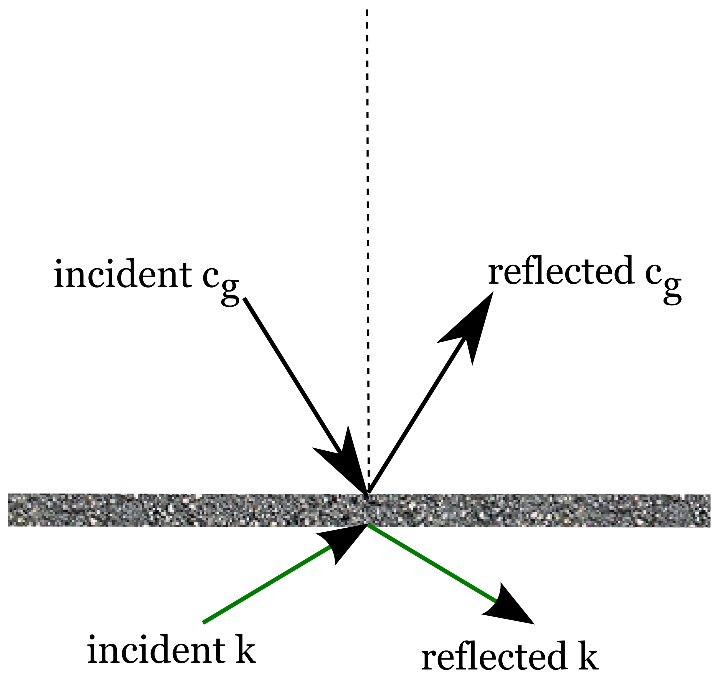
10 Internal Waves: Reflection and Relation to Normal Modes

This lecture is based on a set of notes written by R. Pawlowicz

10.1 Reflection at a Boundary

If we consider waves propagating downward toward a flat bottom, they will reflect upward in a manner that we would expect, except that we must remember that the group speed is down (toward the boundary) means the phase speed (and k) is up. Its easier to draw the k's below the boundary (Figure 10.1).

Figure 10.1 *An incident internal wave reflecting from a flat bottom. Black vectors (in real ocean above the bottom) represent the group speed or direction of energy propagation. Green arrows, shown below the bottom, are the two wave vectors.*

