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Stable reduction to the pole at the magnetic equator

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ABSTRACT

The solution of reduction to the pole (RTP) of magnetic data in the wavenumber domain faces a long standing difficulty of instability when the observed data are acquired at low magnetic latitudes or at the equator. We develop a solution to this problem that allows stable reconstruction of the RTP field with a high fidelity even at the magnetic equator. The solution is obtained by inverting the Fourier transform of the observed magnetic data in the wavenumber domain with explicit regularization. The degree of regularization is chosen according to the estimated error level in the data. The Fourier transform of the RTP field is constructed as a model that is maximally smooth and, at the same time, has a power-spectral decay common to all fields produced by the same source. The applied regularization alleviates the singularity associated with the wavenumber-domain RTP operator, and the imposed power spectral decay ensures that the constructed RTP field has the correct spectral content. As a result, the algorithm can perform the reduction to the pole stably at any magnetic latitude, and the constructed RTP field yields a good representation of the true field at the pole even when the reduction is carried out at the equator.

INTRODUCTION

The reduction-to-the-pole (RTP) operation computes, from the observed magnetic field, the vertical magnetic field that would be observed if the magnetization were also vertical. This operation was first introduced by Baranov (1957) in an effort to rid magnetic anomalies of the complicating patterns due to the oblique angles of magnetization and anomaly projection. RTP has become almost a standard part of data processing, especially for large-scale maps, and there is a plethora of literature on this subject. Following the work of Bhattacharyya (1965) and the advent of the fast Fourier transform (FFT), most work has been treating this problem as a simple filtering operation performed in the wavenumber domain by multiplying the Fourier transform of the observed magnetic field with the RTP operator in the wavenumber domain.

As the magnetic latitude approaches the equator, however, the RTP operator becomes unbounded along the direction of magnetic declination and therefore amplifies the noise in this direction to the extent that the resultant RTP field is dominated by linear features aligned with the direction of declination. Faced with this difficulty, much effort has been spent on formulating stable approximations of the RTP operator (Pearson and Skinner, 1982; Hansen and Pawlowski, 1989; Mendonca and Silva, 1993). Most approaches focussed upon the stabilization of the operator without regards to the actual noise in the data, and thus postreduction processing was sometimes required to suppress the noise. Hansen and Pawlowski (1989), however, developed an elegant method in which a Wiener filtering technique is used to form a regularized RTP operator. The important attribute of their work is that the stabilization of the operator is carried out in accordance with the noise in the observed field in the form of an estimated noise power spectrum. Since this approach, and others in the literature, are based upon filtering, little emphasis is placed upon being able to reproduce the observed data. Keating and Zerbo (1996) emphasize the need of better reproduction of the data as a factor for improved RTP results. They use an ad hoc method of iterative refinement to obtain a better fit between the observed data and the data calculated from the RTP field. Despite much of the effort, the existing RTP procedure still works only with moderately low magnetic latitudes. To our best knowledge, the lowest latitude at which stable reduction is achieved in the published literature is 10°. Stable reduction at the magnetic equator has not been obtained.

In this paper, we propose a comprehensive solution for the RTP operation under a general framework of an inverse formulation. The RTP field is constructed by solving an inverse problem in which a global objective function is minimized subject to fitting the observed magnetic field. The development of this

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method leads to improvements of practical importance over the existing methods. The RTP operation can be performed equally stably at any latitude, including the equator.

INVERSE FORMULATION

Assume that we have a set of observed magnetic data acquired on a planar observation surface, and that they are measured in the direction $\hat{v}$ and are produced by the magnetization in the direction $\hat{u}$. Let T be the Fourier transform of these observed magnetic data, R be the Fourier transform of the vertical magnetic field that would be observed at the same locations when the magnetization is also vertical (the RTP field), and (p, q) be the wavenumbers corresponding to the (x, y) directions of the spatial coordinates. Then the observed field is related to the RTP field by a projection operator $G(p, q)$,

$$T(p,q) = G(p,q)R(p,q) \equiv \frac{(\hat{u}\cdot\mathbf{k})(\hat{v}\cdot\mathbf{k})}{p^2+q^2}R(p,q), \tag{1}$$

where $\mathbf{k} = (ip, iq, \sqrt{p^2+q^2})$ and $i = \sqrt{-1}$. Here, the implicit assumption is that the magnitude of the magnetization remains the same while the direction is changed to being vertical. Theoretically, the inverse of the projection operator gives the RTP operator that transforms the observed data into RTP data:

