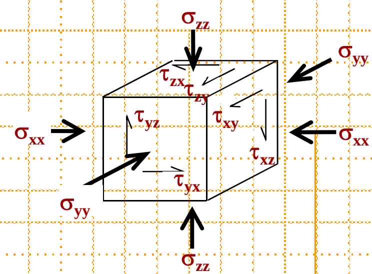
Stress & Strain: A review



Disclaimer before beginning your problem assignment:

Pick up and compare any set of textbooks on rock mechanics, soil mechanics or solid mechanics, and you will find that the discussion on Mohr Circles, stress-strain analysis, matrix math, etc., either uses different conventions or contains a typo that will throw your calculations off. Clockwise is positive, clockwise is negative, mathematical shear strain, engineering shear strain… It all seems rather confusing.

But instead of becoming frustrated or condemning the proof-reader of a given textbook (or these notes), I like to look at it as a good lesson in not relying 100% on something, especially at the expense of your judgement. The notes that follow come from several sources and I have tried to eliminate the errors when I find them. However, when using these notes to complete your problem assignment, try to also use your judgement as to whether the answer you obtain makes sense. If not, consult a different source to double check to see if there was an error.

On that note, if you find an error and/or a source that you would recommend as having given you a clearer understanding of a particular calculation, please let me know.

Understanding Stress

Stress is not familiar: it is a tensor quantity and tensors are not encountered in everyday life.

There is a fundamental difference, both conceptually and mathematically, between a tensor and the more familiar quantities of scalars and vectors:

Scalar: a quantity with magnitude only (e.g. temperature, time, mass).

Vector: a quantity with magnitude and direction (e.g. force, velocity, acceleration).

Tensor: a quantity with magnitude and direction, and with reference to a plane it is acting across (e.g. stress, strain, permeability).

Both mathematical and engineering mistakes are easily made if this crucial difference is not recognized and understood.

The Stress Tensor

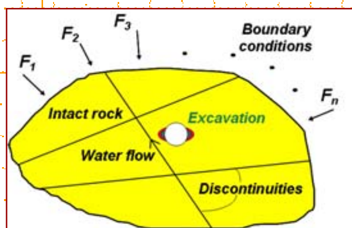
The second-order tensor which we will be examining has:

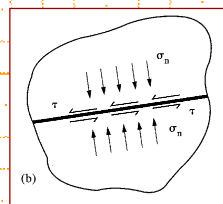
$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

- 9 components of which 6 are independent;
- values which are point properties;
- values which depend on orientation relative to a set of reference axes;
- 6 of the 9 components becoming zero at a particular orientation;
- three principal components;
- complex data reduction requirements because two or more tensors cannot, in general, be averaged by averaging the respective principal stresses.

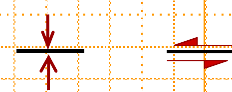
Components of Stress

On a real or imaginary plane through a material, there can be normal forces and shear forces. These forces create the stress tensor. The normal and shear stress components are the normal and shear forces per unit area.





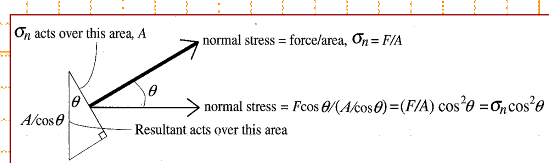
Normal Stress (σ) Shear Stress (τ)



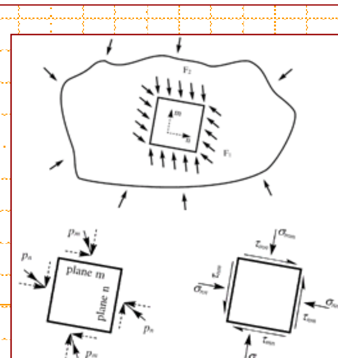
It should be remembered that a solid can sustain a shear force, whereas a liquid or gas cannot. A liquid or gas contains a pressure, which acts equally in all directions and hence is a scalar quantity.

Force and Stress

The reason for this is that it is only the force that is resolved in the first case (i.e. vector), whereas, it is both the force and the area that are resolved in the case of stress (i.e. tensor).



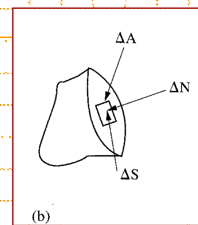
In fact, the strict definition of a second-order tensor is a quantity that obeys certain transformation laws as the planes in question are rotated. This is why the conceptualization of the stress tensor utilizes the idea of magnitude, direction and "the plane in question".



Hudson & Harrison (1997)

Stress as a Point Property

Because the acting forces will vary according to the orientation of ΔA within the slice, it is most useful to consider the normal stress ($\Delta N/\Delta A$) and the shear stress ($\Delta S/\Delta A$) as the area ΔA becomes very small, eventually approaching zero.

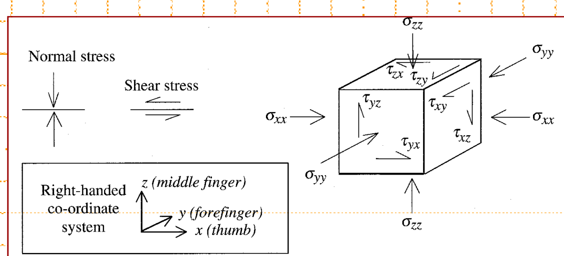


$$\text{normal stress, } \sigma_n = \lim_{\Delta A \to 0} \frac{\Delta N}{\Delta A} \qquad \text{shear stress, } \tau = \lim_{\Delta A \to 0} \frac{\Delta S}{\Delta A}.$$

Although there are practical limitations in reducing the size of the area to zero, it is important to realize that the stress components are defined in this way as mathematical quantities, with the result that stress is a point property.

Stress Components on an Infinitesimal Cube

For convenience, the shear and normal components of stress may be resolved with reference to a given set of axes, usually a rectangular Cartesian x-y-z system. In this case, the body can be considered to be cut at three orientations corresponding to the visible faces of a cube.

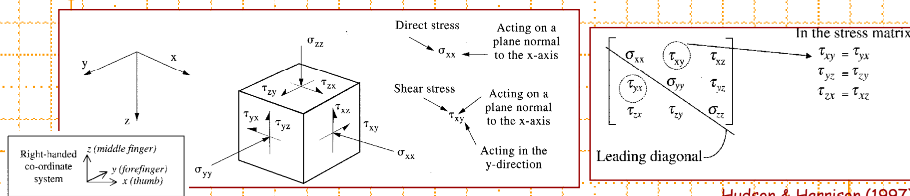


Hudson & Harrison (1997)

To determine all the stress components, we consider the normal and shear stresses on all three planes of this infinitesimal cube.

Stress Tensor Conventions

Thus, we arrive at 9 stress components comprised of 3 normal and 6 shear components.

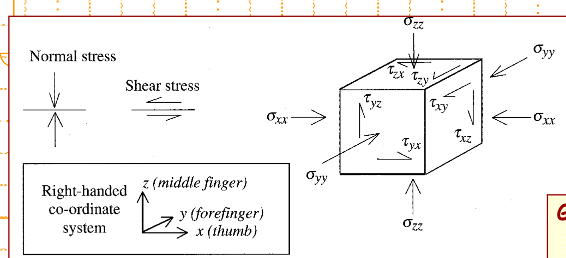


Hudson & Harrison (1997)

The standard convention for denoting these components is that the first subscript refers to the plane on which the stress component acts, and the second subscript denotes the direction in which it acts.

For normal stresses, compression is positive. For shear stresses, positive stresses act in positive directions on negative faces (a negative face is one in which the outward normal to the face points in the negative direction).

Stress Components on a Cube



Geotechnical Engineering
-Right-hand systems
-Compression positive
-Tension negative

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

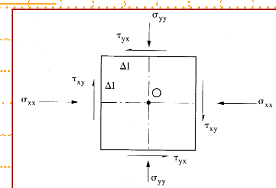
The components in a row are the components acting on a plane; for the first row, the plane on which σ_{xx} acts.

The components in a column are the components acting in one direction; for the first column, the x direction.

Hudson & Harrison (1997)

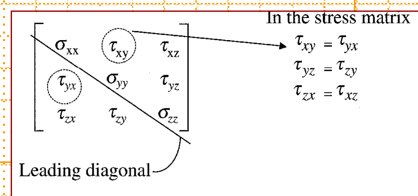
Symmetry in the Stress Matrix

Although we arrive at 9 stress components in the stress matrix, we can assume that the body is in equilibrium. By inspecting the equilibrium of forces at a point in terms of these 9 stress components, we can see that for there to be a resultant moment of zero, then the shear stresses opposite from one another must be equal in magnitude.



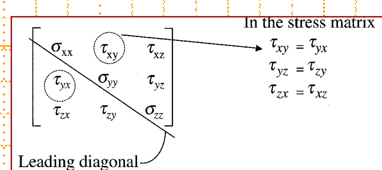
Thus, by considering moment equilibrium around the *x*, *y* and *z* axes, we find that:

$\tau_{xy} = \tau_{yx}$ $\tau_{yz} = \tau_{zy}$ $\tau_{xz} = \tau_{zx}$



Symmetry in the Stress Matrix

If we consider the stress matrix again, we find that it is symmetrical about the leading diagonal.



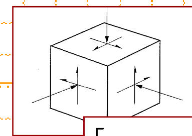
It is clear then that the state of stress at a point is defined completely by six independent components (3 normal and 3 shear).

Remembering back now, it can be noted that a scalar quantity can be completely specified by 1 value and a vector by 3 values, but a tensor requires 6 values.

