EOSC 550: Assignment #3

Due: Wednesday, February 4, 2009

Condition Numbers, Ill-Conditioning, Quadrature Integration

The purpose of this extended question is to examine the practical effects of illconditioning and how the solution of the inverse problem is affected. This provides a sound base for generating practical solutions. The question is extended over two assignments so that we can examine the wonderful aspects of this problem. The numerical questions are based upon the inverse Laplace transform but this typifies difficulties that will be encountered in any inverse problem. We begin with concepts of condition number for a square matrix.

When we generated a minimum norm solution for a linear inverse problem we needed to invert the inner product matrix Γ with elements $\Gamma_{ij} = (g_i, g_j)$. If the kernels are linearly independent then Γ can be determined. In many practical problems however, the kernels become almost linearly dependent. This is reflected in the condition number of the matrix, which for the case of our inner product matrix, is defined as

$$c = \frac{\lambda_{max}}{\lambda_{min}}$$

where λ_{max} and λ_{min} are respectively the largest and smallest eigenvalues of the matrix.

It is insightful to investigate the relationship between condition number of the matrix and the numerical solution obtained from an inversion algorithm. Consider data kernels of the following type

$$g_j(x) = e^{-jx} \qquad j = j_1, ..j_2$$

where x lies in the interval $[0, 1]$.

1. The inner products can be computed analytically. Derive the expression and write a small MATLAB program to generate the matrix corresponding to any set of kernels where $j = j_1, .., j_2$.

2. Plot the eigenvalues for the inner product matrix for $(j = j_1, j_2) = (0, 7)$. What is the condition number for this 8 X 8 matrix?

3. Consider a model

$$m(x) = 1 - \frac{1}{2} cos 2\pi x$$

   Compute the data analytically for $(j = j_1, j_2) = (0, 7)$. You will want to evaluate your expression inside MATLAB so that you can carry enough precision with the data. Tabulate your data using the "long e" format.

4. Calculate the minimum norm model by solving the system $\Gamma\alpha = d$. Plot the recovered model and the true model. Comment on your answer.

5. Develop your own mid-point integration to compute the data for $(j = j_1, j_2) = (0, 7)$. Solve the inverse problem (keeping the same inner product matrix used in part (4). Our goal is to investigate how inaccuracies in the data affect the solution). Plot the recoved and the true model. Comment on your answer. Verify a fundamental formula for the condition number, that is,

$$\frac{||\delta x||}{||x||} < c \frac{||\delta d||}{||d||}$$

6. Repeat the above question but use a numerical integration routine from MATLAB. (use "quad")
7. Repeat the above question but use analytic data that have been truncated at the fifth decimal place.
8. Use the random number generator to add a small amount of noise (0.01%) to your data. That is $\delta d_j = 0.0001 * d_j * randn$. Call your random number generator with a "Seed=1". This allows me to compare final numbers. List the original and new data. Plot the true and constructed models. Verify the inequality in the condition number equation.