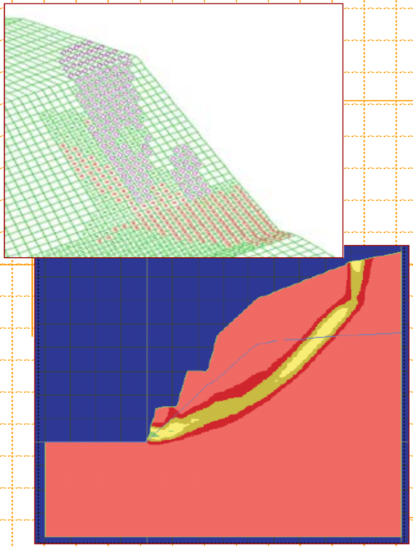


EOSC433/536:

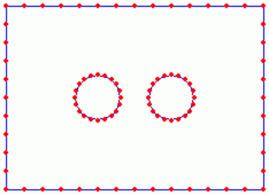
Geological Engineering
Practice I – Rock Engineering

Lecture 10: Deformation Analysis and Elasto-Plastic Yield

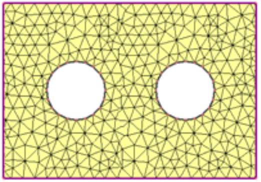
Numerical Modelling

Numerical methods of stress and deformation analysis fall into two categories:

Integral Methods
- incl. boundary-element method
- only problem boundary is defined & discretized
- Pro: more computationally efficient; Con: restricted to elastic analyses



Differential Methods
- incl. finite-element/-difference & distinct-element methods
- problem domain is defined & discretized
- Pro: non-linear & heterogeneous material properties accommodated; Con: longer solution run times



Numerical Analysis – Differential Methods

Differential methods are more difficult and time consuming than boundary analyses (BEM), both in terms of model preparation and solution run times. As such, they require special expertise if they are to be carried out successfully.

… finite-difference method
… finite-element method
} *continuum*

… discrete-element method
… distinct-element method
} *discontinuum*

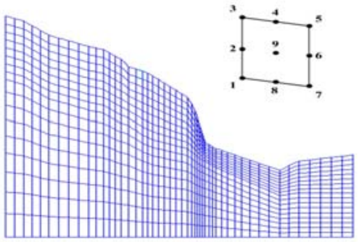
New Considerations (relative to BEM):

- *Division of problem domain (i.e. meshing efficiency & element types).*
- *Selection of appropriate constitutive models (i.e. stress-strain response of elements to applied forces).*
- *Determination of material properties for selected constitutive models (generally derived from lab testing with scaling to field conditions).*
- *Limiting boundary conditions and special loading conditions.*

Numerical Analysis – Differential Methods

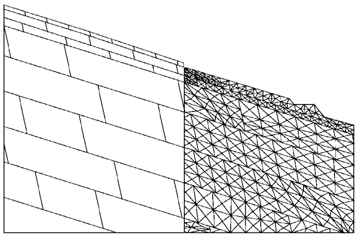
Continuum Methods

- ✓ Rock/soil mass behaviour represented as a continuum.
- ✓ Procedure exploits approximations to the connectivity of elements, and continuity of displacements and stresses between elements.



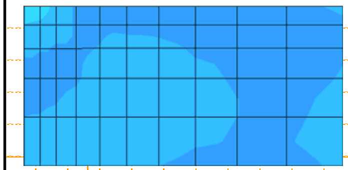
Discontinuum Methods

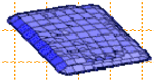
- ✓ Rock mass represented as a assemblage of distinct interacting blocks or bodies.
- ✓ Blocks are subdivided into a deformable finite-difference mesh which follows linear or non-linear stress-strain laws.

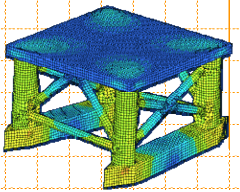


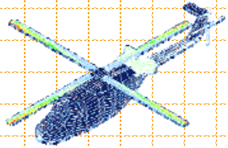
Stead *et al.* (2006)

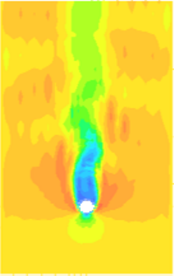
Numerical Analysis – Continuum Methods











Commercial Software:

FLAC (Itasca) - http://www.itascacg.com/
Phase^2 (Rocscience) - http://www.rocscience.com/
DIANA (TNO) - http://www.tnodiana.com/
ELFEN (Rockfield Software Ltd.) - http://www.rockfield.co.uk/
VISAGE (VIPS Ltd.) - http://vips.co.uk/
PLAXIS (PLAXIS BV) - http://www.plaxis.nl/
SVSolid (Soil Vision Systems Ltd.) - http://www.soilvision.com/
ANSYS (ANSYS, Inc.) - http://www.ansys.com/

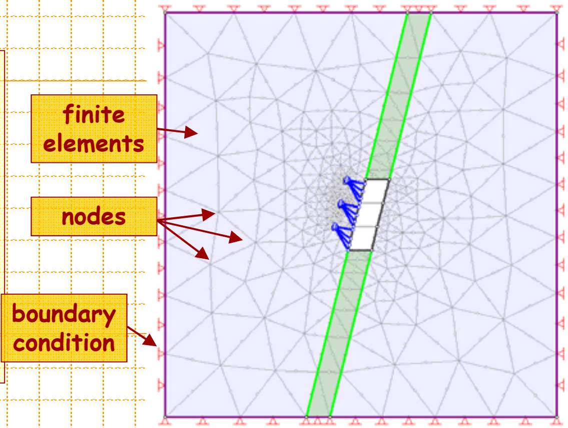
Continuum Analysis – The Basics

Continuum methods divide the rock mass into a set of simple sub-domains called "elements". These elements can be of any geometric shape that allows computation or provides the necessary relation to the values of the solution at selected points called "nodes".

This technique allows:

… accurate representation of complex geometries and inclusion of dissimilar materials.

… accurate representation of the solution within each element, to bring out local effects (e.g. stress or strain concentrations).



Steps in a FEM Solution

Division of the problem domain into parts
(both to represent the geometry
as well as the solution of the problem)

↓

Seek an approximate solution for each part
(using a linear combination of nodal values
and approximation functions)

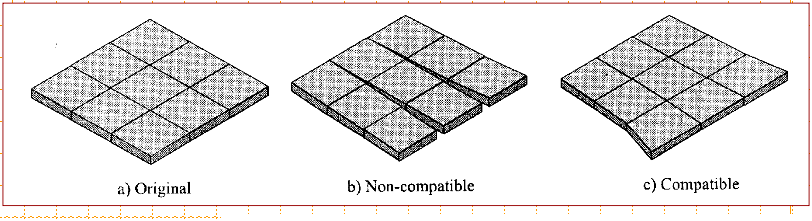
↓

Assemble the parts and solve for the whole
(by deriving the algebraic relations among the nodal values
of the solution over each part)

Basic Formulation of FEM Equations

Piece-wise approximation: The finite-element method has a central requirement that the field quantities (stress, displacement) vary throughout each element in a prescribed fashion using specific functions. As such, the problem domain is represented by an array of small, *interconnected subregions*.