→ Whatever method is used to specify the stress state, there must be 6 independent pieces of information!!

Principal Stresses

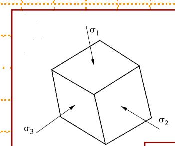
The actual values of the 6 stress components in the stress matrix for a given body subjected to loading will depend on the orientation of the cube in the body itself.



$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

symmetry

If we rotate the cube, it should be possible to find the directions in which the normal stress components take on maximum and minimum values. It is found that in these directions the shear components on all faces of the cube become zero!

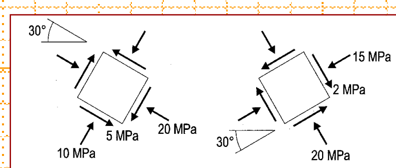


The principal stresses are defined as those normal components of stress that act on planes that have shear stress components with zero magnitude!

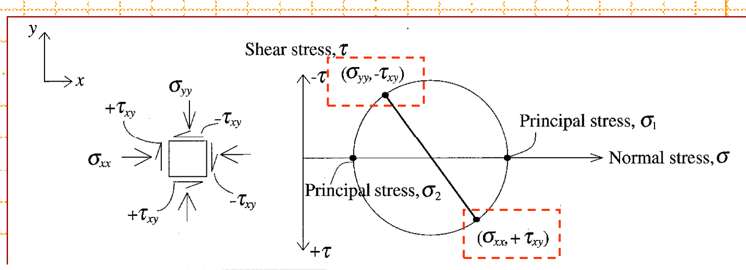
$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

Example #1

Q. Add the following 2-D stress states, and find the principal stresses and directions of the resultant stress state.



A. Hint: Solve the problem graphically using a Mohr's circle plot.

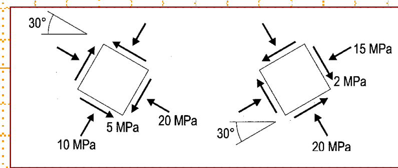


positive shear stresses plot below the σ-axis.

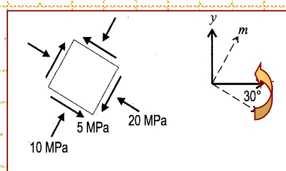
Hudson & Harrison (1997)

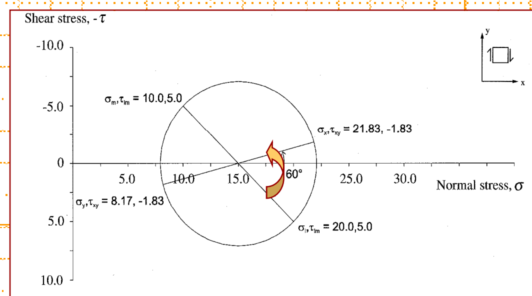
Example #1 (Solution)

Q. Add the following 2-D stress states, and find the principal stresses and directions of the resultant stress state.



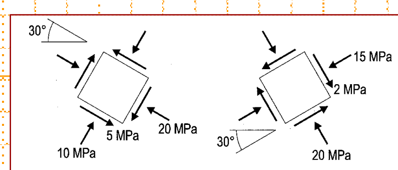
A. <u>Step 1</u>: Draw *xy* and *lm* axes for the first stress state, and then plot the corresponding Mohr circle.



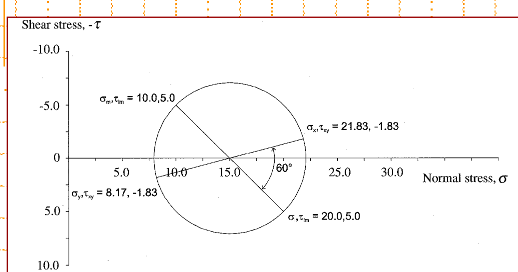


Example #1 (Solution)

Q. Add the following 2-D stress states, and find the principal stresses and directions of the resultant stress state.



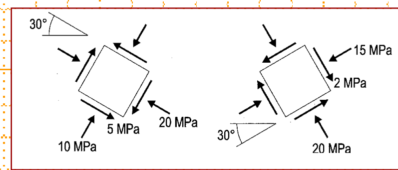
A. The stresses transformed to the *xy* axes are then:



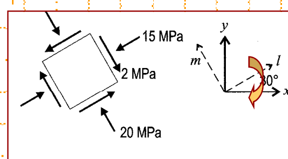
$$\begin{bmatrix} 21.83 & -1.83 \\ -1.83 & 8.17 \end{bmatrix} \text{MPa}$$

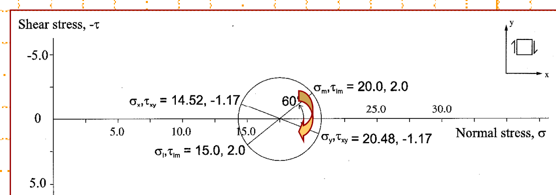
Example #1 (Solution)

Q. Add the following 2-D stress states, and find the principal stresses and directions of the resultant stress state.



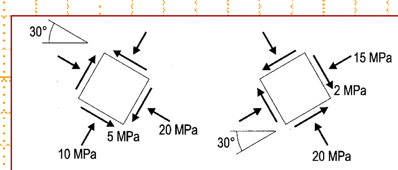
A. **Step 2**: Draw *xy* and *lm* axes for the second stress state, and then plot the corresponding Mohr circle.



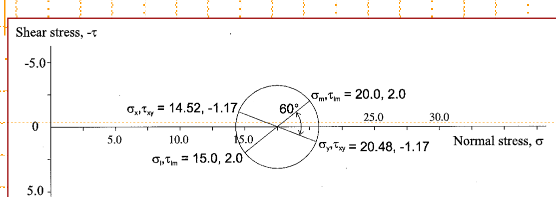


Example #1 (Solution)

Q. Add the following 2-D stress states, and find the principal stresses and directions of the resultant stress state.



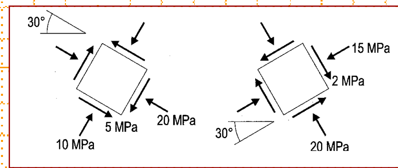
A. The stresses transformed to the *xy* axes are then:



$$\begin{bmatrix} 14.52 & -1.17 \\ -1.17 & 20.48 \end{bmatrix} \text{MPa}$$

Example #1 (Solution)

Q. Add the following 2-D stress states, and find the principal stresses and directions of the resultant stress state.

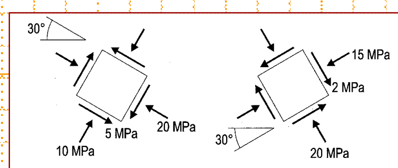


A. Step 3: Adding the two *xy* stress states gives

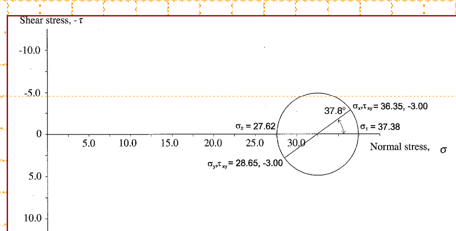
$$\begin{bmatrix} 21.83 & -1.83 \\ -1.83 & 8.17 \end{bmatrix} + \begin{bmatrix} 14.52 & -1.17 \\ -1.17 & 20.48 \end{bmatrix} = \begin{bmatrix} 36.35 & -3.00 \\ -3.00 & 28.65 \end{bmatrix} \text{ MPa}$$

Example #1 (Solution)

Q. Add the following 2-D stress states, and find the principal stresses and directions of the resultant stress state.



A. Step 4: Plotting the Mohr circle for the combined stress state and reading off the principal stresses and the principal directions gives the required values



σ_1 = 37.4 MPa

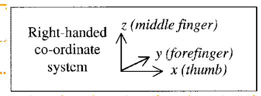
σ_2 = 27.6 MPa

with σ_1 being rotated 19° clockwise from the *x*-direction

Example #2

Q. A stress state has been measured where:

σ_1 = 15 MPa, plunging 35° towards 085°
σ_2 = 10 MPa, plunging 43° towards 217°
σ_3 = 8 MPa, plunging 27° towards 335°



Find the 3-D stress tensor in the right-handed *x-y-z* coordinate system with *x* horizontal to the east, *y* horizontal to the north and *z* vertically upwards.

A. Perhaps before proceeding with this problem it would help to review some matrix math.

Stress Transformation – Step 1

The matrix equation to conduct stress transformation is as follows:

$$\begin{bmatrix} \sigma_l & \tau_{ml} & \tau_{nl} \\ \tau_{lm} & \sigma_m & \tau_{nm} \\ \tau_{ln} & \tau_{mn} & \sigma_n \end{bmatrix} = \begin{bmatrix} l_x & l_y & l_z \\ m_x & m_y & m_z \\ n_x & n_y & n_z \end{bmatrix} \begin{bmatrix} \sigma_x & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \begin{bmatrix} l_x & m_x & n_x \\ l_y & m_y & n_y \\ l_z & m_z & n_z \end{bmatrix}$$

… where the stress components are assumed known in the *x-y-z* coordinate system and are required in another coordinate system *l-m-n* inclined with respect to the first.

The term l_x is the direction cosine of the angle between the *x*-axis and *l*-axis. Physically, it is the projection of a unit vector parallel to *l* on to the *x*-axis, with the other terms similarly defined.

