Lecture 17: Kelvin and Poincaré Waves

Objectives

At the end of this lecture you will be able to:

- Derive and/or sketch the dispersion relation for Poincaré waves
- Describe the fluid motion due to Poincaré waves
- Describe an inertial oscillation

Reading

- Cushman-Roisin Section 6-2, 6-3 OLD
- Cushman-Roisin Section 9-2, 9-3 NEW

Notation change $h \equiv H$

Finish Kelvin Waves

We were considering an ocean ($x > 0$) with a coast at ($x = 0$) in the Northern Hemisphere.

We had a wave with no cross-shore velocity ($u = 0$), an along-shore velocity given by

$$v = v_o \exp(-x/R) \exp[i(\ell y - \omega t)] \tag{1}$$

where $R = (gh)^{1/2}/f$ (note sign change) and ℓ is negative so the wave propagates in the negative y-direction. The dispersion relation is $\omega = -(gh)^{1/2}\ell$. The first momentum equation reduced to geostrophy $-fv = -g\partial\eta/\partial x$ and from this we can calculate η:

$$\eta = \frac{-v_o R f}{g} \exp(-x/R) \exp[i(\ell y - \omega t)] \tag{2}$$

Thus

- A Kelvin wave propagates along a coast in such as way as to keep the coast to the right in the Northern Hemisphere. (to the left in the Southern Hemisphere).
- The wave speed (both phase speed and group speed) is constant at $(gh)^{1/2}$.
- The wave crests and troughs are parallel to the coast
- The wave crests and troughs decay exponentially away from the coast over the length-scale of the Rossby radius, R.
- The force balance perpendicular to the coast is geostrophic
- The force balance along the coast is typical for gravity waves: pressure variations drive flow accelerations.

Three example of Kelvin waves:

Small scale disturbances (100 km) trapped in the atmospheric boundary layer can propagate along coastlines as Kelvin waves.

The El Niño thermocline signal in the ocean propagates up the west coast of North America as a Kelvin wave.

The M2 tide travels as a Kelvin wave up the west coast of Canada and around Alaska to the tip of the Alaskan Islands.

Poincaré Waves

Start with the linear shallow water equations:

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x} \tag{3a}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y} \tag{3b}$$

$$\frac{\partial \eta}{\partial t} = -h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{3c}$$

Assume f and h constant as we did for Kelvin waves. Consider an infinite domain and look for plane waves. Align the y-axis with the crests of the waves, so $\partial/\partial y = 0$.

Then (3b) becomes

$$\frac{\partial v}{\partial t} = -fu \tag{4}$$

and (3c) becomes

$$\frac{\partial \eta}{\partial t} = -h\frac{\partial u}{\partial x} \tag{5}$$

Take $\partial/\partial t$ of (3a):

$$\frac{\partial^2 u}{\partial t^2} - f\frac{\partial v}{\partial t} = -g\frac{\partial}{\partial x}\frac{\partial \eta}{\partial t} \tag{6}$$

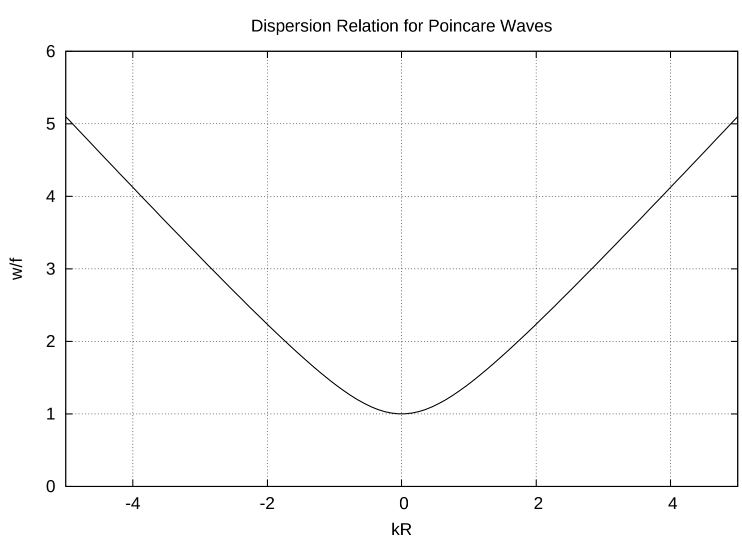
substitute (4) and (5)

$$\frac{\partial^2 u}{\partial t^2} + f^2 u = gh\frac{\partial^2 u}{\partial x^2} \tag{7}$$

Try a wave solution $u = u_o \exp[i(kx - \omega t)]$ and substitute. The variation in x and t cancel leaving the dispersion relation

$$\omega^2 = f^2 + ghk^2 \tag{8}$$

which is plotted below.