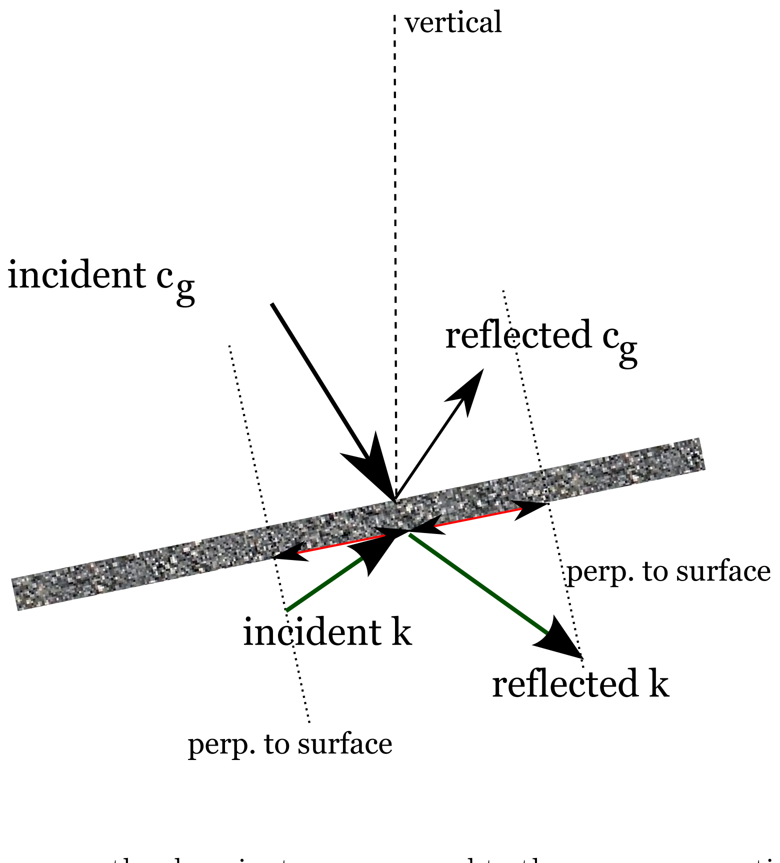


However, if we consider the more realistic case of a slightly tilted bottom, the waves do not reflect around the normal to the surface. The outgoing wave, as it will have the same frequency as the incoming wave, *must* be at the same angle to the vertical as the incoming wave. So the reflection is around a vertical (or horizontal line).

Mathematically one can show that the wavelength along the boundary must match in the outgoing and incoming wave. So the magnitude of $\vec{k}$ projected along the boundary is conserved.

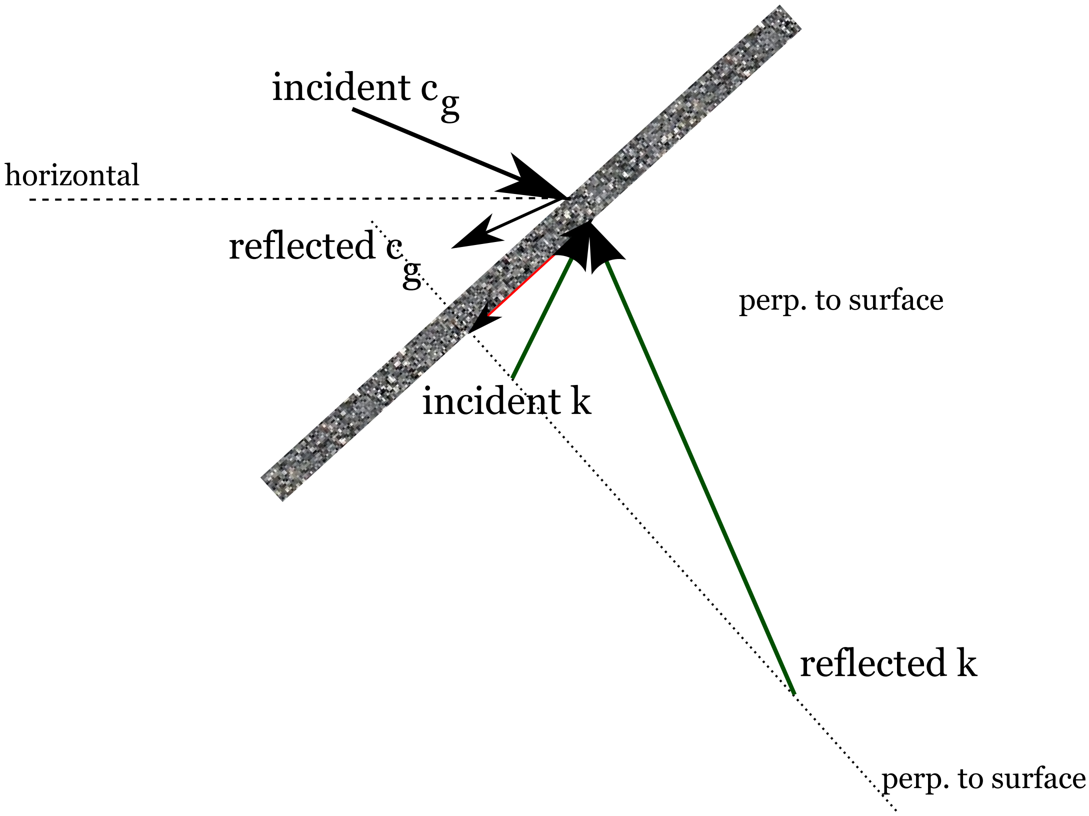
Gentle Slope Waves are *Forward Reflected* and ray in shallower water has a larger $|\vec{k}|$ and is thus the shorter wave (Figure 10.2). It also has a slower group speed.

