consequently should not be expected. Fortunately, just because some unrealistic stresses perhaps appear for some slices does not mean the overall factor of safety is necessarily unacceptable. The greatest caution and care is required when stress concentrations exist in the potential sliding mass due to the slip surface shape or due to soil–structure interaction.

A detailed understanding of the method and its limits leads to greater confidence in the use and in the interpretation of the results. Getting to this position means looking at more than just factors of safety. To use the limit equilibrium method effectively, it is also important to examine the detailed slice forces and the variation of parameters along the slip surface, at least sometime during the course of a project. Looking at a FS versus λ plot, for example, is a great aid in deciding how concerned one needs to be about defining an interslice force function.

Also, limit equilibrium analyses applied in practice should as a minimum use a method that satisfies both force and moment equilibrium such as the Morgenstern–Price or Spencer methods. With the software tools now available, it is just as easy to use one of the mathematically more rigorous methods than to use the simpler methods that only satisfy some of the statics equations.

At this stage the proposed integrated method is likely the most applicable where conventional limit equilibrium methods have numerical difficulties as in vertical or near vertical walls with some kind of reinforcement. The method is perhaps not all that applicable in the stability of natural slopes where it is not easy to accurately determine the stresses in the slope because of the complex geological processes that created the slope. In the analysis of natural slopes where the potential slip surface does not have sharp corners and there are no high stress concentrations, the conventional limit equilibrium method is perhaps more than adequate in spite of its limitations.

The tools required to carry out geotechnical stability analyses based on finite element computed stresses are today readily available. Applying the tools is now not only feasible but also practical. Unforeseen issues will possibly arise in the future, but such details will likely be resolved with time as the method is used more and more in geotechnical engineering practice.

Using finite element computed stresses inside a limit equilibrium framework to analyze the stability of geotechnical structures is a major step forward since it overcomes many of the limitations of traditional limit equilibrium methods and yet provides an anchor to the familiarity of limit equilibrium methods so routinely used in current practice.

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