Lecture 18: Rossby Waves

Objectives

At the end of this lecture you will be able to:

- Describe a β-plane
- Describe the restoring mechanism for Rossby waves
- Derive the dispersion relation for Rossby waves
- State the possible phase and group velocity directions for Rossby waves

Reading

- Cushman-Roisin Section 6-4

The β-plane

The Coriolis parameter $f = 2\Omega \sin\theta$ where Ω is the angular rotation rate of the earth and θ is the latitude. Expand f around $\theta = \theta_o$ using a Taylor series.

$$\begin{aligned} f &= f_o + \left.\frac{\partial f}{\partial \theta}\right|_{\theta_o} \Delta\theta + \left.\frac{\partial^2 f}{\partial \theta^2}\right|_{\theta_o} \frac{\Delta\theta^2}{2} + \ldots && (1) \\ &= 2\Omega \sin\theta_o + 2\Omega \cos\theta_o(\theta - \theta_o) + \mathcal{O}(\Delta\theta^2) && (2) \\ &= f_o + \beta y && (3) \end{aligned}$$

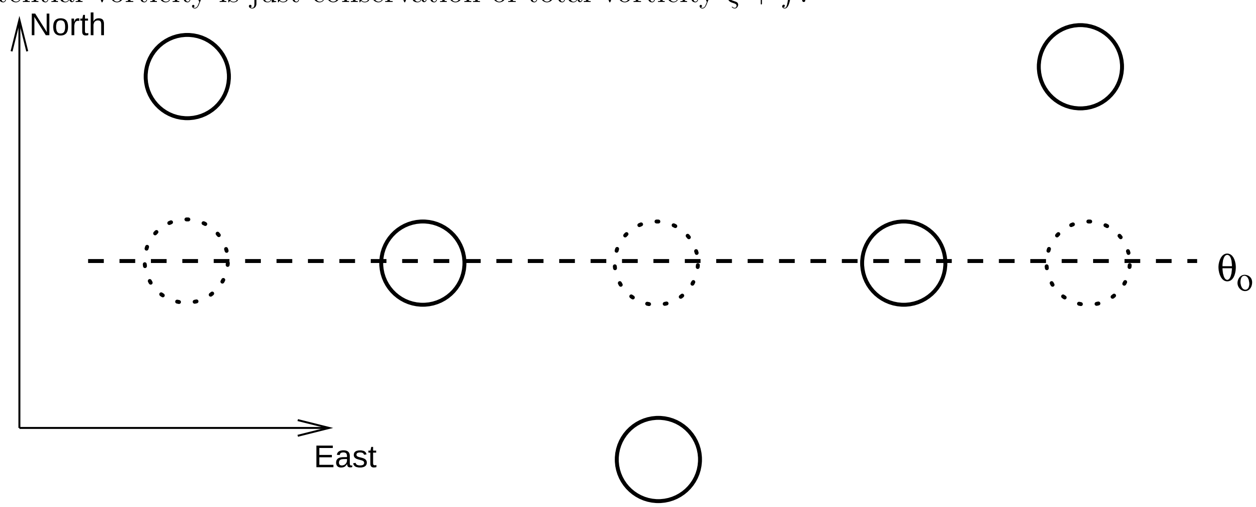
where $f_o = 2\Omega \sin\theta_o$ and $\beta = 2\Omega \cos\theta_o / a$ as $y = a\Delta\theta$ where a is the radius of the earth.

The β-plane approximates f as the first two terms of the Taylor Series. The Coriolis parameter changes linearly with latitude.