Figure 10.2 *An incident internal wave reflecting from a gently sloped bottom. Black vectors (in real ocean above the bottom) represent the group speed or direction of energy propagation. Green arrows, shown below the bottom, are the two wave vectors and red vectors are their projection on the surface.*



Steep Slope Steep slope means the slope is steep compared to the wave propagation angle. Waves are *Back Reflected* and the ray away from the shallow water has a larger $|\vec{k}|$ and is thus the shorter wave (Figure 10.3). It also has a slower group speed.

Figure 10.3 *An incident internal wave reflecting from a steep sloped bottom. Black vectors (in real ocean above the bottom) represent the group speed or direction of energy propagation. Green arrows, shown below the bottom, are the two wave vectors and the red vector is their projection on the surface.*



Summary

- Reflection angles are with horizontal or vertical, not with the slope.
- Incoming and outgoing wave numbers and thus wavelengths are different. So there is an exchange of energy between spatial scales.
- For a given frequency, ω, which implies a given angle of propagation, there is a slope which matches that angle of propagation. Incoming waves are reflected to run directly

along the boundary, wave infinite energy there and zero wavelength. This angle is called the critical angle.

- If a tide or other very long (e.g. nearly zero wavelength) oscillation encounters a critical slope, it can generate internal waves. This slope often occurs at the shelf-break and thus the shelf-break or slope just below the shelf-break is often a source of internal waves at tidal frequencies.

10.2 Internal Waves and Internal Modes

Consider an internal wave bouncing up and down between a bottom and the surface. The two are almost flat but not quite so there will be a bit of scattering. After a while, the energy will almost be uniformly distributed from the top to the bottom of the domain because at each horizontal position, there will be as much energy propagating up as there is propagating down.

So consider an internal wave of frequency ω and wavenumber (k, m) and its counterpart travelling upward, frequency ω and wavenumber $(k, -m)$. So

$$w = w_1 + w_2 = w_o \exp[i(kx + mz - \omega t)] + w_o \exp[i(kx - mz - \omega t + \phi)]$$

where ϕ is a phase shift between the two waves. Now at the surface $z = 0$ the vertical velocity must be zero so:

$$\begin{aligned} 0 &= w_o \exp[i(kx - \omega t)] + w_o \exp[i(kx - \omega t + \phi)] && (1) \\ &= w_o \exp[i(kx - \omega t)]\,[1 + \exp(i\phi)] && (2) \end{aligned}$$

So $\exp(i\phi) = -1$. At the bottom, $z = -H$, the vertical velocity must also be zero so:

$$
\begin{aligned}
0 &= w_o \exp[i(kx - mH - \omega t)] + w_o \exp[i(kx + mH - \omega t + \phi)] && (3)\\
&= w_o \exp[i(kx - \omega t)] \left[\exp(-imH) - \exp(imH)\right] && (4)\\
&= \frac{i}{2} w_o \exp[i(kx - \omega t)] \sin(mH) && (5)
\end{aligned}
$$

So $mH = n\pi$ Substitute this into the internal wave dispersion relation

$$
\begin{aligned}
\omega^2 &= \frac{N^2(k^2 + \ell^2) + f^2 m^2}{k^2 + \ell^2 + m^2} && (6)\\
&= \frac{f^2 + a_n^2 k^2}{1 + \left(\frac{kH}{n\pi}\right)} && (7)\\
&\approx f^2 + a_n^2 k^2 && (8)
\end{aligned}
$$

where we have neglected ℓ and assumed that the horizontal wavelength is large compared to the depth. $a_n = NH/n\pi$ which is the nth baroclinic Rossby radius. Thus our dispersion relation is the dispersion relation for internal Poinare waves.

If we bounce internal waves up and down long enough, we distribute the energy uniformly with depth and we get vertical normal modes! Mode 1 waves look like Plate 10.1.

Plate 10.1: Movie showing particle movement in internal, mode 1 waves. Copyright M. Tomczak. Available at `http://www.es.flinders.edu.au/~mattom/IntroOc/notes/figures/animations/fig10a7c.html`.