Compatibility → Material Behaviour → Equilibrium


a) Original b) Non-compatible c) Compatible

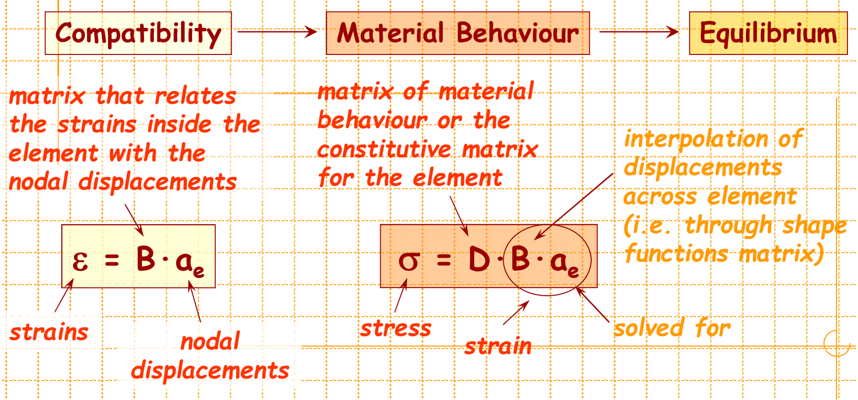
Potts & Zdravković (1999)

Unknowns: 6 stresses + 6 strains + 3 displacements = 15
Equations: 3 equilibrium + 6 compatibility = 9

To obtain a solution therefore requires 6 more equations. These come from the constitutive relationships.

Basic Formulation of FEM Equations

FEM does not solve for a single element, it is assembled and solved as a whole (*FDM, on the other hand, sweeps through a mesh and solves implicitly, element by element*).



Compatibility → **Material Behaviour** → **Equilibrium**

matrix that relates the strains inside the element with the nodal displacements

matrix of material behaviour or the constitutive matrix for the element

interpolation of displacements across element (i.e. through shape functions matrix)

$$\varepsilon = \mathbf{B} \cdot \mathbf{a}_e$$

$$\sigma = \mathbf{D} \cdot \mathbf{B} \cdot \mathbf{a}_e$$

strains

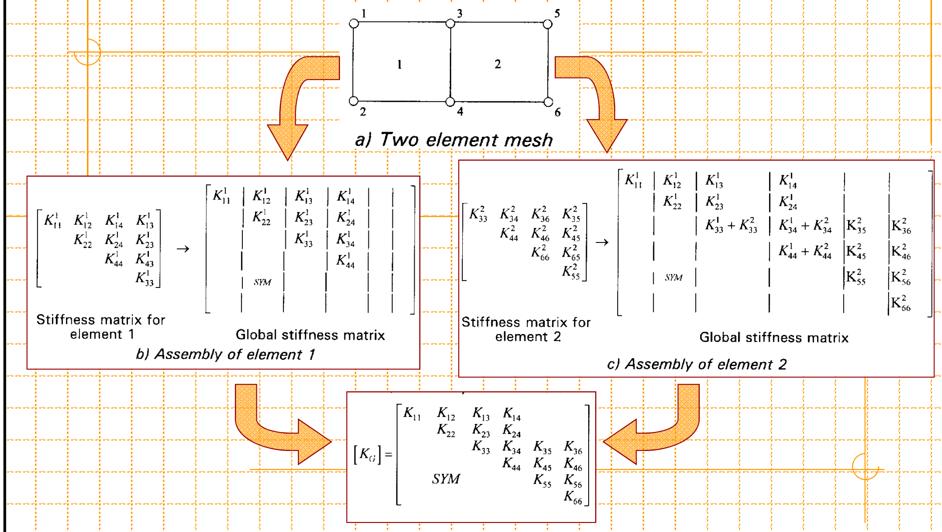
nodal displacements

stress

strain

solved for

Finite-Element Matrix Assembly



a) Two element mesh

$$\begin{bmatrix} K^1_{11} & K^1_{12} & K^1_{14} & K^1_{13} \\ & K^1_{22} & K^1_{24} & K^1_{23} \\ & & K^1_{44} & K^1_{43} \\ & & & K^1_{33} \end{bmatrix} \rightarrow \begin{bmatrix} K^1_{11} & K^1_{12} & K^1_{13} & K^1_{14} & & \\ & K^1_{22} & K^1_{23} & K^1_{24} & & \\ & & K^1_{33} & K^1_{34} & & \\ & & & K^1_{44} & & \\ & SYM & & & & \\ & & & & & \end{bmatrix}$$

Stiffness matrix for element 1

Global stiffness matrix

b) Assembly of element 1

$$\begin{bmatrix} K^2_{33} & K^2_{34} & K^2_{36} & K^2_{35} \\ & K^2_{44} & K^2_{46} & K^2_{45} \\ & & K^2_{66} & K^2_{65} \\ & & & K^2_{55} \end{bmatrix} \rightarrow \begin{bmatrix} K^1_{11} & K^1_{12} & K^1_{13} & K^1_{14} & & \\ & K^1_{22} & K^1_{23} & K^1_{24} & & \\ & & K^1_{33} + K^2_{33} & K^1_{34} + K^2_{34} & K^2_{35} & K^2_{36} \\ & & & K^1_{44} + K^2_{44} & K^2_{45} & K^2_{46} \\ & SYM & & & K^2_{55} & K^2_{56} \\ & & & & & K^2_{66} \end{bmatrix}$$

Stiffness matrix for element 2

Global stiffness matrix

c) Assembly of element 2

$$\left[K_G\right] = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & & \\ & K_{22} & K_{23} & K_{24} & & \\ & & K_{33} & K_{34} & K_{35} & K_{36} \\ & & & K_{44} & K_{45} & K_{46} \\ & SYM & & & K_{55} & K_{56} \\ & & & & & K_{66} \end{bmatrix}$$

Potts & Zdravković (1999)

Numerical Analysis – Continuum Methods

In geotechnical engineering, there are two key continuum-based differential approaches used (to find an approximate solution to a set of partial differential equations):

Finite-Difference

- method is an approximation to the differential equation
- solves a problem on a set of points that form a grid
- easier to implement, but approximation between grid points can be problematic.