$$R(p,q) = G^{-1}(p,q)T(p,q). \tag{2}$$

At high magnetic latitudes (for example, greater than 30°), the RTP operator is well defined and the resulting RTP data are of satisfactory quality. At low magnetic latitude (or when the anomaly projection direction approaches being horizontal), the factor $\hat{u}\cdot\mathbf{k}$ (or $\hat{v}\cdot\mathbf{k}$) approaches zero in the neighbourhood of $u_x p + u_y q = 0$ (or $v_x p + v_y q = 0$), which is along the direction of the declination, since u_z (or v_z) approaches zero. This causes the projection operator $G(p, q)$ to approach zero and the RTP operator to be unbounded. Consequently, the RTP process is unstable at low latitudes. Physically, a nearly zero projection operator means that the Fourier transform, $T(p, q)$, of the observed field approaches zero along the declination direction no matter what the value of $R(p, q)$ is at those frequency points. The RTP process attempts to recover a finite value of $R(p, q)$ from nearly zero values of $T(p, q)$ by applying a large amplification, and any noise that is present in the data will be amplified by the same degree. As a result, the RTP process becomes unstable at low magnetic latitudes and undefined at the equator.

To overcome this difficulty, the existing work has focussed upon constructing a stable approximation of the RTP operator near these regions. Since the resulting approximation is one that still operates pointwise on the Fourier transform $T(p, q)$ of the observation, it is a local approximation. The derivation of $R(p, q)$ at any point (p, q) is decoupled from all other points and, hence, the global information embedded in $T(p, q)$ as a function is not used. With this realization, we take a more global approach to derive $R(p, q)$.

In practical applications, the Fourier transform T is obtained through a discrete Fourier transform (typically an FFT). Therefore T is defined over a set of discrete points (p_i, q_i) within a rectangular domain. Correspondingly, we seek to derive R over the identical set of points. Under these assumptions, equation (1) can be written in matrix format as

$$\mathbf{d} = \mathbf{G}\mathbf{m}, \tag{3}$$

where

$$\begin{aligned} \mathbf{d} &= (\ldots, T(p_i, q_i), \ldots)^T, \\ \mathbf{m} &= (\ldots, R(p_i, q_i), \ldots)^T, \\ \mathbf{G} &= \mathrm{diag}\{\ldots, G(p_i, q_i), \ldots\}. \end{aligned} \tag{4}$$

Here the superscript T denotes transpose. Let N be the total number of discrete points in the Fourier domain. Then both $\mathbf{d}$ and $\mathbf{m}$ are N-length vectors and $\mathbf{G}$ is an $N \times N$ matrix. In this form, when $G(p_i, q_i) \to 0$, it means that the matrix $\mathbf{G}$ is singular. Thus we have a classical inverse problem involving a singular operator, and it is therefore mathematically sensible to invert it by regularization.

To apply inverse theory, we formally specify the Fourier transform of the RTP field, $\mathbf{m}$, as the model to be recovered, and $\mathbf{d}$ as the data to be inverted. The elements of matrix $\mathbf{G}$ are defined analytically in equation (1) but, for practical applications, they are evaluated more appropriately by a numerical method (see Appendix). The model, data, and the matrix are all complex. There are the same number of model parameters to be recovered as there are data and, although the problem is even-determined, it is still ill-posed and the solution is nonunique. Thus we take the approach of minimizing a model objective function subject to fitting the data.

We introduce a weighted model objective function of the form

$$\phi_m = \int_p \int_q \left[\alpha_s |s(\omega)R(p,q)|^2 + \alpha_p \left| \frac{\partial s(\omega)R(p,q)}{\partial p} \right|^2 + \alpha_q \left| \frac{\partial s(\omega)R(p,q)}{\partial q} \right|^2 \right] dp\,dq, \tag{5}$$

where α_s, α_p, and α_q are positive constants controling the relative importance of the smallest model term and the two flatness terms. The requirement of flatness for the RTP spectrum is reasonable since the Fourier spectrum of the magnetic field produced by subsurface sources is generally a smooth function with no abrupt changes. The function $s(\omega)$ is a weighting function that depends only on the radial wavenumber $\omega = \sqrt{p^2+q^2}$. The radially symmetric weighting function $s(\omega)$ is used to incorporate desired spectral decay of $R(p, q)$ into the inversion. The choice of the weighting function will be discussed in detail later.