Stress Transformation – Step 2

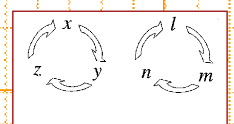
Expanding this matrix equation in order to obtain expressions for the normal component of stress in the *l*-direction and shear on the *l*-face in the *m*-direction gives:

$$\begin{bmatrix} \sigma_l & \tau_{ml} & \tau_{nl} \\ \tau_{lm} & \sigma_m & \tau_{nm} \\ \tau_{ln} & \tau_{mn} & \sigma_n \end{bmatrix} = \begin{bmatrix} l_x & l_y & l_z \\ m_x & m_y & m_z \\ n_x & n_y & n_z \end{bmatrix} \begin{bmatrix} \sigma_x & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \begin{bmatrix} l_x & m_x & n_x \\ l_y & m_y & n_y \\ l_z & m_z & n_z \end{bmatrix}$$

$$\sigma_l = l_x^2\sigma_{xx} + l_y^2\sigma_{yy} + l_z^2\sigma_{zz} + 2(l_x l_y \sigma_{xy} + l_y l_z \sigma_{yz} + l_z l_x \sigma_{zx})$$

$$\sigma_{lm} = l_x m_x \sigma_{xx} + l_y m_y \sigma_{yy} + l_z m_z \sigma_{zz} + (l_x m_y + l_y m_x)\,\sigma_{xy} + (l_y m_z + l_z m_y)\,\sigma_{yz} + (l_z m_x + l_x m_z)\,\sigma_{zx}$$

The other four necessary equations (i.e. for σ_y, σ_z, τ_{yz} and τ_{zx}) are found using cyclic permutation of the subscripts in the equations above.



Stress Transformation – Step 3

It is generally most convenient to refer to the orientation of a plane on which the components of stress are required using dip direction/dip angle notation (α, β). The dip direction is measured clockwise bearing from North and the dip angle is measured downwards from the horizontal plane.

If we use a right-handed coordinate system with *x* = north, *y* = east and *z* = down, and take *n* as the normal to the desired plane, then:

$$n_x = \cos\alpha_n \cos\beta_n; \;\; n_y = \sin\alpha_n \cos\beta_n; \;\; n_z = \sin\beta_n.$$

And the rotation matrix becomes:

$$\begin{bmatrix} l_x & l_y & l_z \\ m_x & m_y & m_z \\ n_x & n_y & n_z \end{bmatrix} = \begin{bmatrix} \cos\alpha_l \cos\beta_l & \sin\alpha_l \cos\beta_l & \sin\beta_l \\ \cos\alpha_m \cos\beta_m & \sin\alpha_m \cos\beta_m & \sin\beta_m \\ \cos\alpha_n \cos\beta_n & \sin\alpha_n \cos\beta_n & \sin\beta_n \end{bmatrix}$$

Note that right-handed systems are always used for mathematical work. There are two obvious choices for a right-handed system of axes: *x* East, *y* North and *z* up; or *x* North, *y* East and *z* down. There are advantages to both, and so being adept with both is important.

Example #2 (Solution)

Q. A stress state has been measured where:

σ_1 = 15 MPa, plunging 35° towards 085°
σ_2 = 10 MPa, plunging 43° towards 217°
σ_3 = 8 MPa, plunging 27° towards 335°

Find the 3-D stress tensor in the right-handed *xyz* co-ordinate system with *x*, horizontal to the east; *y*, horizontal to the north; and *z*, vertically upwards.

A. Step 1: The stress transformation equations are given by

$$\begin{bmatrix} \sigma_l & \tau_{ml} & \tau_{nl} \\ \tau_{lm} & \sigma_m & \tau_{nm} \\ \tau_{ln} & \tau_{mn} & \sigma_n \end{bmatrix} = \begin{bmatrix} l_x & l_y & l_z \\ m_x & m_y & m_z \\ n_x & n_y & n_z \end{bmatrix} \begin{bmatrix} \sigma_x & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \begin{bmatrix} l_x & m_x & n_x \\ l_y & m_y & n_y \\ l_z & m_z & n_z \end{bmatrix}$$

which can be written as $\sigma_{lmn} = \mathbf{R}\sigma_{xyz}\mathbf{R}^T$

Example #2 (Solution)

Q. A stress state has been measured where:

σ_1 = 15 MPa, plunging 35° towards 085°
σ_2 = 10 MPa, plunging 43° towards 217°
σ_3 = 8 MPa, plunging 27° towards 335°

Find the 3-D stress tensor (where *x*=east, *y*=north, *z*=up).

A. The equation $\sigma_{lmn} = \mathbf{R}\sigma_{xyz}\mathbf{R}^T$ means that, if we know the stresses relative to the *xyz* axes (i.e. σ_{xyz}) and the orientation of the *lmn* axes relative to the *xyz* axes (i.e. **R**), we can then compute the stresses relative to the *lmn* axes (i.e. σ_{lmn}).

However, in this problem we have been given the principal stresses, which is a stress state relative to some *lmn* system (i.e. σ_{lmn}), where the *lmn* axes correspond to the principal directions.

Example #2 (Solution)

Q. A stress state has been measured where:

σ_1 = 15 MPa, plunging 35° towards 085°
σ_2 = 10 MPa, plunging 43° towards 217°
σ_3 = 8 MPa, plunging 27° towards 335°

Find the 3-D stress tensor (where x=east, y=north, z=up).

A. As we know the principal directions relative to the *xyz* axes, we are able to compute **R**. Thus, we need to evaluate σ_{lxyz}, and we do this using the inverse of the stress transformation equations:

$$\sigma_{xyz} = \mathbf{R}^T \sigma_{lmn} \mathbf{R}$$

Notice that since the rotation matrix is orthogonal, we do not need to use the inverse of **R**, i.e. $\mathbf{R}^{-1}$, and thus $\mathbf{R}^{-1} = \mathbf{R}^T$.

Example #2 (Solution)

Q. A stress state has been measured where:

σ_1 = 15 MPa, plunging 35° towards 085°
σ_2 = 10 MPa, plunging 43° towards 217°
σ_3 = 8 MPa, plunging 27° towards 335°

Find the 3-D stress tensor.

A. Step 2: With the given data for the principal directions …

$$\alpha_l = 85^\circ \quad \alpha_m = 217^\circ \quad \alpha_n = 335^\circ$$
$$\beta_l = 35^\circ \quad \beta_m = 43^\circ \quad \beta_n = 27^\circ$$

… the matrix **R** is computed as:

$$\mathbf{R} = \begin{bmatrix} \cos\alpha_l \cos\beta_l & \sin\alpha_l \cos\beta_l & \sin\beta_l \\ \cos\alpha_m \cos\beta_m & \sin\alpha_m \cos\beta_m & \sin\beta_m \\ \cos\alpha_n \cos\beta_n & \sin\alpha_n \cos\beta_n & \sin\beta_n \end{bmatrix} = \begin{bmatrix} 0.071 & 0.816 & 0.574 \\ -0.584 & -0.440 & 0.682 \\ 0.808 & -0.377 & 0.454 \end{bmatrix}$$

Example #2 (Solution)

Q. A stress state has been measured where:

σ_1 = 15 MPa, plunging 35° towards 085°
σ_2 = 10 MPa, plunging 43° towards 217°
σ_3 = 8 MPa, plunging 27° towards 335°

Find the 3-D stress tensor (where *x*=east, *y*=north, *z*=up).

A. Step 3: Since the matrix σ_{lmn} is given by …

$$\sigma_{lmn} = \begin{bmatrix} 15 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 8 \end{bmatrix} \text{MPa}$$

… the 3-D stress tensor, $\sigma_{xyz} = R^T \sigma_{lmn} R$ solves as:

$$\sigma_{xyz} = \begin{bmatrix} 8.70 & 1.01 & -0.44 \\ 1.01 & 13.06 & 2.65 \\ -0.44 & 2.65 & 11.23 \end{bmatrix} \text{MPa.}$$

Example #2 (Solution)

Q. A stress state has been measured where:

σ_1 = 15 MPa, plunging 35° towards 085°
σ_2 = 10 MPa, plunging 43° towards 217°
σ_3 = 8 MPa, plunging 27° towards 335°

Find the 3-D stress tensor (where *x*=east, *y*=north, *z*=up).

A. … or: if orientation and matrix values are not rounded, a more accurate answer may be obtained…

… the 3-D stress tensor, $\sigma_{xyz} = R^T \sigma_{lmn} R$ then solves more exactly as:

$$\sigma_{xyz} = \begin{bmatrix} 8.71 & 0.88 & -0.54 \\ 0.88 & 13.04 & 2.69 \\ -0.54 & 2.69 & 11.24 \end{bmatrix} \text{MPa.}$$

Example #3

Q. For our previously given rock mass with the stress state:

σ_1 = 15 MPa, plunging 35° towards 085°
σ_2 = 10 MPa, plunging 43° towards 217°
σ_3 = 8 MPa, plunging 27° towards 335°

… a fault has been mapped with an orientation of 295°/50°. Determine the stress components in a local coordinate system aligned with the fault. Assume for this question that the presence of the fault does not affect the stress field.

A. Hint: Here we use the same methodology to find the 3-D stress tensor in an *lmn* coordinate system where the *n*-axis coincides with the normal to the fault and the *l*-axis coincides with the strike of the fault.

Example #3 (Solution)

Q. Determine the stress components in a local coordinate system aligned with the fault.

σ_1 = 15 MPa, plunging 35° towards 085°
σ_2 = 10 MPa, plunging 43° towards 217°
σ_3 = 8 MPa, plunging 27° towards 335°
fault orientation = 295°/50°.

