Mohr's circle of stress

This is a graphical method of transforming the stress tensor. It is easy to use and remember, and is the best way of remembering the transformation equations.

If we choose the global x- and y-axes to coincide with the principal directions (and because we can choose the axes arbitrarily there is nothing to prevent this), then the transformation equations become

$$\sigma_{x'} = \sigma_1 \cos^2\theta + \sigma_2 \sin^2\theta \qquad \text{(A)}$$

$$\sigma_{y'} = \sigma_1 \sin^2\theta + \sigma_2 \cos^2\theta \qquad \text{(B)}$$

$$\tau_{x'y'} = -(\sigma_1 - \sigma_2)\cos\theta \sin\theta \qquad \text{(C)}$$

where σ_1 and σ_2 are now the principal stresses, and θ is measured anticlockwise from the principal direction x to the local direction x'.

These new equations can be simplified still further, by making use of trigonometric identities.

Let

$$\phi = 2\theta = \theta + \theta$$

then

$$\sin\phi = \sin(\theta + \theta) = \sin\theta\cos\theta + \cos\theta\sin\theta = 2\sin\theta\cos\theta$$

$$\therefore \underline{\underline{\cos\theta\sin\theta = \tfrac{1}{2}\sin\phi}} \qquad \text{(D)}$$

and

$$\cos\phi = \cos(\theta + \theta) = \cos\theta\cos\theta - \sin\theta\sin\theta = \cos^2\theta - \sin^2\theta$$

but

$$\cos^2\theta + \sin^2\theta = 1$$

so

$$\cos\phi = \cos^2\theta - (1 - \cos^2\theta) = 2\cos^2\theta - 1$$

from which

$$\underline{\underline{\cos^2\theta = \tfrac{1}{2}(1 + \cos\phi)}} \qquad \text{(E)}$$

or

$$\cos\phi = (1 - \sin^2\theta) - \sin^2\theta = 1 - 2\sin^2\theta$$

from which

$$\underline{\underline{\sin^2\theta = \tfrac{1}{2}(1 + \cos\phi)}}. \qquad \text{(F)}$$

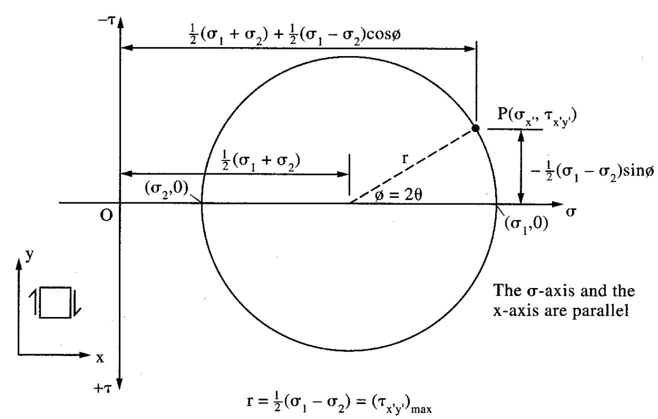
Substituting (D), (E) and (F) into equations (A), (B) and (C):

$$\sigma_{x'} = \sigma_1(\tfrac{1}{2}(1 + \cos\phi)) + \sigma_2(\tfrac{1}{2}(1 + \cos\phi))$$

i.e.

$$\boxed{\begin{aligned} \sigma_{x'} &= \tfrac{1}{2}(\sigma_1 + \sigma_2) + \tfrac{1}{2}(\sigma_1 + \sigma_2)\cos\phi \\ \tau_{x'y'} &= -\tfrac{1}{2}(\sigma_1 + \sigma_2)\sin\phi. \end{aligned}}$$

These two equations are simply the equations of a circle centred at $\tfrac{1}{2}(\sigma_1 + \sigma_2)$ on the σ-axis in σ–τ space:



To use Mohr's circle you must **understand** and **remember**:

1. Positive shear stresses and positive rotations have been used in developing the equations for a point (σ, τ), but the τ co-ordinate is negative. This means the τ-axis is upside down:

from Hudson & Harrison (1997)

positive shear stresses plot below the σ-axis.

2. The trigonometric relations used to simplify the equations resulted in $\phi = 2\theta$:

whatever rotation takes place in real life, twice the rotation takes place on Mohr's circle.

3. Each point on the circumference of the circle represents the (σ, τ) stress state on a plane of specific orientation. The points where the circle intersects the σ-axis represent planes on which $\tau = 0$: the principal planes. The associated σ-values are the principal stresses. Mohr's circle shows

the principal stresses are the maximum and minimum values of normal stress in the body.

4. The points representing the principal planes lie at opposite ends of a diameter: in real life planes are perpendicular.

Two perpendicular planes are represented on the circle by points at opposite ends of a diameter.

5. The maximum shear stress is given by ½ $(\sigma_1 - \sigma_2)$ and occurs when $\phi = 90°$ (i.e. $\theta = 45°$). Thus

the planes of maximum shear stress are orientated at 45° to the principal planes.