Finite-Element

- method is an approximation to the solution of the differential equation
- solves a problem on the interiors of the grid cells (elements) and for the grid points
- can more easily handle complex geometries

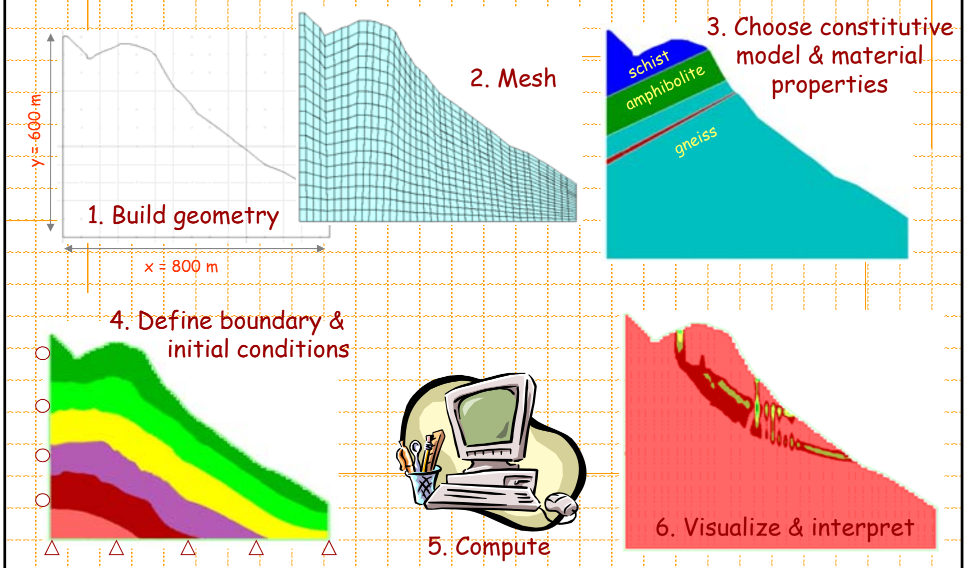
FLAC – Fast Lagrangian Analysis of Continua (by Itasca)

finite difference

Phase^2 (by RocScience)

finite element

Numerical Problem Solving



1. Build geometry

2. Mesh

3. Choose constitutive model & material properties

4. Define boundary & initial conditions

5. Compute

6. Visualize & interpret

Analysis in Geotechnical Design

Geotechnical analyses involve complex systems! Often, field data required for model input (e.g. *in situ* stresses, material properties, geological structure, etc.) are not available or can never be known completely/exactly. This creates *uncertainty*, preventing the models from being used to provide design data (e.g. expected displacements).

Such models, however, may prove useful in providing a picture of the *mechanisms* acting in a particular system. In this role, the model may be used to aid intuition/judgement providing a series of cause-and-effect examples.

Situation	❖ complicated geology ❖ inaccessible ❖ no testing budget	⟺	❖ simple geology ❖ $$$ spent on site investigation
Data	none	⟺	complete (?)
Approach	investigation of failure mechanism(s)	⟺	predictive (design use)

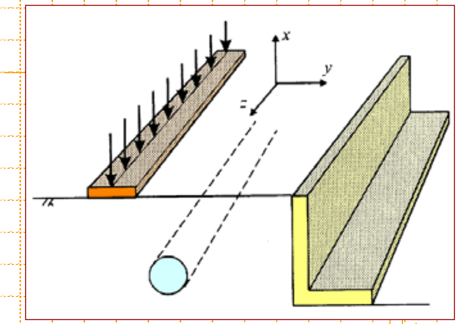
Problem Solving: 2-D or 3-D?

Many geotechnical problems can be assumed to be plane strain (2-D assumption) without significant loss of accuracy of the solution.

… in plane strain, one dimension must be considerably longer than the other two;

… strains along the out-of-plane direction can be assumed to be zero;

… as such, we only have to solve for strains in one 2-D plane.

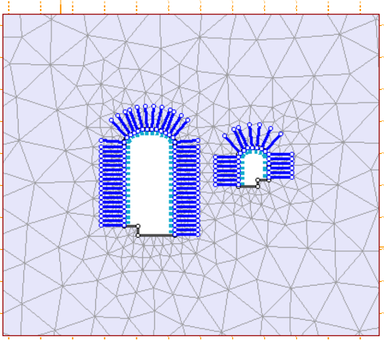


Potts & Zdravković (1999)

Problem Solving: Meshing

The intention of grid generation is to fit the model grid to the physical domain under study. When deciding on the geometric extent of the grid and the number of elements to specify, the following two aspects must be considered:

1. How will the location of the grid boundaries influence model results?
2. What density of zoning is required for an accurate solution in the region of interest?



Do's and Don'ts

- The density of elements should be highest in regions of high stress or strain gradients.

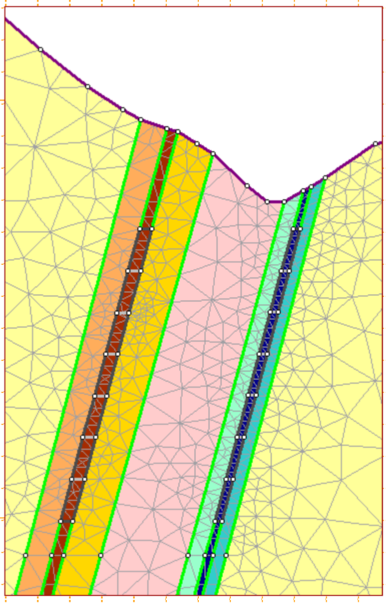
- The greatest accuracy is achieved when the element's aspect ratio is near unity; anything above 10:1 is potentially inaccurate (5:1 for FDM).

- The ratio between adjacent elements should not exceed 4:1 (using a smooth transition to zone from fine to coarse mesh).

Material Properties

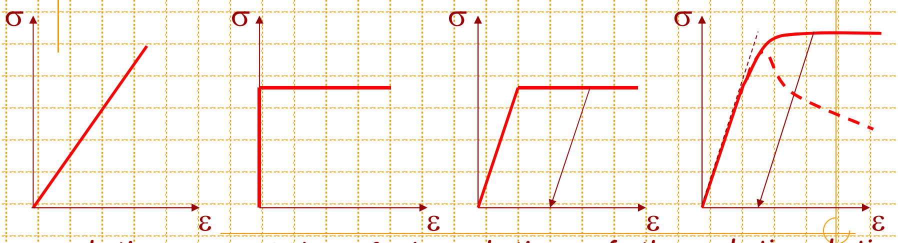
A key advantage of differential methods over integral methods is that by discretizing the problem domain, the assignment of varying material properties throughout a heterogeneous rock mass is permitted.

Material properties required by the chosen constitutive stress-strain relationship are generally derived from laboratory testing programs. Laboratory values should be extrapolated to closely correlate with the actual *in situ* conditions.