A. Step 1: We therefore need to determine σ_{lmn}, where *lmn* are given by the orientation of the fault. With the *l*-axis parallel to the strike of the plane and the *n*-axis normal to the plane, the *m*-axis becomes the dip. The tend and plunge of each axes are then as follows:

$$\alpha_l = 205^\circ \quad \alpha_m = 295^\circ \quad \alpha_n = 115^\circ$$
$$\beta_l = 0^\circ \quad \beta_m = 50^\circ \quad \beta_n = 40^\circ$$

Example #3 (Solution)

Q. Determine the stress components in a local coordinate system aligned with the fault.

σ_1 = 15 MPa, plunging 35° towards 085°
σ_2 = 10 MPa, plunging 43° towards 217°
σ_3 = 8 MPa, plunging 27° towards 335°
fault orientation = 295°/50°.

A. Step 2: The matrix R, then computes as:

$$\mathbf{R} = \begin{bmatrix} \cos\alpha_l \cos\beta_l & \sin\alpha_l \cos\beta_l & \sin\beta_l \\ \cos\alpha_m \cos\beta_m & \sin\alpha_m \cos\beta_m & \sin\beta_m \\ \cos\alpha_n \cos\beta_n & \sin\alpha_n \cos\beta_n & \sin\beta_n \end{bmatrix} \rightarrow \begin{bmatrix} -0.906 & -0.423 & 0.000 \\ 0.272 & -0.583 & 0.766 \\ -0.324 & 0.694 & 0.643 \end{bmatrix}$$

Example #3 (Solution)

Q. Determine the stress components in a local coordinate system aligned with the fault.

σ_1 = 15 MPa, plunging 35° towards 085°
σ_2 = 10 MPa, plunging 43° towards 217°
σ_3 = 8 MPa, plunging 27° towards 335°
fault orientation = 295°/50°.

A. Step 3: From Q #2, the matrix σ_{xyz} is:

$$\begin{bmatrix} 8.70 & 1.01 & -0.44 \\ 1.01 & 13.06 & 2.65 \\ -0.44 & 2.65 & 11.23 \end{bmatrix}.$$

Step 4: As a result, the matrix $\sigma_{lmn} = R\sigma_{xyz}R^T$, the solution to which is:

$$\begin{bmatrix} 10.26 & 0.94 & -2.24 \\ 0.94 & 8.80 & 0.32 \\ -2.24 & 0.32 & 13.94 \end{bmatrix} \text{MPa.}$$

Example #3 (Solution)

Q. Determine the stress components in a local coordinate system aligned with the fault.

σ_1 = 15 MPa, plunging 35° towards 085°
σ_2 = 10 MPa, plunging 43° towards 217°
σ_3 = 8 MPa, plunging 27° towards 335°
fault orientation = 295°/50°.

A. … or: if orientation and matrix values are not rounded, a more accurate answer may be obtained. the 3-D stress tensor, $\sigma_{lmn} = \mathbf{R}\sigma_{xyz}\mathbf{R}^T$ then solves more exactly as:

$$\sigma_{lmn} = \mathbf{R}\begin{bmatrix} 8.71 & 0.88 & -0.54 \\ 0.88 & 13.04 & 2.69 \\ -0.54 & 2.69 & 11.24 \end{bmatrix}\mathbf{R}^T = \begin{bmatrix} 10.16 & 0.93 & -2.12 \\ 0.93 & 8.76 & 0.29 \\ -2.12 & 0.29 & 14.07 \end{bmatrix} \text{MPa.}$$

Example #4

Q. Six components of stress are measured at a point:

σ_{xx} = 14.0 MPa τ_{xy} = -0.6 MPa
σ_{yy} = 34.8 MPa τ_{yz} = 6.0 MPa
σ_{zz} = 16.1 MPa τ_{xz} = -2.1 MPa

Determine the principal stresses and their direction cosines.

A. Before proceeding with this problem, we must define the invariants of stress.

Stress Invariants – Step 1

When the stress tensor is expressed with reference to sets of axes oriented in different directions, the components of the tensor change. However, certain functions of the components do not change. These are known as stress invariants, expressed as I_1, I_2, I_3, where:

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$$

$$I_2 = \sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2$$

$$I_3 = \sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_{xx}\tau_{yz}^2 - \sigma_{yy}\tau_{zx}^2 - \sigma_{zz}\tau_{xy}^2$$

The expression for the first invariant, I_1, indicates that for a given stress state, whatever its orientation, the values of the three normal stresses will add up to the same value I_1.

Stress Invariants – Step 2

When the principal stresses have to be calculated from the components of the stress tensor, a cubic equation can be used for finding the three values σ_1, σ_2, σ_3:

$$\sigma^3 - (\sigma_{xx} + \sigma_{yy} + \sigma_{zz})\sigma^2 + (\sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{xx}\sigma_{zz} - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2)\sigma -$$
$$(\sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_{xx}\tau_{yz}^2 - \sigma_{yy}\tau_{zx}^2 - \sigma_{zz}\tau_{xy}^2) = 0$$

or

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

Because the values of the principal stresses must be independent of the choice of axes, the coefficients I_1, I_2, I_3 must be invariant with respect to the orientation of the axes. It can also be noted from the first invariant that:

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = \sigma_1 + \sigma_2 + \sigma_3$$

Stress Invariants – Step 3

Each principal stress is related to a principal stress axis, whose direction cosines can be obtained, for example for σ_1, through a set of simultaneous, homogeneous equations in λ_{x1}, λ_{y1}, λ_{z1}, based on the dot product theorem of vector analysis:

$$\frac{\lambda_{x1}}{A} = \frac{\lambda_{y1}}{B} = \frac{\lambda_{z1}}{C} = K$$

Where:

$$A = \begin{vmatrix} \sigma_{yy}-\sigma_1 & \sigma_{yz} \\ \sigma_{yz} & \sigma_{zz}-\sigma_1 \end{vmatrix} \qquad B = -\begin{vmatrix} \sigma_{xy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zz}-\sigma_1 \end{vmatrix}$$

$$C = \begin{vmatrix} \sigma_{xy} & \sigma_{yy}-\sigma_1 \\ \sigma_{zx} & \sigma_{yz} \end{vmatrix}$$

Substituting for λ_{x1}, λ_{y1}, λ_{z1} in the dot product relation for any unit vector gives:

$$\lambda_x^2 + \lambda_y^2 + \lambda_z^2 = 1$$

$$\lambda_{x1} = A/(A^2+B^2+C^2)^{1/2}$$

$$\lambda_{y1} = B/(A^2+B^2+C^2)^{1/2}$$

$$\lambda_{z1} = C/(A^2+B^2+C^2)^{1/2}$$

Brady & Brown (1993)

Stress Invariants – Step 4

Proceeding in a similar way, the vectors of direction cosines for the intermediate and minor principal stresses axes, i.e. (λ_{x2}, λ_{y2}, λ_{z2}) and (λ_{x3}, λ_{y3}, λ_{z3}) are obtained by repeating the calculations but substituting σ_2 and σ_3.

$$\frac{\lambda_{x2}}{D} = \frac{\lambda_{y2}}{E} = \frac{\lambda_{z2}}{F} = K$$

Where:

$$D = -\begin{vmatrix} \sigma_{xy} & \sigma_{zx} \\ \sigma_{yz} & \sigma_{zz}-\sigma_2 \end{vmatrix} \qquad E = \begin{vmatrix} \sigma_{xx}-\sigma_2 & \sigma_{zx} \\ \sigma_{zx} & \sigma_{zz}-\sigma_2 \end{vmatrix}$$

$$F = -\begin{vmatrix} \sigma_{xx}-\sigma_2 & \sigma_{xy} \\ \sigma_{zx} & \sigma_{yz} \end{vmatrix}$$

$$\frac{\lambda_{x3}}{G} = \frac{\lambda_{y3}}{H} = \frac{\lambda_{z3}}{I} = K$$

Where:

$$G = \begin{vmatrix} \sigma_{xy} & \sigma_{zx} \\ \sigma_{yy}-\sigma_3 & \sigma_{yz} \end{vmatrix} \qquad H = -\begin{vmatrix} \sigma_{xx}-\sigma_3 & \sigma_{zx} \\ \sigma_{xy} & \sigma_{yz} \end{vmatrix}$$

$$I = \begin{vmatrix} \sigma_{xx}-\sigma_3 & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy}-\sigma_3 \end{vmatrix}$$

Stress Invariants – Step 5

The procedure for calculating the principal stresses and the orientations of the principal stress axes is simply the determination of the eigenvalues of the stress matrix, and the eigenvector for each eigenvalue. Thus, some simple checks can be performed to assess the correctness of the solution:

Invariance of the sum of the normal stresses requires that:

$$\sigma_1 + \sigma_2 + \sigma_3 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$$

The condition of orthogonality requires that each of the three dot products of the vectors of the direction cosines must vanish:

$$\lambda_{x1}\,\lambda_{x2} + \lambda_{y1}\,\lambda_{y2} + \lambda_{z1}\,\lambda_{z2} = 0$$

Example #4 (Solution)

Q. Six components of stress are measured at a point:

σ_{xx} = 14.0 MPa τ_{xy} = -0.6 MPa
σ_{yy} = 34.8 MPa τ_{yz} = 6.0 MPa
σ_{zz} = 16.1 MPa τ_{xz} = -2.1 MPa

Determine the principal stresses and their direction cosines.

