EOSC 550: Assignment #2

Due Wednesday, January 28, 2009

This problem set is designed to refresh your memory and skill about those basics of linear algebra that are relevant to solving inverse problems. The first part of the assignment considers vectors in $\mathcal{R}^N$. These are "vectors" which we are used to dealing with and for which we have a great deal of physical intuition. The second part of the assignment asks parallel questions but this time the "vectors" are functions. The duality of the questions should increase your physical intuition regarding the concepts of functions as vectors.
You can work the problems by hand or you can use MATLAB. If you use MATLAB then make sure you hand in your *.m file and also write your answers and methodology on a separate paper so that it can be marked in a straight-forward manner.

Part I: Euclidean Vectors

Consider two vectors

$$v_1 = (1,3)$$
$$v_2 = (2,2)$$

1. Use the definition for linear dependence to show that these vectors are linearly independent.

2. Since v_1 and v_2 are independent, they can be used as a basis for E^2, that is, any vector in E^2 can be written as

$$x = \sum_{i=1}^{2} \alpha_i \, v_i$$

   What are the coefficients α_i when $x = (2, 18)$

3. Use the Gram-Schmidt procedure to construct an orthonormal set of vectors e_1, e_2 which span E^2. Verify by direct calculation that vectors are orthonormal.

4. Find the expansion coefficients of the vector $x = (2, 18)$ in terms of the new basis. That is find β_i in

$$x = \sum_{i=1}^{2} \beta_i \, e_i$$

5. Suppose that there exists another vector $m = (m_1, m_2)$ in E^2. An experiment is designed to measure the projection of m onto certain directions. We measure

$$(m, v_i) = d_i \qquad i = 1, N$$

   The vectors v_i are successively: $v_1 = (1,3)$, $v_2 = (2,2)$, $v_3 = (1,-1)$. What can be said about m under the following four scenarios:

   a. N=1: d=(28)

b. N=2: d=(28,24)
c. N=3: d=(28,24,1)
d. N=3: d=(28,24,-4)

6. The model m lies in E^2 and since v_1, v_2 form a basis for that space we can write

$$m = \sum_{i=1}^{2} \alpha_i \, v_i$$

Substitute this form for the model into the data equations $d_i = (m, v_i)$ to form the system of equations $\Gamma\alpha = d$ where $\Gamma_{ij} = (v_i, v_j)$ is called the inner product matrix.

a. What are the numerical values of Γ
b. Solve the system of equations to find m when $d = (28, 24)$.

7. Your problem can also be solved in the following manner. Rather than solving $\Gamma\alpha = d$ directly, carry out a decomposition $\Gamma = R\Lambda R^T$. Then

$$\begin{aligned} \Gamma\alpha &= d \\ R\Lambda R^T \alpha &= d \\ \Lambda\hat{\alpha} &= \hat{d} \qquad \hat{\alpha} = R^T\alpha \quad \hat{d} = R^T d \\ \hat{\alpha} &= \Lambda^{-1}\hat{d} \\ \alpha &= R\hat{\alpha} \end{aligned}$$

a. What are eigenvalues and eigenvectors of Γ and thus the decomposition $\Gamma = R\Lambda R^T$
b. The matrix R is an orthonormal matrix. It has the following properties:

- det $|R| = \pm 1$
- $RR^T = R^T R = I$
- $||R^T x|| = ||x||$

Show that the above properties hold for the orthonormal matrix you have just computed.

c. Solve the problem for $d = (28, 24)$ using the decomposition methodology outlined above.

Part II: Functions as Vectors

Consider the Hilbert space $L_2[0,1]$ with inner product between two vectors $x(t)$ and $y(t)$ defined as

$$(x, y) = \int_0^1 x(t)\, y(t)\, dt$$

Consider two vectors $v_1 = 1$ and $v_2 = t$.

1. Show that the two vectors (functions) are linearly independent.

2. Use the Gram-Schmidt procedure to construct an orthonormal set of basis vectors $e_i = e_i(t) \quad i = 1,2$

3. What are the expansion coefficients of $y = t^2$ with respect to the basis in (2). If the expansion coefficients are β_i, plot

   a. $\beta_1 e_1$
   b. $\beta_1 e_1 + \beta_2 e_2$
   c. t^2

4. Another way to approach the problem of fitting is to use a least squares minimization. That is, find the coefficients γ_i such that

$$\left\| t^2 - \sum_{i=1}^{2} \gamma_i \, e_i(t) \right\|^2$$

   is a minimum. Carry out the minimization, compute the γ_i, and compare them with the β_i found in (3).

5. The minimization can also be done using the initial vectors v_i to span the space. Compute the coefficients α_i which minimizes

$$\phi = \left\| t^2 - \sum_{i=1}^{2} \alpha_i \, v_i(t) \right\|^2$$

   Compute and compare $\sum \alpha_i v_i(t)$ and $\sum \gamma_i e_i(t)$.

6. An experiment is carried out which yielded the following data:

$$d_1 = (m, v_1)$$
$$d_2 = (m, v_2)$$

   The results of the experiment were $d = (1/3,\ 1/4)$. Compute that portion of the model which lies in the subspace generated by v_1, v_2. That is, consider

$$m(t) = \sum_{i=1}^{2} \alpha_i \, v_i(t)$$

   Substitute this expression into the data equations to get a set of matrix equations $\Gamma\alpha = d$. Solve for α and form $m(t)$.

7. Solve the equations in (6) by generating a decomposition of Γ and carrying out the steps used in section (7) when the vectors were Euclidean vectors.