Problem Solving: Constitutive Models

During deformation, solid materials undergoes irreversible strains relating to slips at grain/crack boundaries and the opening/closing of pore space/cracks through particle movements. Constitutive relations act to describe, in terms of phenomenological laws, the stress-strain behaviour of these particles in terms of a collective behaviour within a continuum.



elastic

rigid – perfectly plastic

elastic – perfectly plastic

elastic – plastic (strain hardening/softening)

Constitutive Models

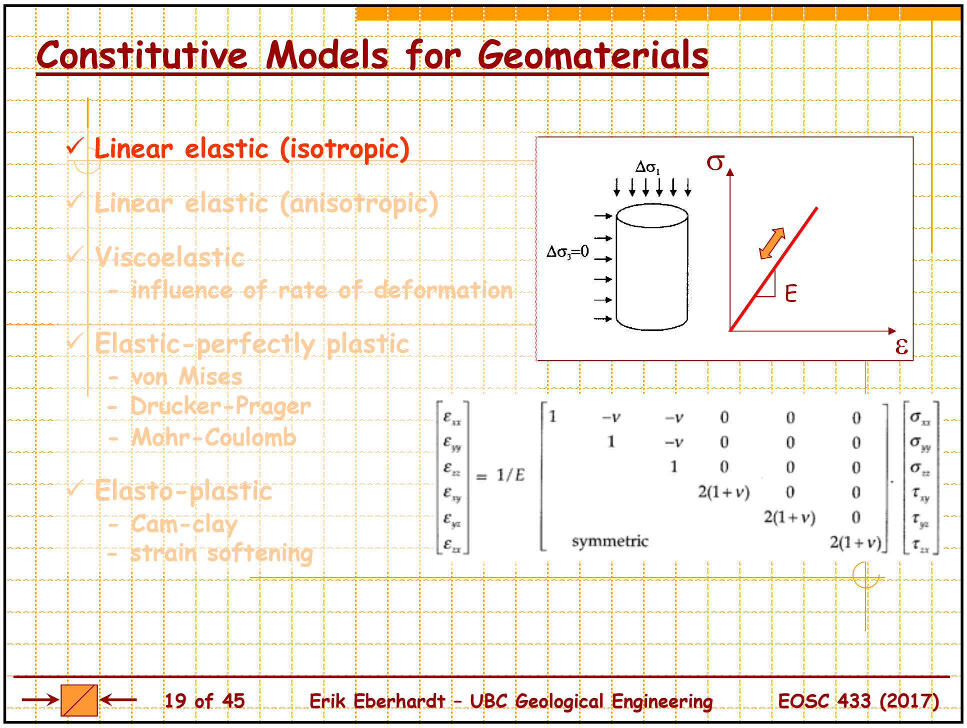
"Most fundamental ideas of science are essentially simple and may, as a rule, be expressed in a language comprehensible to everyone".

Einstein

… the more complex the constitutive model, the more the number of input parameters it requires and the harder it gets to determine these parameters without extensive, high quality (and of course, expensive) laboratory testing;

… as such, one should always begin by using the simplest model that can represent the key behaviour of the problem, and increase the complexity as required.

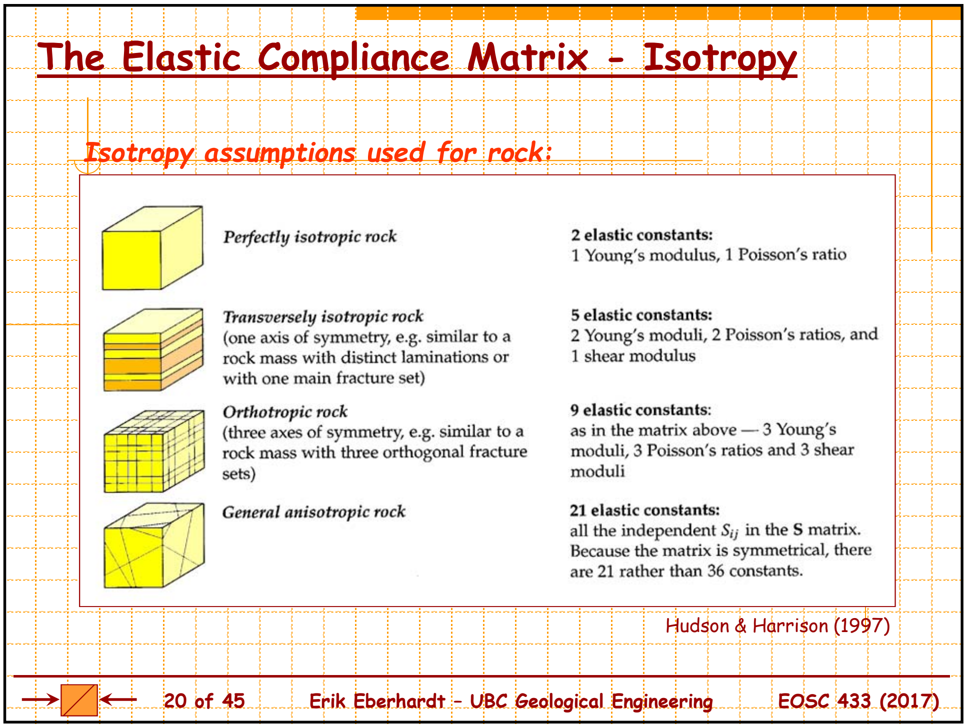
> *"Everything should be made as simple as possible… <u>but not simpler</u>".*
>
> Einstein



Constitutive Models for Geomaterials

- ✓ Linear elastic (isotropic)
- ✓ Linear elastic (anisotropic)
- ✓ Viscoelastic
  - influence of rate of deformation
- ✓ Elastic-perfectly plastic
  - von Mises
  - Drucker-Prager
  - Mohr-Coulomb
- ✓ Elasto-plastic
  - Cam-clay
  - strain softening

$$\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \varepsilon_{xy}\\ \varepsilon_{yz}\\ \varepsilon_{zx}\end{bmatrix} = 1/E \begin{bmatrix}1 & -\nu & -\nu & 0 & 0 & 0\\ & 1 & -\nu & 0 & 0 & 0\\ & & 1 & 0 & 0 & 0\\ & & & 2(1+\nu) & 0 & 0\\ & & & & 2(1+\nu) & 0\\ & \text{symmetric} & & & & 2(1+\nu)\end{bmatrix} \cdot \begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \tau_{xy}\\ \tau_{yz}\\ \tau_{zx}\end{bmatrix}$$



The Elastic Compliance Matrix - Isotropy

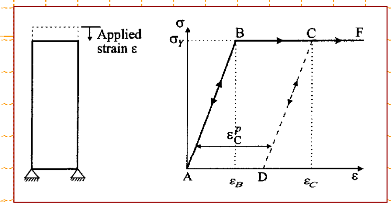
Isotropy assumptions used for rock:

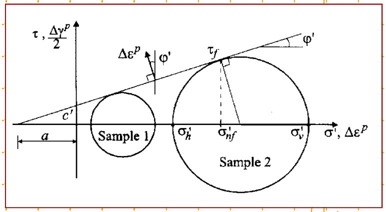
▨	*Perfectly isotropic rock*	**2 elastic constants:** 1 Young's modulus, 1 Poisson's ratio
▨	*Transversely isotropic rock* (one axis of symmetry, e.g. similar to a rock mass with distinct laminations or with one main fracture set)	**5 elastic constants:** 2 Young's moduli, 2 Poisson's ratios, and 1 shear modulus
▨	*Orthotropic rock* (three axes of symmetry, e.g. similar to a rock mass with three orthogonal fracture sets)	**9 elastic constants:** as in the matrix above — 3 Young's moduli, 3 Poisson's ratios and 3 shear moduli
▨	*General anisotropic rock*	**21 elastic constants:** all the independent S_{ij} in the **S** matrix. Because the matrix is symmetrical, there are 21 rather than 36 constants.