A. Step 1: Solving the stress invariants we get:

$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = 14.0 + 34.8 + 16.1 = \mathbf{64.9\ MPa}$

$I_2 = \sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2$
$= (14.0)(34.8) + (34.8)(16.1) + (16.1)(14.0) - (-0.6)^2 - (6.0)^2 - (-2.1)^2$
= 1232.1 MPa

$I_3 = \sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_{xx}\tau_{yz}^2 - \sigma_{yy}\tau_{zx}^2 - \sigma_{zz}\tau_{xy}^2$
$= (14.0)(34.8)(16.1) + 2(-0.6)(6.0)(-2.1) - (14.0)(6.0)^2 - (34.8)(-2.1)^2 - (16.1)(-0.6)^2$
= 7195.8 MPa

Example #4 (Solution)

Q. Six components of stress are measured at a point:

σ_{xx} = 14.0 MPa τ_{xy} = -0.6 MPa
σ_{yy} = 34.8 MPa τ_{yz} = 6.0 MPa
σ_{zz} = 16.1 MPa τ_{xz} = -2.1 MPa

Determine the principal stresses and their direction cosines.

A. Step 2: Substituting these values into the cubic equation we get:

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

$$\sigma^3 - 64.9\sigma^2 + 1232.1\sigma - 7195.8 = 0$$

Step 3: Solving the cubic equation gives: $\sigma = \begin{bmatrix} 36.6 \\ 16.0 \\ 12.3 \end{bmatrix}$ MPa

Thus: σ_1 = 36.6 MPa σ_2 = 16.0 MPa σ_3 = 12.3 MPa

Example #4 (Solution)

Q. Six components of stress are measured at a point:

σ_{xx} = 14.0 MPa τ_{xy} = -0.6 MPa
σ_{yy} = 34.8 MPa τ_{yz} = 6.0 MPa
σ_{zz} = 16.1 MPa τ_{xz} = -2.1 MPa

Determine the principal stresses and their direction cosines.

A. Step 4: Obtain the direction cosines (direction σ_1) by first solving for the determinates:

$$A = \begin{vmatrix} \sigma_{yy}-\sigma_1 & \sigma_{yz} \\ \sigma_{yz} & \sigma_{zz}-\sigma_1 \end{vmatrix} \Rightarrow \begin{vmatrix} 34.8-36.6 & 6.0 \\ 6.0 & 16.1-36.6 \end{vmatrix} \Rightarrow A = 0.90$$

$$B = -\begin{vmatrix} \sigma_{xy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zz}-\sigma_1 \end{vmatrix} \Rightarrow -\begin{vmatrix} -0.6 & 6.0 \\ -2.1 & 16.1-36.6 \end{vmatrix} \Rightarrow B = -24.90$$

$$C = \begin{vmatrix} \sigma_{xy} & \sigma_{yy}-\sigma_1 \\ \sigma_{zx} & \sigma_{yz} \end{vmatrix} \Rightarrow \begin{vmatrix} -0.6 & 34.8-36.6 \\ -2.1 & 6.0 \end{vmatrix} \Rightarrow C = -7.38$$

Example #4 (Solution)

Q. Six components of stress are measured at a point:

σ_{xx} = 14.0 MPa $\quad$ τ_{xy} = -0.6 MPa
σ_{yy} = 34.8 MPa $\quad$ τ_{yz} = 6.0 MPa
σ_{zz} = 16.1 MPa $\quad$ τ_{xz} = -2.1 MPa

Determine the principal stresses and their direction cosines.

A. Step 5: Substituting the determinates into the equations for the direction cosines for σ_1 gives:

$$\lambda_{x1} = A/(A^2+B^2+C^2)^{1/2}$$
$$\lambda_{y1} = B/(A^2+B^2+C^2)^{1/2}$$
$$\lambda_{z1} = C/(A^2+B^2+C^2)^{1/2}$$

$$\lambda_{x1} = 0.90 \Big/ \left((0.90)^2 + (-24.90)^2 + (-7.38)^2\right)^{0.5} = \boxed{0.035}$$
$$\lambda_{y1} = -24.90 \Big/ \left((0.90)^2 + (-24.90)^2 + (-7.38)^2\right)^{0.5} = \boxed{-0.958}$$
$$\lambda_{z1} = -7.38 \Big/ \left((0.90)^2 + (-24.90)^2 + (-7.38)^2\right)^{0.5} = \boxed{-0.284}$$

Example #4 (Solution)

Q. Six components of stress are measured at a point:

σ_{xx} = 14.0 MPa $\quad$ τ_{xy} = -0.6 MPa
σ_{yy} = 34.8 MPa $\quad$ τ_{yz} = 6.0 MPa
σ_{zz} = 16.1 MPa $\quad$ τ_{xz} = -2.1 MPa

Determine the principal stresses and their direction cosines.

A. Step 6: Repeating for σ_2 and σ_3, gives the direction cosines:

λ_{x2} = -0.668 $\quad$ λ_{x3} = 0.741
λ_{y2} = -0.246 $\quad$ λ_{y3} = -0.154
λ_{z2} = 0.702 $\quad$ λ_{z3} = 0.653

Example #4 (Solution)

Q. Six components of stress are measured at a point:

σ_{xx} = 14.0 MPa τ_{xy} = -0.6 MPa
σ_{yy} = 34.8 MPa τ_{yz} = 6.0 MPa
σ_{zz} = 16.1 MPa τ_{xz} = -2.1 MPa

Determine the principal stresses and their direction cosines.

A. Thus:

σ_1 = 36.6 MPa	σ_2 = 16.0 MPa	σ_3 = 12.3 MPa
λ_{x1} = 0.035	λ_{x2} = -0.668	λ_{x3} = 0.741
λ_{y1} = -0.958	λ_{y2} = -0.246	λ_{y3} = -0.154
λ_{z1} = -0.284	λ_{z2} = 0.702	λ_{z3} = 0.653

Strain

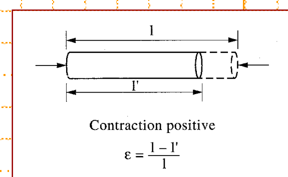
Strain is a change in the relative configuration of points within a solid. One can study finite strain or infinitesimal strain – both are relevant to the deformations that occur in the context of stressed rock.

Large-scale strain is experienced when severe deformations occur. When such displacements are very small, one can utilize the concept of infinitesimal strain and develop a strain tensor analogous to the stress tensor.

Finite-Strain

Strains may be regarded as normalized displacements. If a structure is subjected to a stress state, it will deform. However, the magnitude of the deformation is dependent on the size of the structure as well as the magnitude of the stress applied. In order to render the displacement a scale-independent parameter, the concept of strain is utilized.

Strain → in its simplest form, strain is the ratio of displacement to the undeformed length.

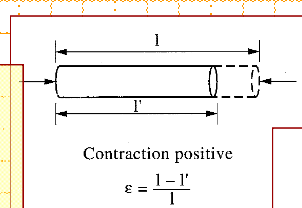


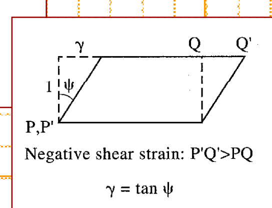
Hudson & Harrison (1997)

Finite-Strain

It should be noted that strain is a 3-D phenomenon that requires reference to all three Cartesian coordinate axes. However, it is instructive to start with 2-D strain, and then once the basic concepts have been introduced, 3-D strain follows as a natural progression.

It is easier to grasp the concept of normal strain than shear strain. This is because the normal displacement and the associated strain occur along one axis. In the case of shear strain, the quantity of strain involves an interaction between two (or three) axes.



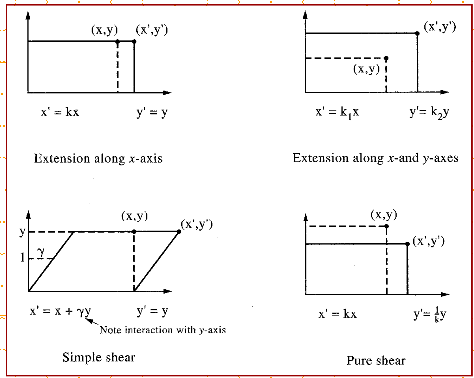


Hudson & Harrison (1997)

Homogeneous Finite-Strain

One convenient simplification that can be introduced is the concept of homogeneous strain which occurs when the state of strain is the same throughout the solid (i.e. straight lines remain straight, circles are deformed into ellipses, etc.).

… in each of the examples, equations are given relating new positions (e.g. x´) in terms of their original positions (e.g. x). The coefficients k and γ indicate the magnitudes of the normal and shear strains, respectively.

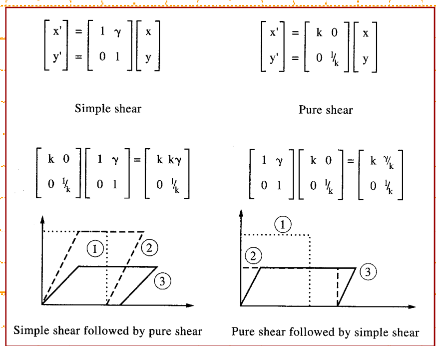


Hudson & Harrison (1997)

Homogeneous Finite-Strain

During geological history, a rock mass may experience successive phases of deformation. Thus, in decoding such compound deformation into its constituent parts, we need to know whether strain phases are commutative, i.e. if there are two deformation phases, A and B, is the final result of A followed by B the same as B followed by A?

… the answer is generally NO! The final state of strain is dependent on the straining sequence in those circumstances where shear strains are involved. This can be seen in the off-diagonal terms in the strain matrix.