Discretizing equation (5) using a finite difference approximation within the rectangular domain of (p, q) gives the discrete form of the model objective function:

$$\begin{aligned} \phi_m &= \mathbf{m}^H \mathbf{S}^T \left(\alpha_s \mathbf{I} + \alpha_p \mathbf{D}_p^T \mathbf{D}_p + \alpha_q \mathbf{D}_q^T \mathbf{D}_q \right) \mathbf{S}\mathbf{m} \\ &\equiv \mathbf{m}^H \mathbf{S}^T \mathbf{W}^T \mathbf{W} \mathbf{S} \mathbf{m}, \end{aligned} \tag{6}$$

where superscript H denotes the complex conjugate transpose, $\mathbf{S}$ is a diagonal matrix whose elements are the discretized weighting function, and $\mathbf{W}^T\mathbf{W}$ is the weighting matrix defined by the difference operators, $\mathbf{D}_p$ and $\mathbf{D}_q$, in the p- and q-directions.

The data misfit function is defined as

$$\phi_d = \sum_{i=1}^{N} \left| T(p_i, q_i) - T^{pre}(p_i, q_i) \right|^2, \quad (7)$$

where T^{pre} are the predicted data calculated using equation (3). Assume that the observed data in the spatial domain all have independent Gaussian noise with zero mean and standard deviations σ_i, then the expected value ϕ_d^* of ϕ_d is

$$\phi_d^* = \sum_{i=1}^{N} \sigma_i^2. \quad (8)$$

This value is the expected power spectrum of the random noise contaminating the original observations. If the standard deviations of the error associated with the original observations are known, then the target misfit can be calculated by equation (8). Otherwise, a constant noise power spectrum can be estimated from the data power spectrum $|T(p, q)|^2$ in the wavenumber domain and used as the target misfit.

The inverse problem of reconstructing the RTP field spectrum is then solved by minimizing a total objective function, ϕ, consisting of a weighted sum of the model objective function and data misfit,

$$\phi = \phi_d + \mu\phi_m, \quad (9)$$

where μ is the regularization parameter. An appropriate value is chosen for μ such that the target misfit is achieved. This usually involves a line search.

Minimizing equation (9) with respect to the complex model **m** using a variational principle yields

$$(\mathbf{G}^H\mathbf{G} + \mu\mathbf{S}^T\mathbf{W}^T\mathbf{W}\mathbf{S})\mathbf{m} = \mathbf{G}^H\mathbf{d}. \quad (10)$$

Here the term $\mathbf{G}^H\mathbf{G}$ is a real, diagonal matrix since $\mathbf{G}$ is diagonal (the superscript H denotes complex conjugate transpose). Thus, the matrix in the parenthesis is extremely sparse, we can employ a conjugate gradient (CG) technique to solve the system. Since the matrix is real, the solution can be obtained either by applying a complex CG solver to the whole system, or by applying a real CG solver separately to the real and imaginary parts of the system.

SPECTRAL WEIGHTING FUNCTION

The power spectrum of surface magnetic field data generally exhibits a decay with the increasing wavenumber, and there is a large discrepancy in amplitude of the model **m** at low and high wavenumbers. A simple model objective function without the spectral weighting $s(\omega)$ will not be an adequate measure for a "good" RTP model. Without the spectral weighting $s(\omega)$, the model objective function in equation (5) would favor a recovered RTP spectrum that has equal amplitude over the entire wavenumber band. It is therefore necessary to incorporate the knowledge about the spectral decay through the use of the weighting function. Suppose that the spectrum of the RTP field is dependent upon the radial wavenumber ω only, then a spectral weighting $s(\omega)$ that is inversely proportional to the amplitude spectrum of the $R(p, q)$ would make the product $s(\omega)R(p, q)$ constant. Consequently, the derivative terms in the objective function in equation (5) would approach zero, and the inversion constructs a constant model of $s(\omega)R(p, q)$ that minimizes the smallness term. The recovered RTP spectrum $R(p, q)$ would then have the correct decay with the wavenumber.

The radially averaged power spectrum of the potential field data provides an overall description of this decay. Thus, we choose to use the desired power spectrum of the RTP field to form the weighting. If $P_R(\omega)$ is the radial power spectrum of $R(p, q)$, then the $s(\omega) = P_R^{-1/2}(\omega)$ provides the right degree of weighting; this is small at low wavenumbers and large at high wavenumbers.