Hudson & Harrison (1997)

Constitutive Models for Geomaterials

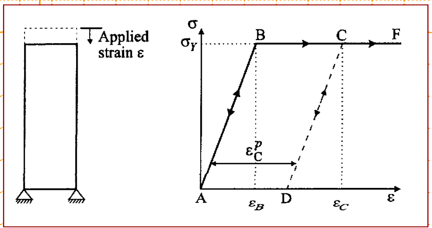
- ✓ Linear elastic (isotropic)
- ✓ Linear elastic (anisotropic)
- ✓ Viscoelastic
  - influence of rate of deformation
- ✓ Elastic-perfectly plastic
  - von Mises
  - Drucker-Prager
  - Mohr-Coulomb
- ✓ Elasto-plastic
  - Cam-clay
  - strain softening





Plasticity: An Introduction

Elastic materials have a unique stress-strain relationship given by the generalized Hooke's law. For many materials, the overall stress-strain response is <u>not</u> unique. Many states of strains can correspond to one state of stress and vice-versa. Such materials are called inelastic or plastic.



… when load is increased, material behaves elastically up to point B, and regains its original state upon unloading.

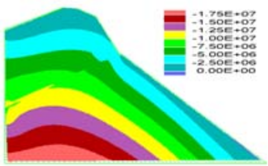
… if the material is stressed beyond point B up to C, and then unloaded, there will be some permanent or irrecoverable deformations in the body, and the material is said to have undergone plastic deformations.

Plasticity & Yield

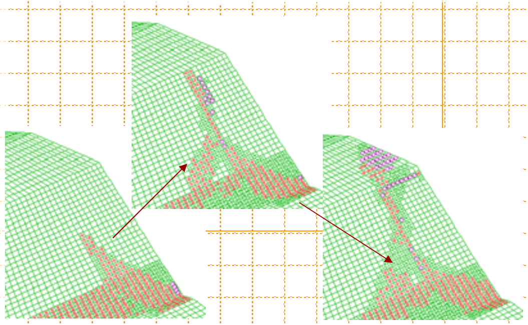
When run elastically, yield and/or failure within the model are not considered/enabled.

An elasto-plastic model allows and solves for yielding within the model (and the resulting displacements that arise). All plastic models potentially involve some degree of permanent, path-dependent deformations. Once an element has reached it's yield state, further increases in stress must be supported by neighbouring elements, which in turn may yield, setting off a chain reaction leading to localization and catastrophic failure.

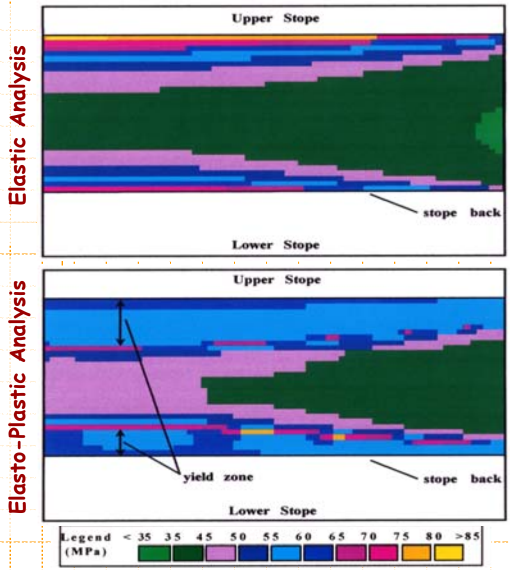
Solution Step 1 - Initial stress state.

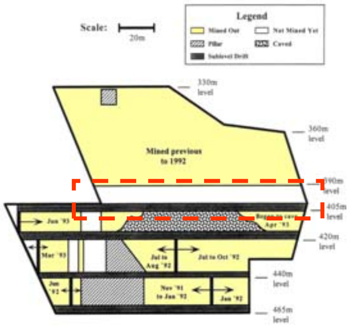


Solution Step 2 - Disequilibrium condition.

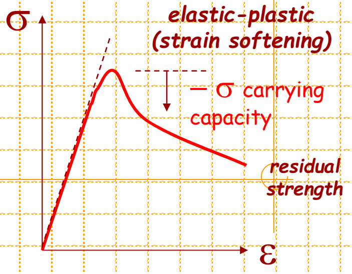


Plasticity & Yield

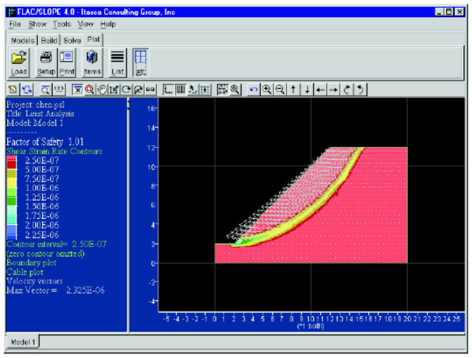




Eberhardt et al. (1997)



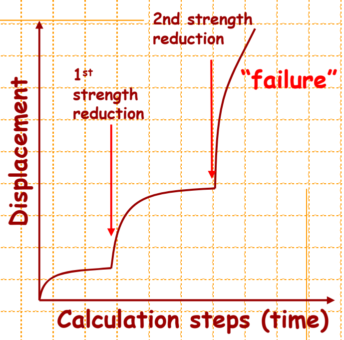
Shear Strength Reduction



Project: chen.psl
Title: Limit Analysis
Model: Model 1

Factor of Safety 1.01
Shear Strain Rate Contours
2.50E-07
5.00E-07
7.50E-07
1.00E-06
1.25E-06
1.50E-06
1.75E-06
2.00E-06
2.25E-06
Contour interval= 2.50E-07
(zero contour omitted)
Boundary plot
Cable plot
Velocity vectors
Max Vector = 2.325E-06

(Itasca - FLAC\Slope)



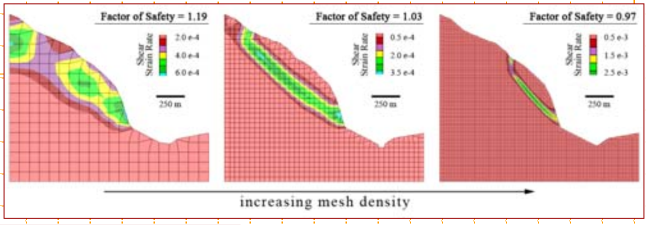
Strength reduction:

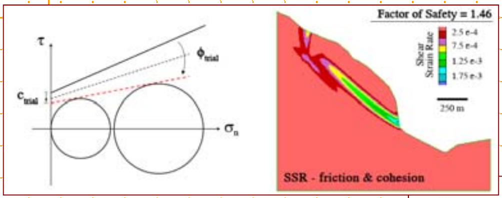
$c_{mob} = c/F \quad \varphi_{mob} = \varphi/F$

The shear strength is reduced until collapse occurs, from which a factor of safety is produced by comparing the estimated shear strength of the material to the reduced/increased shear strength at failure.