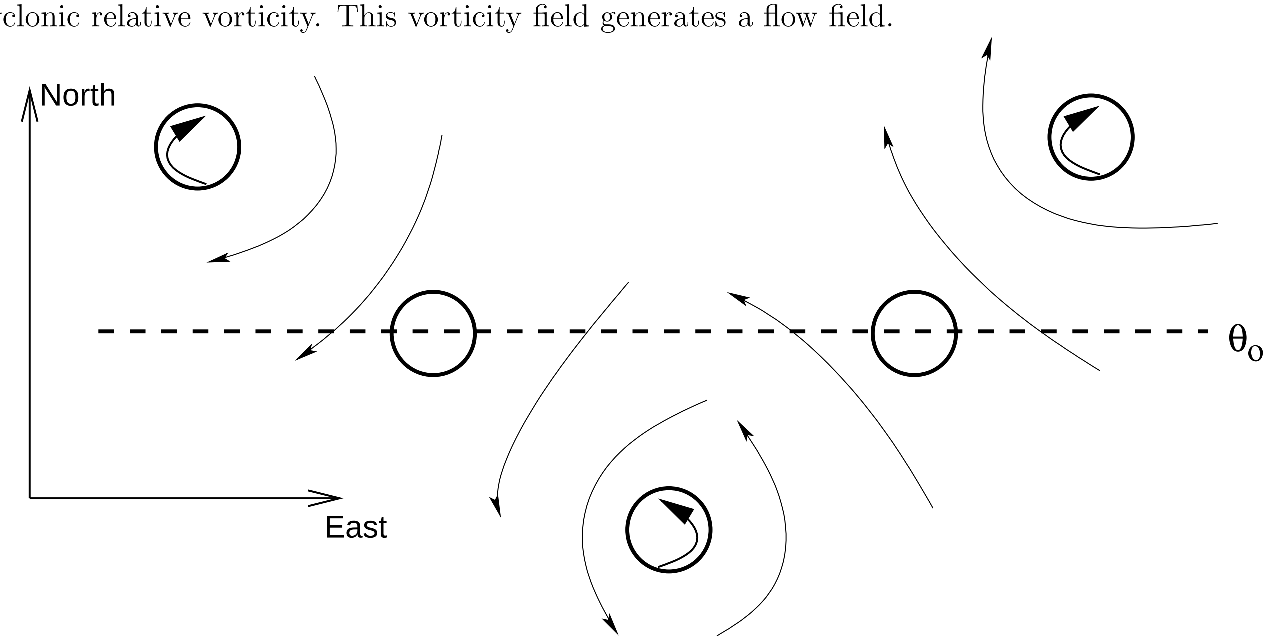
The f-plane approximates f as the first term of the Taylor Series; f is taken as a constant.

Mechanism of Rossby Wave Propagation

Consider five columns of water all initially at the same latitude, on a β-plane. Move two the columns north and one to south, conserving potential vorticity. As we are assuming the water depth is constant, and thus the fluid column lengths remain constant, conservation of potential vorticity is just conservation of total vorticity $\zeta + f$.