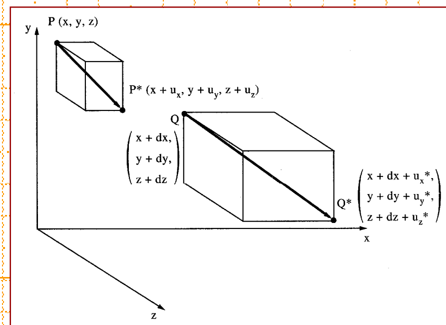


Hudson & Harrison (1997)

Infinitesimal Strain

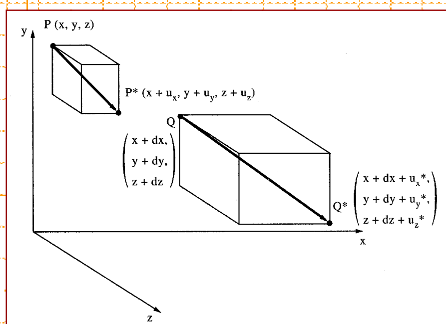
Infinitesimal strain is homogeneous strain over a vanishingly small element of a finite strained body. To find the components of the strain matrix, we need to consider the variation in coordinates of the ends of an imaginary line inside a body as the body is strained.

… the point P with coordinates (x,y,z) moves when the body is strained, to a point P with coordinates ($x+u_x$, $y+u_y$, $z+u_z$). The components of movement u_x, u_y and u_z, may vary with location within the body, and so are regarded as functions of x, y and z.*



Hudson & Harrison (1997)

Infinitesimal Strain



… similarly, the point Q (which is a small distance from P), with coordinates ($x+\delta x$, $y+\delta y$, $z+\delta z$), is strained to Q which has coordinates ($x+\delta x+u_x^*$, $y+\delta y+u_y^*$, $z+\delta z+u_z^*$).*

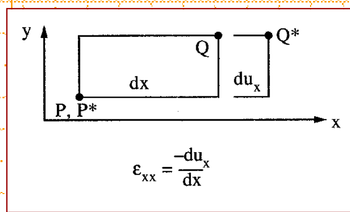
If we now consider holding P in a constant position and Q being strained to Q*, the normal and shear components of strain can be isolated.

Infinitesimal Strain

The infinitesimal longitudinal strain can now be considered in the x-direction. Because strain is 'normalized displacement', if it is assumed that u_x is a function of x only, then:

$$\varepsilon_{xx} = du_x/dx \quad \text{and hence} \quad du_x = \varepsilon_{xx}dx.$$

Considering similar deformations in the y- and z-directions, the normal components of strain can be generated.

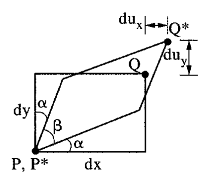


$$\begin{bmatrix} du_x \\ du_y \\ du_z \end{bmatrix} = \begin{bmatrix} -\varepsilon_{xx} & 0 & 0 \\ 0 & -\varepsilon_{yy} & 0 \\ 0 & 0 & -\varepsilon_{zz} \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}$$

Hudson & Harrison (1997)

Infinitesimal Strain

Derivation of the expressions for the shear strains follows a similar course, except that instead of assuming that simple shear occurs parallel to one of the coordinate axes, the assumption is made initially that shear strain (expressed as a change in angle) is equally distributed between both coordinate axes, i.e. $du=du_y$ if $dx=dy$.



$$\gamma_{xy} = (\tfrac{\pi}{2} - 2\alpha) = \beta \qquad e_{xy} = \gamma_{xy}/2, \text{ etc.}$$

$$\alpha = \frac{\gamma_{xy}}{2}$$

$$\begin{bmatrix} du_x \\ du_y \\ du_z \end{bmatrix} = \begin{bmatrix} 0 & \gamma_{xy}/2 & \gamma_{xz}/2 \\ \gamma_{xy}/2 & 0 & \gamma_{yz}/2 \\ \gamma_{zx}/2 & \gamma_{zy}/2 & 0 \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} \qquad \begin{bmatrix} du_x \\ du_y \\ du_z \end{bmatrix} = \begin{bmatrix} 0 & e_{xy} & e_{xz} \\ e_{yx} & 0 & e_{yz} \\ e_{zx} & e_{zy} & 0 \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}$$

… it should be noted that the term γ_{xy}, i.e. 2α, is known as the engineering shear strain, whereas the term $\gamma_{xy}/2$, i.e. α, is known as the tensorial shear strain. It is the tensorial shear strain that appears as the off-diagonal components in the strain matrix.

The Strain Tensor

By combining the longitudinal and shear strain components, we can now present the complete strain tensor – which is a second order tensor directly analogous to the stress tensor.

$$\begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix}.$$

… note that this matrix is symmetrical and hence has six independent components- with its properties being the same as the stress matrix (because they are both second-order tensors).

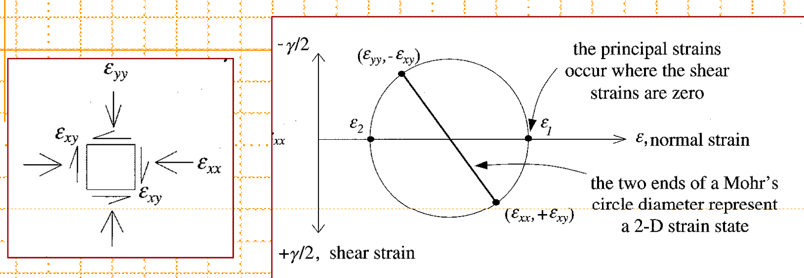
… for example, at an orientation of the infinitesimal cube for which there are no shear strains, we have principal values as the three leading diagonal strain components.

$$\begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix}.$$

$$\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \text{a constant.}$$

The Strain Tensor

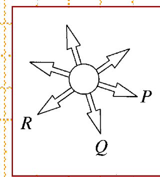
The strain component transformation equations are also directly analogous to the stress transformation equations and so the Mohr's circle representation can be utilized directly for relating normal and shear strains on planes at different orientations.



Hudson & Harrison (1997)

Example #5

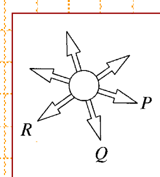
Q. Assume that strains measured by a strain gauge rosette are ε_P=43.0e-6, ε_Q=7.8e-6 and ε_R=17.0e-6, and that the gauges make the following angles to the x-direction: θ_P=20°, θ_Q=80° and θ_R=140°. Determine the principal strains and their orientations.



A. In order to use the strain transformation equations to determine the 2-D state of strain from measurements made with strain gauges, we firstly determine the angle each gauge makes to the x-axis: say, for gauges P, Q and R, these are θ_P, θ_Q, and θ_R. The strains measured by the gauges are ε_P, ε_Q and ε_R.

Example #5 (Solution)

Q. Assume that strains measured by a strain gauge rosette are ε_P=43.0e-6, ε_Q=7.8e-6 and ε_R=17.0e-6, and that the gauges make the following angles to the x-direction: θ_P=20°, θ_Q=80° and θ_R=140°. Determine the principal strains and their orientations.



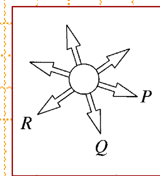
A. Step 1: Remembering our stress transformation equation:

$$\sigma_x' = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$

we can derive our strain transformation equations in the same way.

Example #5 (Solution)

Q. Assume that strains measured by a strain gauge rosette are ε_P=43.0e-6, ε_Q=7.8e-6 and ε_R=17.0e-6, and that the gauges make the following angles to the x-direction: θ_P=20°, θ_Q=80° and θ_R=140°. Determine the principal strains and their orientations.



A. In doing so, the 2-D strain transformation equation linking each of the measured strains ε_P, ε_Q and ε_R to the 2-D strains ε_x, ε_y and γ_{xy} are:

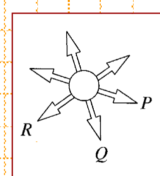
$$\begin{aligned}
\varepsilon_P &= \varepsilon_x \cos^2\theta_P + \varepsilon_y \sin^2\theta_P + \gamma_{xy}\sin\theta_P\cos\theta_P\\
\varepsilon_Q &= \varepsilon_x \cos^2\theta_Q + \varepsilon_y \sin^2\theta_Q + \gamma_{xy}\sin\theta_Q\cos\theta_Q\\
\varepsilon_R &= \varepsilon_x \cos^2\theta_R + \varepsilon_y \sin^2\theta_R + \gamma_{xy}\sin\theta_R\cos\theta_R
\end{aligned}$$

or, in matrix form:

$$\begin{bmatrix}\varepsilon_P\\ \varepsilon_Q\\ \varepsilon_R\end{bmatrix} = \begin{bmatrix}\cos^2\theta_P & \sin^2\theta_P & \sin\theta_P\cos\theta_P\\ \cos^2\theta_Q & \sin^2\theta_Q & \sin\theta_Q\cos\theta_Q\\ \cos^2\theta_R & \sin^2\theta_R & \sin\theta_R\cos\theta_R\end{bmatrix}\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix}$$

Example #5 (Solution)

Q. Assume that strains measured by a strain gauge rosette are ε_P=43.0e-6, ε_Q=7.8e-6 and ε_R=17.0e-6, and that the gauges make the following angles to the x-direction: θ_P=20°, θ_Q=80° and θ_R=140°. Determine the principal strains and their orientations.

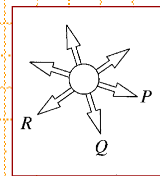


A. Step 2: We invert these equations to find the strains ε_x, ε_y and γ_{xy}, as

$$\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix} = \begin{bmatrix}\cos^2\theta_P & \sin^2\theta_P & \sin\theta_P\cos\theta_P\\ \cos^2\theta_Q & \sin^2\theta_Q & \sin\theta_Q\cos\theta_Q\\ \cos^2\theta_R & \sin^2\theta_R & \sin\theta_R\cos\theta_R\end{bmatrix}^{-1}\begin{bmatrix}\varepsilon_P\\ \varepsilon_Q\\ \varepsilon_R\end{bmatrix}$$

The solution to the problem can then be found by solving the matrix where we have θ_P=20°, θ_Q=80° and θ_R=140°.