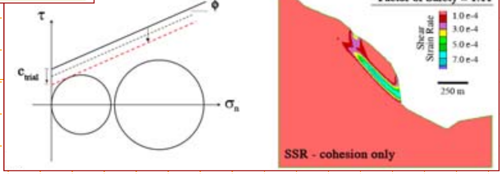
Understanding Shear Strength Reduction

mesh dependency





Eberhardt (2008)



modelling "failure"

Good Modelling Practice

The modelling of geomechanical processes involves special considerations and a design philosophy different from that in other fields of applied mechanics. This is because situations in earth materials often involve limited amounts of input data.

As such, the model should never be considered as a "black box" that accepts data input at one end and produces a prediction at the other. The model should instead be prepared carefully and tested several times in progression of increasing difficulty to gain a full understanding of the problem.

In order to perform a successful numerical study, several steps are recommended:

Step 1 - Define the objectives of the model analysis.
Step 2 - Create a conceptual picture of the physical system.
Step 3 - Construct and run idealized models.
Step 4 - Assemble problem-specific data.
Step 5 - Prepare a series of detailed runs.
Step 6 - Perform the model calculations.
Step 7 - Present results for interpretation.

Input & Assumptions

The fundamental requirement for a meaningful modelling study should include the following steps of data collection/evaluation:

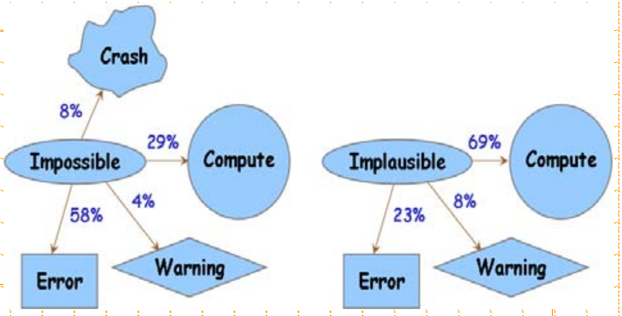
- site characterization (geological and hydrogeological conditions);
- groundwater conditions (pore pressure model);
- geotechnical parameters (strength, deformability, permeability);
- instability mechanisms (kinematics or potential failure modes).

> "if you do not know what you are looking for, you are not likely to find much of value"
>
> *R. Glossop, 8th Rankine Lecture, 1968*

Good Modelling Practice

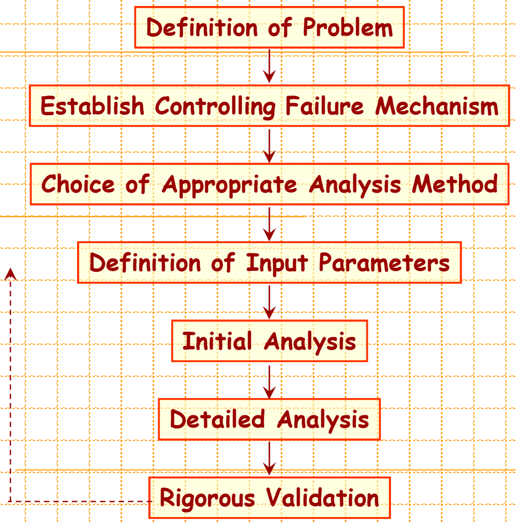
> "numerical modelling should not be used as a substitute for thinking, but as an aid to thought "

… results of a survey of nine commonly used geotechnical modelling programs and their response to impossible (e.g. $E<0$) and implausible ($E_{soil} > E_{rock}$) input data.



Crilly (1993)

Good Modelling Practice



Definition of Problem

Establish Controlling Failure Mechanism

Choice of Appropriate Analysis Method

Definition of Input Parameters

Initial Analysis

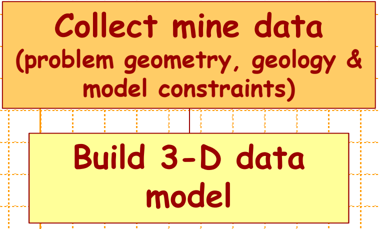
Detailed Analysis

Rigorous Validation

Case Study: Model Verification & Validation

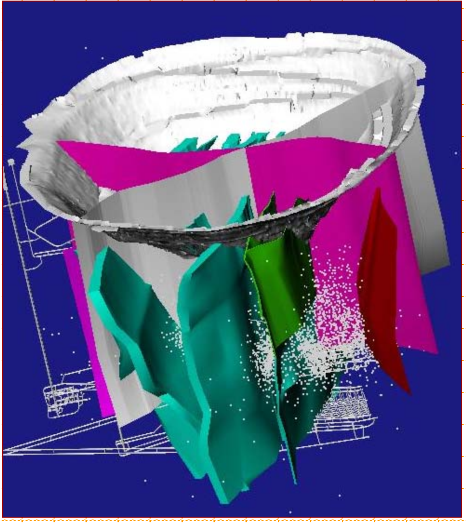
Palabora: Managing geo-risk through improved data integration, model input and constraint of 3-D model uncertainty.

Case Study: Model Verification & Validation



Collect mine data
(problem geometry, geology & model constraints)

Build 3-D data model

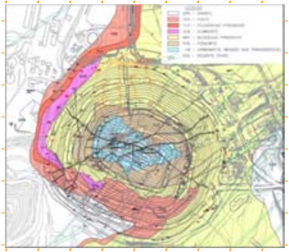


Case Study: Model Verification & Validation

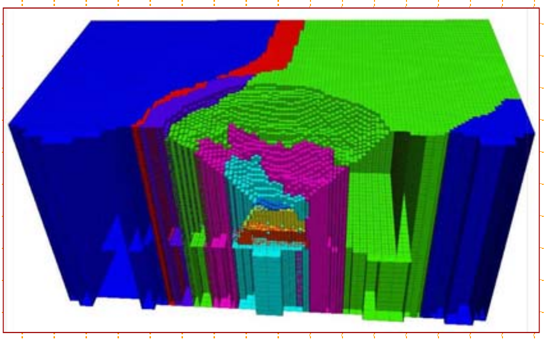
Collect mine data
(problem geometry, geology & model constraints)

Build 3-D data model

Build 3-D numerical model



Woo et al. (2011)