Of course, the power spectrum $P_R(\omega)$ is unknown and we must therefore estimate it prior to inversion. We have observed that the radially averaged power spectra of the magnetic fields produced by the same causative body at different magnetic latitudes are similar, and their decay with the radial wavenumber can be approximated with a single function. Thus the expected power spectrum of RTP field can be estimated directly from the data **d**. We first illustrate this property with a synthetic model, and then present a method of implementation.

The synthetic model consists of a single prism in a nonsusceptible background. The prism has a width of 20 m in both horizontal directions and a thickness of 2 m, and the depth to the top is 1 m. Total field anomalies are calculated over a 64×64 grid with a 1-m interval in each direction. [This model was first used by Hansen and Pawlowski (1989) to illustrate their RTP algorithm and has since been used by others in their work, and so it is a good example for comparative studies.] We have calculated the total field anomaly at different magnetic latitudes ranging from the equator to the pole at a 10° increment. The declination is 0° in all cases. Figure 1 shows the total field anomalies at the equator and the pole. From these accurate data, we have calculated the radially averaged power spectra which are shown in Figure 2. It is clear that they cluster within a narrow zone on a log-log plot. Therefore, the radial power spectrum estimated from data taken at a lower latitude can be used to represent the radial power spectrum of the corresponding RTP field.

Clearly, the power spectra have much structure in it. However, we only need the general decay to define a spectra weighting function. A simple functional approximation should therefore suffice. We have found that a function of the form

$$f(\omega) = P_0\left[1 + \left(\frac{\omega}{\omega_0}\right)^2\right]^{-\beta} \quad (11)$$

well approximates the radially averaged power spectra like those in Figure 2, where P_0, ω_0, and β are constants to be determined. Only ω_0 and β are related to the decay of the power spectrum. This empirical functional form provides a generally good description of power spectra that vary slowly in near-zero wavenumber band and then exhibit a straight line decay as the wavenumber increases. We can fit this representation to the calculated power spectrum of data to determine the three unknown constants. Once this is done, the spectral weighting is specified by using two parameters,

$$s(\omega) = \left[1 + \left(\frac{\omega}{\omega_0}\right)^2\right]^{\beta/2}. \quad (12)$$

Since we are only concerned with the spectral decay, the parameter P_0 is not needed to define the weighting function.

When the data are contaminated by noise, the radial power spectrum will plateau in the high wavenumber band instead of exhibiting the continued decay as in Figure 2. This is because the power spectrum of the contaminating noise is not decaying, and it dominates at high wavenumbers. A commonly-used assumption is that the noise is not correlated with the magnetic anomaly and that it has a flat spectrum. This is the case when the noise is Gaussian and independent. Under this assumption, the power spectrum of the noisy data can be represented as

$$f_n(\omega) = P_0\left[1 + \left(\frac{\omega}{\omega_0}\right)^2\right]^{-\beta} + P_n, \qquad (13)$$

where P_n is the power spectrum of the noise. We note that four parameters are needed in equation (13) to represent the spectrum. Nevertheless, the weighting function $s(\omega)$ used in the model objective function, needs only ω_0 and β in accordance with equation (12).

To determine these parameters, we perform a nonlinear least-squares fitting on the log-log scale by minimizing the following quantity,

$$\psi(\omega_0, \beta, P_n) = \int_{\omega_1}^{\omega_2}\left[\ln\frac{P_T(\omega)}{f_n(\omega)}\right]^2 d\ln\omega, \qquad (14)$$

where $P_T(\omega)$ is the radially averaged power spectrum of the data, ω_1 is the smallest nonzero value of the discrete wavenumber, and ω_2 is the highest radial wavenumber at which the $P_T(\omega)$ is calculated. Since $P_T(\omega)$ is defined over a set of discrete wavenumbers, the integral is evaluated using a trapezoid rule. We employ a downhill simplex method to carry out the minimization. If the standard deviations of the noise are known, P_n can be calculated and the minimization needs only to determine ω_0 and β. If the standard deviations are unknown, then all three parameters are determined from the least-squares fitting. The resulting P_n can also be used to define the target misfit ϕ_d^*.