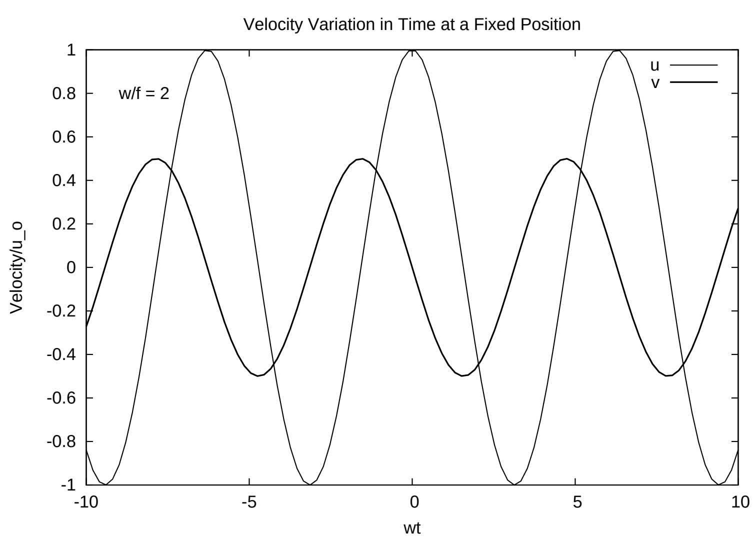
Remember that the phase speed of a wave is ω/k and the group speed of the wave is $\partial\omega/\partial k$. Thus Poincaré waves are dispersive. Long waves have zero group speed and infinite phase speed. Short waves asymptote to a phase speed and group speed of $(gh)^{1/2}$.

Taking u_o to be real, the real part of the u velocity is

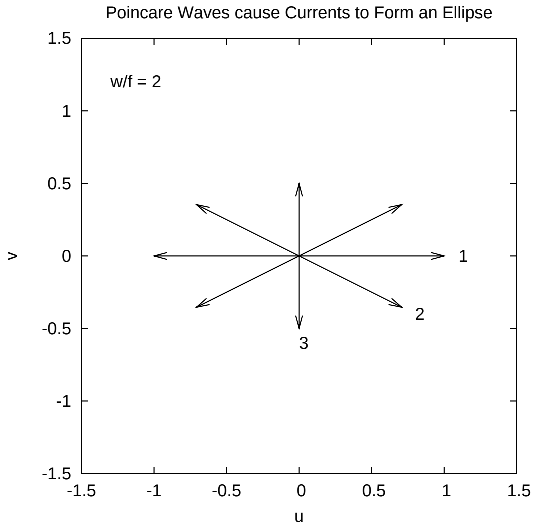
$$u = u_o \cos(kx - \omega t) \tag{9}$$

From (4) we can find $\partial v/\partial t$ and integrate to find v as

$$v = \frac{f u_o}{\omega} \sin(kx - \omega t) \tag{10}$$



So at a fixed position, the velocity traces out an anti-cyclonic ellipse.



From (5) and integrating with respect to time you can find η as

$$\eta = \frac{khu_o}{\omega}\cos(kx - \omega t) \tag{11}$$

Thus

- A Poincaré wave propagates in the open ocean in any direction.
- It is a dispersive wave.
- For a plane wave crests and troughs have uniform height.
- The flow vector turns anti-cyclonically
- Short waves act like non-rotating shallow water gravity waves
- The force balance in the direction the wave propagates includes pressure gradients and flow accelerations but also the Coriolis force due to flow velocity along the crests/troughs
- The force balance in the direction parallel to the crests and troughs is acceleration driven by the Coriolis force due to flow velocity in the direction the wave moves.

Inertial Oscillations

If one looks at a spectrum of wave frequencies observed in the ocean, besides peaks at the various tidal frequencies, one also sees a peak at the inertial frequency, f. At this limit Poincaré waves become inertial oscillations.

Note that for $\omega = f$, $k = 0$. Thus our waves become oscillations with

$$u = u_o \cos(\omega t) \tag{12a}$$

$$v = -u_o \sin(\omega t) \tag{12b}$$

$$\eta = 0 \tag{12c}$$

There are no pressure gradients! The flow goes around in a circle driven simply by the Coriolis force. These are large scale oscillations of the ocean (usually generated near the surface by storm winds). In a way, this is the natural oscillation frequency of the ocean... easy to excite.