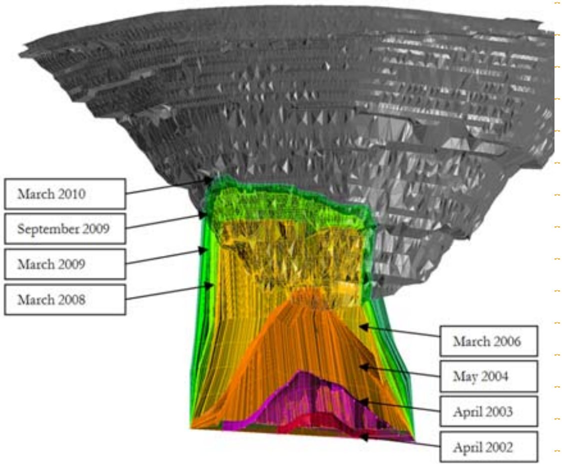
Case Study: Model Verification & Validation

Woo et al. (2011)

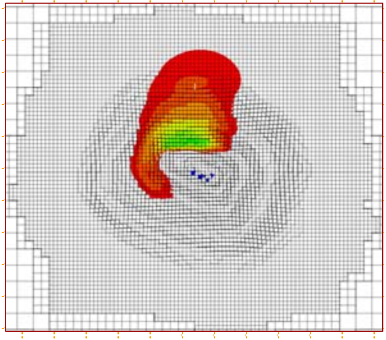
production data

microseismic data

satellite imagery (volume balance)

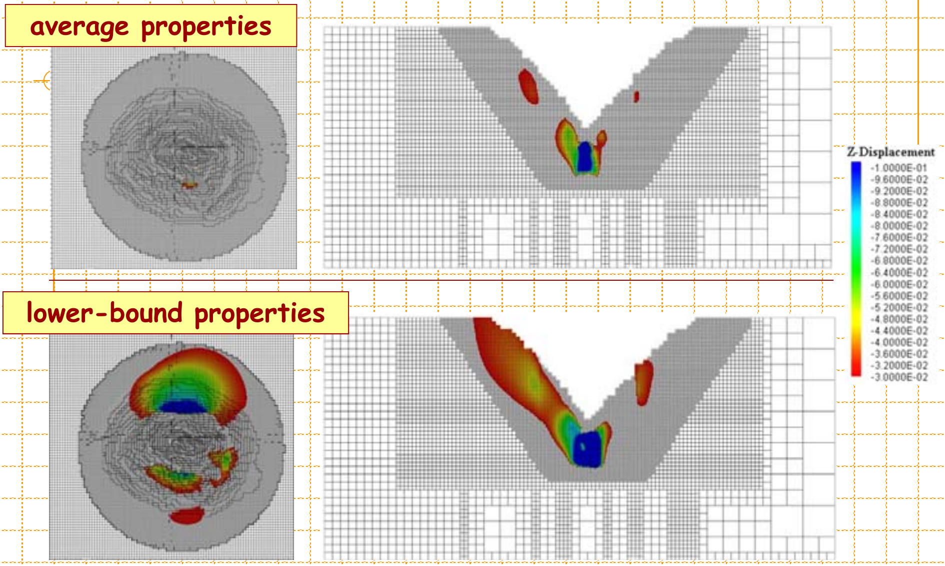


Case Study: Model Verification & Validation



Woo et al. (2011)

Case Study: Model Verification & Validation



average properties

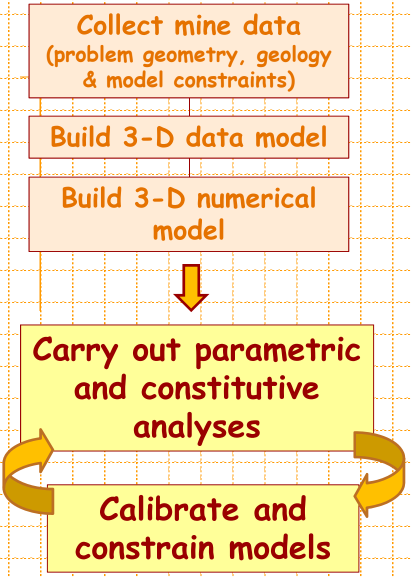
lower-bound properties

Woo et al. (2011)

Case Study: Model Verification & Validation

Model Inputs	Input Confidence	
Surface topography	Good	Digital mine plans
Model boundaries	Good	
Material boundaries	Good	
Cave geometry	Marginal	Limited data
Mesh controls	Good	Sensitivity testing
Constitutive model (for each rock unit)	Poor	Geological uncertainty
Rock mass properties (for each rock unit)	Poor	
In-situ stresses	Marginal	Inconclusive field data

Case Study: Model Verification & Validation



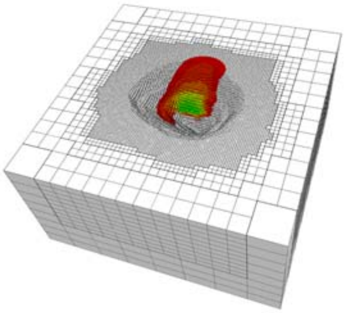
Collect mine data (problem geometry, geology & model constraints)

Build 3-D data model

Build 3-D numerical model

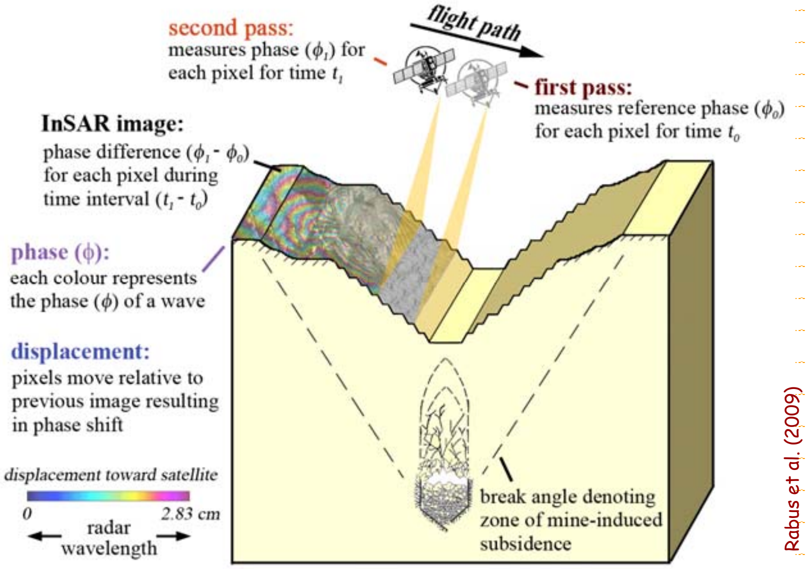
Carry out parametric and constitutive analyses