Example #5 (Solution)



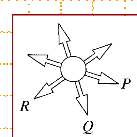
Q. Assume that strains measured by a strain gauge rosette are ε_P=43.0e-6, ε_Q=7.8e-6 and ε_R=17.0e-6, and that the gauges make the following angles to the x-direction: θ_P=20°, θ_Q=80° and θ_R=140°. Determine the principal strains and their orientations.

A.

$$\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix} = \begin{bmatrix}0.883 & 0.117 & 0.321\\ 0.030 & 0.970 & 0.171\\ 0.587 & 0.413 & -0.492\end{bmatrix}^{-1} \cdot \begin{bmatrix}43.0\times10^{-6}\\ 7.8\times10^{-6}\\ 17.0\times10^{-6}\end{bmatrix}$$

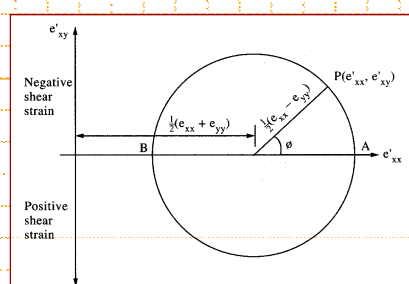
$$= \begin{bmatrix}0.884 & -0.293 & 0.449\\ -0.177 & 0.960 & 0.218\\ 0.857 & 0.456 & -1.313\end{bmatrix} \cdot \begin{bmatrix}43.6\times10^{-6}\\ 7.9\times10^{-6}\\ 17.0\times10^{-6}\end{bmatrix} = \begin{bmatrix}41.6\times10^{-6}\\ 3.6\times10^{-6}\\ 18.1\times10^{-6}\end{bmatrix}.$$

Example #5 (Solution)



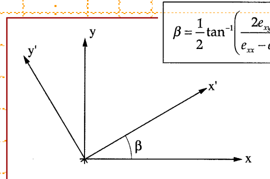
Q. Assume that strains measured by a strain gauge rosette are ε_P=43.0e-6, ε_Q=7.8e-6 and ε_R=17.0e-6, and that the gauges make the following angles to the x-direction: θ_P=20°, θ_Q=80° and θ_R=140°. Determine the principal strains and their orientations.

A. Step 3: Because our problem is restricted to a 2-D plane, we can solve for the principal strains using a Mohr circle construction.



Hudson & Harrison (1997)

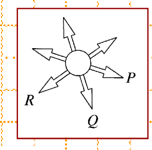
$$\beta = \frac{1}{2}\tan^{-1}\left(\frac{2e_{xy}}{e_{xx}-e_{yy}}\right).$$



From the Mohr circle, we also obtain a mathematical relationship for the angle of the principal strain.

Example #5 (Solution)

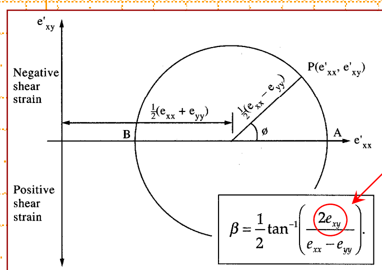
Q. Assume that strains measured by a strain gauge rosette are ε_P=43.0e-6, ε_Q=7.8e-6 and ε_R=17.0e-6, and that the gauges make the following angles to the x-direction: θ_P=20°, θ_Q=80° and θ_R=140°. Determine the principal strains and their orientations.



A. Step 4: Remembering:

$e_{xx} = \varepsilon_x$, $e_{yy} = \varepsilon_y$, and $e_{xy} = ½\, \gamma_{xy}$

γ_{xy} is referred to as engineering shear strain.
e_{xy} is referred to as mathematical shear strain.



$$\beta = \frac{1}{2}\tan^{-1}\left(\frac{2e_{xy}}{e_{xx}-e_{yy}}\right).$$

$$\varepsilon_1 = 0.5\left(\varepsilon_{xx}+\varepsilon_{yy}\right) + \left[0.25\left(\varepsilon_{xx}-\varepsilon_{yy}\right)^2 + \varepsilon_{xy}^{\ 2}\right]^{0.5}$$

$$\varepsilon_2 = 0.5\left(\varepsilon_{xx}+\varepsilon_{yy}\right) - \left[0.25\left(\varepsilon_{xx}-\varepsilon_{yy}\right)^2 + \varepsilon_{xy}^{\ 2}\right]^{0.5}$$

Solving, we calculate ε_1 = 43.7e-6 and ε_2 = 1.52e-6, with the angle between the *x*-direction and the major principal strain being 12.7°.

The Elastic Compliance Matrix

Given the mathematical similarities between the structure of the strain matrix with that of the stress matrix, it may seem fitting to find a means to link the two matrices together. Clearly, this would be of great benefit for engineering, because we would be able to predict either the strains (and associated displacements) from a knowledge of the applied stresses or vice versa.

A simple way to begin would be to assume that each component of the strain tensor is a linear combination of all the components of the stress tensor, i.e. each stress component contributes to the magnitude of each strain component. For example, in the case of the ε_{xx} component, we can express this relation as:

$$\varepsilon_{xx} = S_{11}\sigma_{xx} + S_{12}\sigma_{yy} + S_{13}\sigma_{zz} + S_{14}\tau_{xy} + S_{15}\tau_{yz} + S_{16}\tau_{zx}.$$

The Elastic Compliance Matrix

Because there are six independent components of the strain matrix, there will be six equations of this type. If we considered that the strain in the *x*-direction were only due to stress in the *x*-direction, the previous equation would reduce to:

$$\varepsilon_{xx} = S_{11}\sigma_{xx} + S_{12}\sigma_{yy} + S_{13}\sigma_{zz} + S_{14}\tau_{xy} + S_{15}\tau_{yz} + S_{16}\tau_{zx}. \longrightarrow \varepsilon_{xx} = S_{11}\sigma_{xx}$$

or: $\sigma_{xx} = \varepsilon_{xx}/S_{11} = E\varepsilon_{xx}$, where $E = 1/S_{11}$.

The theory of elasticity, in the form of the generalized Hooke's Law, relates all the components of the strain matrix to all the components of the stress matrix.

$$\begin{aligned}
\varepsilon_{xx} &= S_{11}\sigma_{xx} + S_{12}\sigma_{yy} + S_{13}\sigma_{zz} + S_{14}\tau_{xy} + S_{15}\tau_{yz} + S_{16}\tau_{zx}\\
\varepsilon_{yy} &= S_{21}\sigma_{xx} + S_{22}\sigma_{yy} + S_{23}\sigma_{zz} + S_{24}\tau_{xy} + S_{25}\tau_{yz} + S_{26}\tau_{zx}\\
\varepsilon_{zz} &= S_{31}\sigma_{xx} + S_{32}\sigma_{yy} + S_{33}\sigma_{zz} + S_{34}\tau_{xy} + S_{35}\tau_{yz} + S_{36}\tau_{zx}\\
\varepsilon_{xy} &= S_{41}\sigma_{xx} + S_{42}\sigma_{yy} + S_{43}\sigma_{zz} + S_{44}\tau_{xy} + S_{45}\tau_{yz} + S_{46}\tau_{zx}\\
\varepsilon_{yz} &= S_{51}\sigma_{xx} + S_{52}\sigma_{yy} + S_{53}\sigma_{zz} + S_{54}\tau_{xy} + S_{55}\tau_{yz} + S_{56}\tau_{zx}\\
\varepsilon_{zx} &= S_{61}\sigma_{xx} + S_{62}\sigma_{yy} + S_{63}\sigma_{zz} + S_{64}\tau_{xy} + S_{65}\tau_{yz} + S_{66}\tau_{zx}.
\end{aligned}$$

The Elastic Compliance Matrix

It is not necessary to write these equations in full. An accepted convention is to use matrix notation:

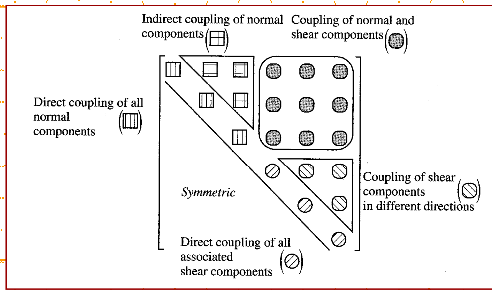
$$[\varepsilon] = [S][\sigma] \quad \text{where } [\varepsilon] = \begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \varepsilon_{xy}\\ \varepsilon_{yz}\\ \varepsilon_{zx}\end{bmatrix} \text{ and } [\sigma] = \begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \tau_{xy}\\ \tau_{yz}\\ \tau_{zx}\end{bmatrix} \text{ and } [S] = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16}\\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26}\\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36}\\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46}\\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56}\\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66}\end{bmatrix}.$$

The [S] matrix is known as the compliance matrix. In general, the higher the magnitudes of a specific element in this matrix, the greater will be the contribution to the strain, representing an increasingly compliant material. 'Compliance' is a form of 'flexibility', and is the inverse of 'stiffness'.

The Elastic Compliance Matrix

The compliance matrix contains 36 elements, but through considerations of conservation of energy, is symmetrical. Therefore, in the context that each strain component is a linear combination of the six stress components, we need 21 independent elastic constants to completely characterize a material that follows the generalized Hooke's law.