Using Mohr's circle to determine principal stresses

1. Draw x–y-axes on the element, draw an element with positive normal and shear stresses on it, and so write down (σ_x, τ_{xy}) and (σ_y, τ_{yx}).
2. Draw σ–τ-axes (same scale on each) with the σ-axis parallel to, and in the same direction as, σ_x. Plot (σ_x, τ_{xy}) bearing in mind the positive shear stresses plot below the σ-axis. Then plot (σ_y, τ_{yx}) **on the other side of the** σ**-axis.** Draw the diameter between the two points, and then draw the circle.
3. Calculate the radius as

$$\frac{1}{2}\sqrt{\left(\sigma_x - \sigma_y\right)^2 + \left(2\tau_{xy}\right)^2}$$

and the σ-value of the centre as $\frac{1}{2}(\sigma_x + \sigma_y)$.

4. Calculate the principal stresses and the maximum shear stress:

$$\sigma_1 = c + r, \quad \sigma_2 = c - r, \quad \tau_{max} = r.$$

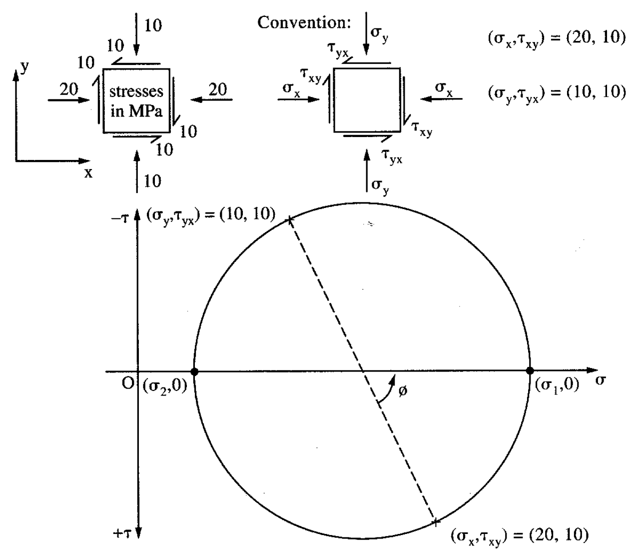
5. Calculate the rotation angle and direction from σ_x to σ_1. Remember that rotations on the circle are twice real life rotations (ϕ = positive rotation).

$$\phi = \tan^{-1}\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)$$

but be aware that $0° < \phi < 180°$.

6. Finally, draw the element on which the principal stresses act, in the correct orientation.

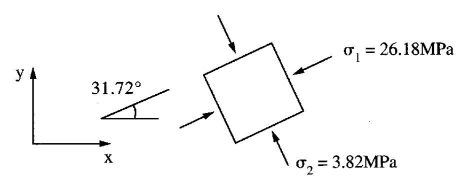
Example



radius $= \frac{1}{2}\sqrt{(20-10)^2 + (2 \times 10)^2} = \underline{11.18\text{MPa}}$ centre $= \frac{1}{2}(20 + 10) = \underline{15\text{MPa}}$

$\sigma_1 = 15 + 11.18 = \underline{26.18\text{MPa}}$ $\sigma_2 = 15 - 11.18 = \underline{3.82\text{MPa}}$ $\tau_{max} = \underline{11.18\text{MPa}}$

$\phi = \tan^{-1}\frac{2 \times 10}{20 - 10} = 63.43°$ $\therefore \underline{\theta = 31.72°}$



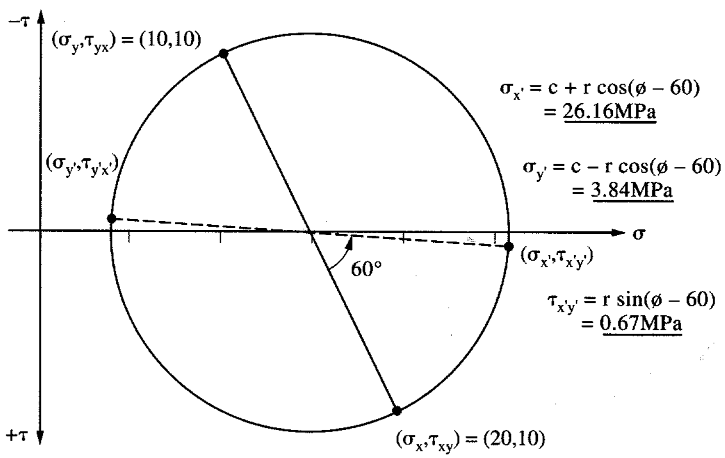
from Hudson & Harrison (1997)

Using Mohr's circle to determine stresses on a plane

Follow points 1, 2, 3 and 5 of the method for determining principal stresses, then:

4. Draw an element of the correct orientation relative to the x–y-axes, and mark on positive $\sigma_{x'}$, $\sigma_{y'}$, $\tau_{x'y'}$ and $\tau_{y'x'}$. Write down the sense (positive anticlockwise) and the magnitude of the rotation x-axis to x'-axis.
5. Mark this rotation on the circle, measuring from the (σ_x, τ_{xy}) point, remembering that you do twice as much on the circle as you do in real life.
6. The new point is ($\sigma_{x'}$, $\tau_{x'y'}$). Draw the diameter to determine ($\sigma_{y'}$, $\tau_{y'x'}$).

Example. What are the stresses on an element rotated 30° anticlockwise relative to the element in the previous example?



from Hudson & Harrison (1997)