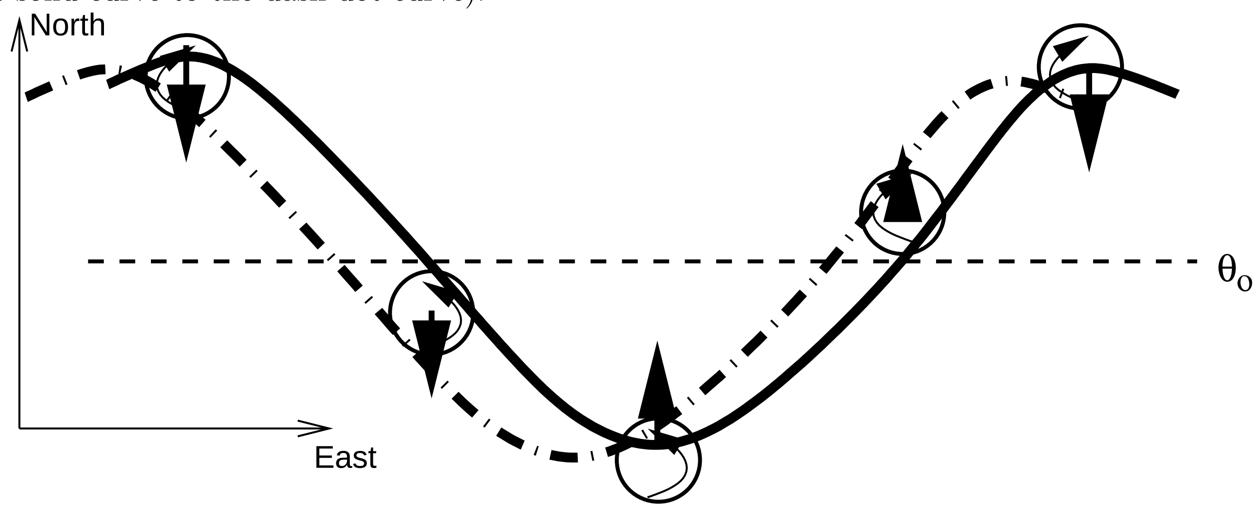


The two columns moved north have increased planetary vorticity and so get anti-cyclonic relative vorticity. The column moved south has decreased planetary vorticity and thus gets cyclonic relative vorticity. This vorticity field generates a flow field.



This flow field tends to move the other two columns. But as soon as they are displaced, they too have vorticity. This vorticity tends to restore the initial three columns back to their original position.

Also note that the second column from the left moves toward deep water whereas the fourth column moves toward shallow water. Thus the pattern is moving to the left (from the solid curve to the dash-dot curve).



Dispersion Relation for Rossby Waves

Assume a homogenous fluid and $Ro \ll 1$, $Ek \ll 1$ and $Ro_t < 1$. The x-momentum equation is

$$\frac{\partial u}{\partial t} - (f_o + \beta y)v = -g\frac{\partial \eta}{\partial x} \tag{4}$$

Remember that the basic balance is geostrophic (the second term on the left balances the right-hand side) and that the other terms are expected to be smaller.

Rearranging this equation we can write

$$fv = g\frac{\partial \eta}{\partial x} + \left(\frac{\partial u}{\partial t} - \beta y v\right) \tag{5a}$$

and similarly for y-momentum equation

$$fu = -g\frac{\partial\eta}{\partial y} - \left(\frac{\partial v}{\partial t} + \beta yu\right) \qquad (5b)$$

Substitute (5b) into the first term of (4) and (5a) into the third term of (4) to give:

$$\frac{1}{f_o}\frac{\partial}{\partial t}\left[-g\frac{\partial\eta}{\partial y} - \left(\frac{\partial v}{\partial t} + \beta yu\right)\right] - f_o v - \frac{\beta y}{f_o}\left[g\frac{\partial\eta}{\partial x} + \left(\frac{\partial u}{\partial t} - \beta yv\right)\right] = -g\frac{\partial\eta}{\partial x} \qquad (6)$$

The scale of $1/f_o\partial/\partial t$ is $Ro_t = 1/(f_oT)$. Considering periods of 3 days, with $f = 1.0\times10^{-4}\mathrm{s}^{-1}$ gives $Ro_t = 0.03$.

The scale of $\beta y/f_o$ for $\beta = 1.6\times10^{-11}$ m^{-1} s^{-1} and and wavelength of 100 km is 0.016.

So in the following equation the largest terms (geostrophic balance) appears in bold face. The terms written small are small terms multiplied by small terms and thus are negligible.

$$\frac{1}{f_o}\frac{\partial}{\partial t}\left[-g\frac{\partial\eta}{\partial y} - \left(\tfrac{\partial v}{\partial t}+\beta yu\right)\right] \mathbf{-f_o v} - \frac{\beta y}{f_o}\left[g\frac{\partial\eta}{\partial x} + \left(\tfrac{\partial u}{\partial t}-\beta yv\right)\right] = \mathbf{-g\frac{\partial\eta}{\partial x}} \qquad (7)$$

So keeping the large terms and the next order terms we get

$$\frac{-g}{f_o}\frac{\partial^2\eta}{\partial t\partial y} - f_o v - \frac{\beta g}{f_o}y\frac{\partial\eta}{\partial x} = -g\frac{\partial\eta}{\partial x} \qquad (8a)$$

and if we had similarly substituted (5b) and (5a) into the y-momentum equation and kept the large terms and next order terms we would get

$$\frac{g}{f_o}\frac{\partial^2\eta}{\partial t\partial x} + f_o u - \frac{\beta g}{f_o}y\frac{\partial\eta}{\partial y} = -g\frac{\partial\eta}{\partial y} \qquad (8b)$$

The first of these equations can be simply solved to give v as a function of η and the second to give u as a function of η.