Calibrate and constrain models



ELASTIC
↓
ELASTO-PLASTIC
↓
UBIQUITOUS JOINT
↓
STRAIN SOFTENING

Case Study: Model Verification & Validation



second pass: measures phase (ϕ_1) for each pixel for time t_1

flight path

first pass: measures reference phase (ϕ_0) for each pixel for time t_0

InSAR image: phase difference ($\phi_1 - \phi_0$) for each pixel during time interval ($t_1 - t_0$)

phase (ϕ): each colour represents the phase (ϕ) of a wave

displacement: pixels move relative to previous image resulting in phase shift

displacement toward satellite

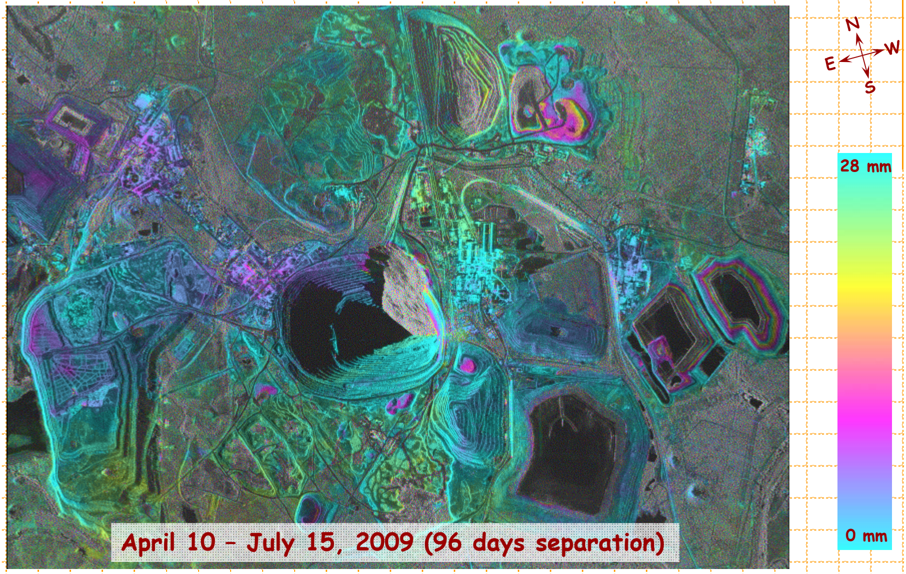
0 *2.83 cm*

radar wavelength

break angle denoting zone of mine-induced subsidence

Rabus et al. (2009)

Case Study: Model Verification & Validation



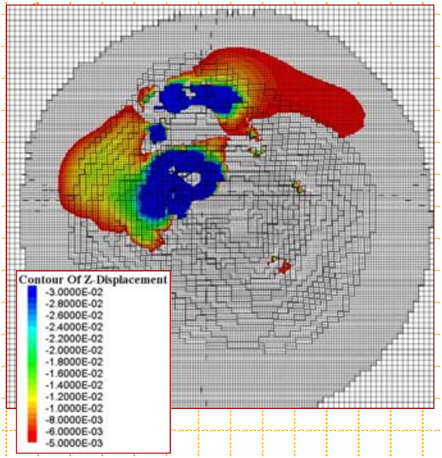
April 10 – July 15, 2009 (96 days separation)

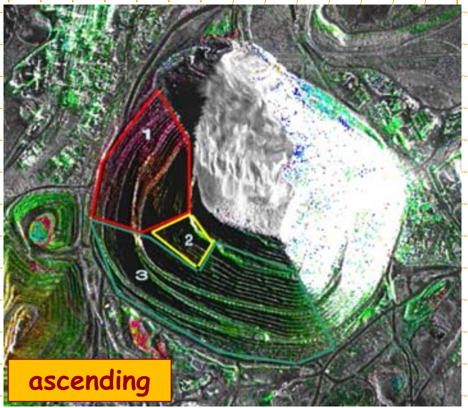
Data courtesy of MDA

Case Study: Model Verification & Validation

Forward Analysis: Predictive Model (2009-2010)

Woo et al. (2011)





10-40mm (*red*)

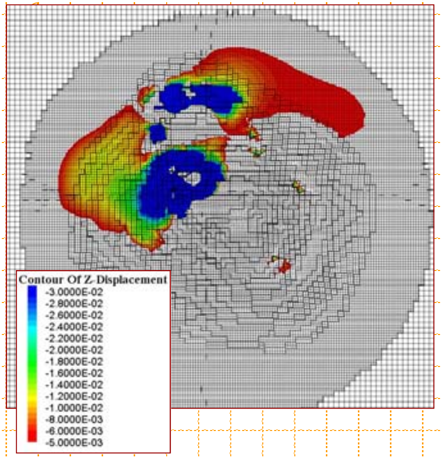
10-20mm (*yellow*)

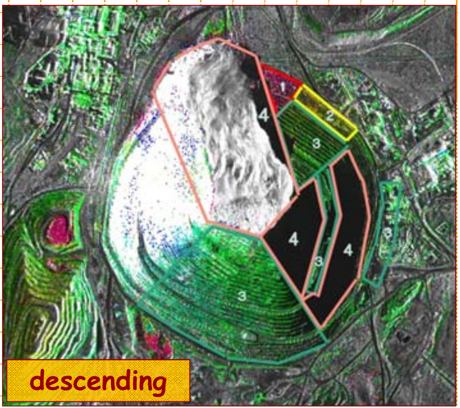
0-10mm (*green*)

Case Study: Model Verification & Validation

Forward Analysis: Predictive Model (2009-2010)

Woo et al. (2011)





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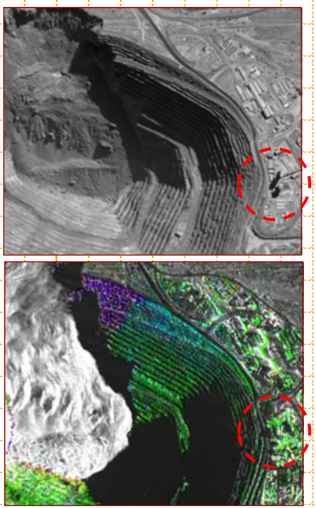
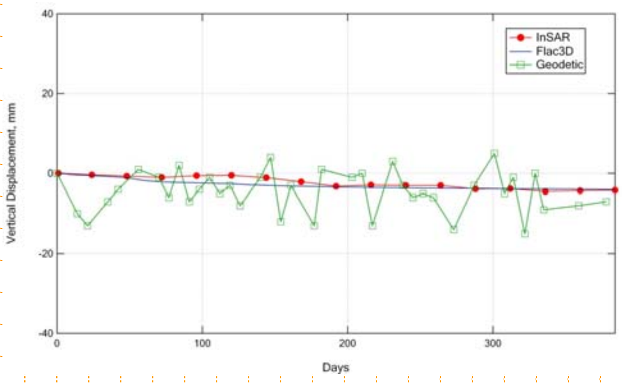
Case Study: Model Verification & Validation



Forward Analysis:
Predictive Model (2009-2010)



Woo et al. (2011)

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