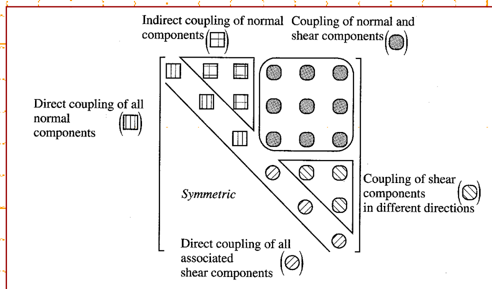
It is necessary, for practical applications, to consider to what extent we can reduce the number of non-zero elements of the matrix.



Hudson & Harrison (1997)

The Elastic Compliance Matrix

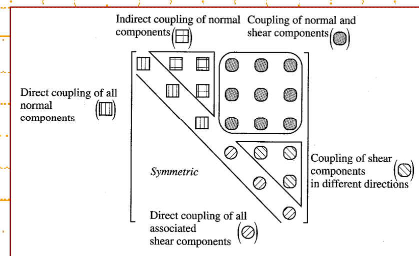
For typical engineering materials, there will be non-zero terms along the leading diagonal because longitudinal stresses must lead to longitudinal strains and shear stresses must lead to shear strains.



The isotropy of the material is directly specified by the interaction terms, i.e. whether a normal or shear strain may result from a shear or normal stress, respectively.

The Elastic Compliance Matrix - Isotropy

For example, omitting all shear linkages not on the leading diagonal – which means assuming that any contributions made by shearing stress components in a given direction to normal or shear strain components in other directions are negligible – causes all off-diagonal shear linkages to become zero.

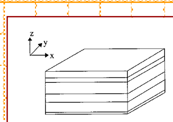


The compliance matrix then reduces to one with nine material properties, which is the case for an orthotropic material.

$$\begin{bmatrix} 1/E_1 & -\nu_{21}/E_2 & -\nu_{31}/E_3 & 0 & 0 & 0 \\ & 1/E_2 & -\nu_{32}/E_3 & 0 & 0 & 0 \\ & & 1/E_3 & 0 & 0 & 0 \\ & & & 1/G_{12} & 0 & 0 \\ & & & & 1/G_{23} & 0 \\ & \text{symmetric} & & & & 1/G_{31} \end{bmatrix}.$$

The Elastic Compliance Matrix

We can reduce the elastic compliance matrix even further by considering the case of transverse isotropy. This is manifested by a rock mass with a laminated fabric or one set of parallel discontinuities. In the case when the plane of isotropy is parallel to the plane containing Cartesian axes 1 and 2, we can say that:



$$E_1 = E_2 = E \quad \text{and} \quad E_3 = E'$$

$$\nu_{12} = \nu_{21} = \nu \quad \text{and} \quad \nu_{13} = \nu_{23} = \nu'$$

$$G_{12} \neq G_{23} \quad \text{and} \quad G_{23} = G_{31} = G'.$$

$$\begin{bmatrix} 1/E & -\nu/E & -\nu'/E' & 0 & 0 & 0 \\ & 1/E & -\nu'/E' & 0 & 0 & 0 \\ & & 1/E' & 0 & 0 & 0 \\ & & & \frac{2(1+\nu)}{E} & 0 & 0 \\ & & & & 1/G' & 0 \\ & \text{symmetric} & & & & 1/G' \end{bmatrix}$$

… thus, the number of independent elastic constants for a transversely isotropic material is five.

The Elastic Compliance Matrix - Isotropy

The final reduction that can be made to the compliance matrix is to assume complete isotropy, where:

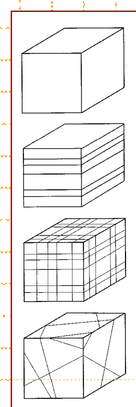
$$E_1 = E_2 = E_3 = E$$
$$\nu_{12} = \nu_{23} = \nu_{31} = \nu$$
$$G_{12} = G_{23} = G_{31} = G.$$

Note that, because we now have complete isotropy, the subscripts can be dispensed with, the shear modulus *G* is implicit and the factor 1/*E* is common to all terms and can be brought outside the matrix. This ultimate reduction results in two elastic constants for the case of a perfectly isotropic material.

$$1/E\begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ & 1 & -\nu & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 2(1+\nu) & 0 & 0 \\ & & & & 2(1+\nu) & 0 \\ & \text{symmetric} & & & & 2(1+\nu) \end{bmatrix}.$$

The Elastic Compliance Matrix - Isotropy

General anisotropic rock	**21 elastic constants:** all the independent S_{ij} in the **S** matrix. Because the matrix is symmetrical, there are 21 rather than 36 constants.
Orthotropic rock (three axes of symmetry, e.g. similar to a rock mass with three orthogonal fracture sets)	**9 elastic constants:** as in the matrix above — 3 Young's moduli, 3 Poisson's ratios and 3 shear moduli
Transversely isotropic rock (one axis of symmetry, e.g. similar to a rock mass with distinct laminations or with one main fracture set)	**5 elastic constants:** 2 Young's moduli, 2 Poisson's ratios, and 1 shear modulus (see Q5.4)
Perfectly isotropic rock	**2 elastic constants:** 1 Young's modulus, 1 Poisson's ratio

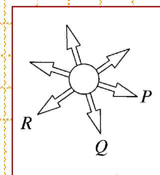


… *typical isotropy assumptions used for rock.*

Hudson & Harrison (1997)

Example #6

Q. For the strains found in the previous problem, and using values for the elastic constants of E = 150 GPa and ν = 0.30, determine the principal stresses and their orientations.



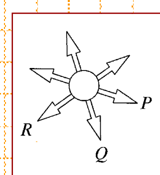
A. **Step 1**: Remembering back to the previous example, in 'Step 2' we had inverted the strain transformation matrix to find the strains ε_x, ε_y and γ_{xy}:

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} 0.883 & 0.117 & 0.321 \\ 0.030 & 0.970 & 0.171 \\ 0.587 & 0.413 & -0.492 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 43.0 \times 10^{-6} \\ 7.8 \times 10^{-6} \\ 17.0 \times 10^{-6} \end{bmatrix}$$

$$= \begin{bmatrix} 0.884 & -0.293 & 0.449 \\ -0.177 & 0.960 & 0.218 \\ 0.857 & 0.456 & -1.313 \end{bmatrix} \cdot \begin{bmatrix} 43.6 \times 10^{-6} \\ 7.9 \times 10^{-6} \\ 17.0 \times 10^{-6} \end{bmatrix} = \begin{bmatrix} 41.6 \times 10^{-6} \\ 3.6 \times 10^{-6} \\ 18.1 \times 10^{-6} \end{bmatrix}.$$

Example #6 (solution)

Q. For the principal strains found in the previous problem, and using values for the elastic constants of E = 150 GPa and ν = 0.30, determine the principal stresses and their orientations.



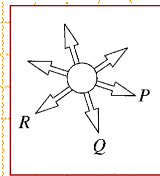
A. **Step 2**: To compute the stress state from the strain state we use the stress-strain relations for an isotropic material, i.e.:

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \cdot \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}$$

Which when inverted gives:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{\nu^2 - 1} \begin{bmatrix} -1 & -\nu & 0 \\ -\nu & -1 & 0 \\ 0 & 0 & -\frac{1}{2}(1-\nu) \end{bmatrix} \cdot \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}.$$

Example #6 (Solution)



Q. For the principal strains found in the previous problem, and using values for the elastic constants of E = 150 GPa and ν = 0.30, determine the principal stresses and their orientations.

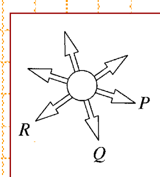
A. Step 3: Solving

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{\nu^2 - 1} \begin{bmatrix} -1 & -\nu & 0 \\ -\nu & -1 & 0 \\ 0 & 0 & -\frac{1}{2}(1-\nu) \end{bmatrix} \cdot \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}$$

gives

σ_x = 7.04 MPa
σ_y = 2.65 MPa
τ_{xy} = 1.04 MPa

Example #6 (Solution)



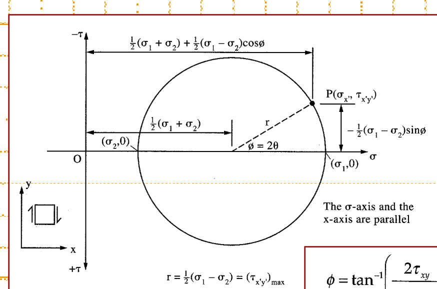
Q. For the principal strains found in the previous problem, and using values for the elastic constants of E = 150 GPa and ν = 0.30, determine the principal stresses and their orientations.

A. Step 4: Similar to our Mohr circle construction for the principal strains, we can solve for the 2-D principal stresses, where:

Calculate the radius as

$$\frac{1}{2}\sqrt{\left(\sigma_x - \sigma_y\right)^2 + \left(2\tau_{xy}\right)^2}$$

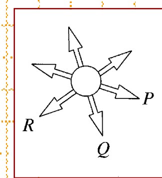
and the σ-value of the centre as ½(σ_x + σ_y).



$$\phi = \tan^{-1}\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)$$

Example #6 (Solution)

Q. For the principal strains found in the previous problem, and using values for the elastic constants of E = 150 GPa and ν = 0.30, determine the principal stresses and their orientations.



A. Computing the principal stresses and their orientations from these values gives σ_1 = **7.28 MPa** and σ_2 = **2.41 MPa**, with the angle between the *x*-direction and the major principal stress being **12.7°**.

Notice that because this is an isotropic material, the orientations of the principal stresses and the principal strains are identical.