The conservation of mass equation is

$$\frac{\partial \eta}{\partial t} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \tag{9}$$

Substituting the expression for u and v one get the following single partial differential equation of η.

$$\frac{\partial \eta}{\partial t} - R^2\frac{\partial}{\partial t}\left(\nabla_H^2\eta\right) - \beta R^2\frac{\partial \eta}{\partial x} = 0 \tag{10}$$

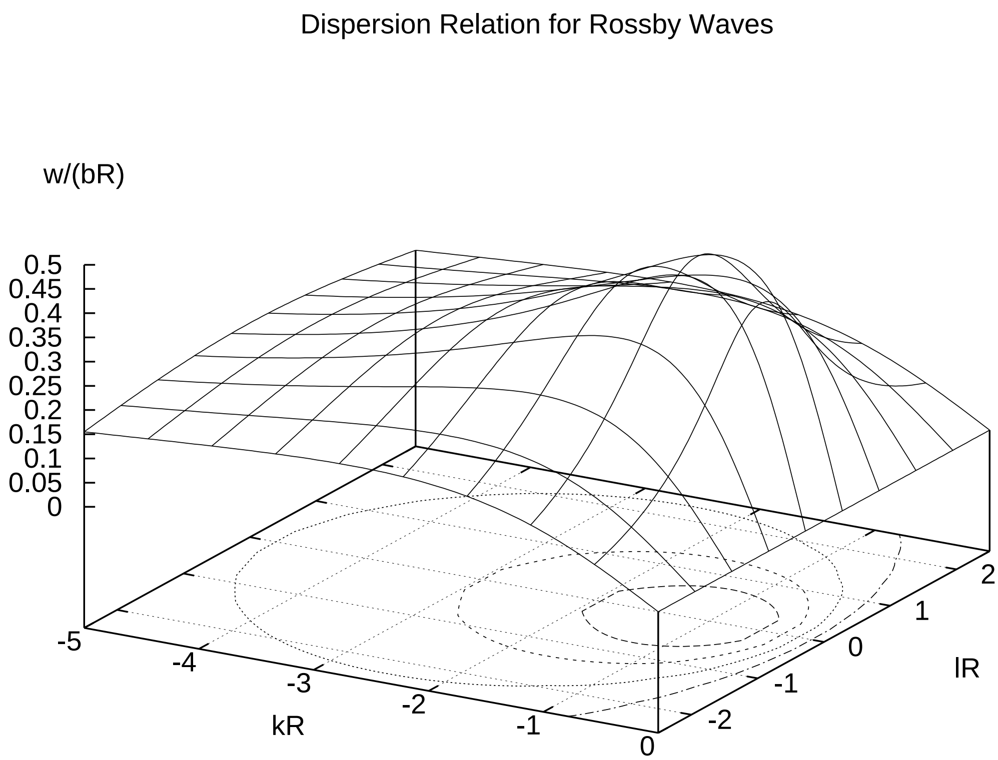
where $R^2 = gh/f_o^2$ is the Rossby radius squared.

Assuming a wave solution $\eta = \eta_o \exp[i(kx + \ell y - \omega t)]$ and substituting gives:

$$-i\omega - R^2(-i\omega)(-k^2 - \ell^2) - \beta R^2 ik = 0 \tag{11}$$

or

$$\omega = \frac{-\beta k}{k^2 + \ell^2 + 1/R^2} \tag{12}$$



Properties of Rossby Waves

- $k < 0$ which means that Rossby waves have phase speed west (not east) or south-west or north-west
- the group velocity can be either east or west
- long waves have group velocity to the west
- short waves have group velocity to the east
- for given values of R and β there is maximum value of ω. If you move North, this value decreases, so waves can be trapped around the equator.
- these waves are always sub-inertial $\omega < f$.