SYNTHETIC EXAMPLES

As the first example, we reduce the total field anomaly at the equator (shown in Figure 1a) to the pole after adding random noise to it. The contaminating noise is uncorrelated Gaussian random numbers with zero mean and a standard deviation of 1 nT. The resultant noisy anomaly is shown in Figure 3a. Figure 3b shows the radially averaged power spectrum of the noisy anomaly and the corresponding representation by f_n derived through the least-squares fit, which produced values of $\beta = 2.26$ and $\omega_0 = 0.228$.

Using the calculated parameter ω_0 and β to generate the spectral weighting according to equation (12) and carrying out the reduction process, we obtained the RTP result shown in Figure 4b. It clearly shows the central high above the causative body and four negative sidelobes. There is little striping along

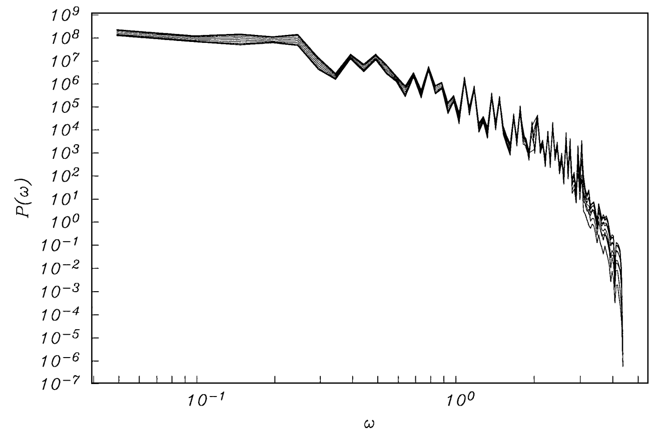

FIG. 2. The radially average power spectra of total field anomalies calculated at ten different magnetic latitudes with a declination of 0°. The latitudes range from the equator to the pole in increments of 10°.

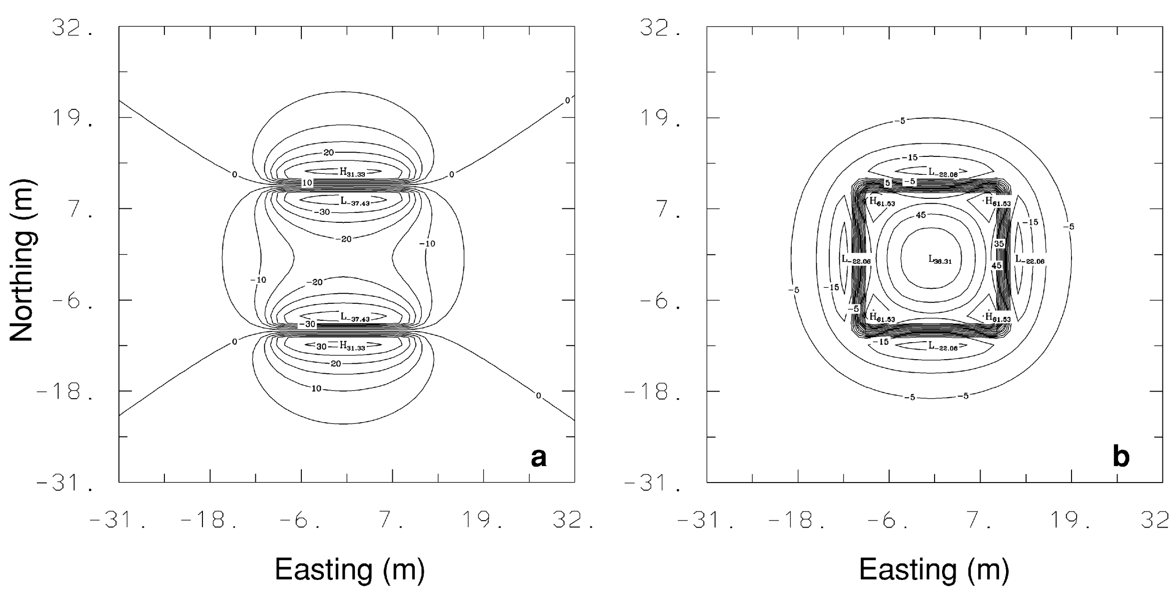

FIG. 1. Total field anomalies at the equator (a) and at the pole (b). The causative body is a prism that has a width of 20 m in both horizontal directions, a thickness of 2 m, and a depth to the top of 1 m. The declination is 0° in both cases. The contour interval is 5 nT.

the direction of declination, and the amplitude agrees well with the true value. Gradients in the east-west direction are weaker than their true values. The overall correspondence between the recovered RTP field and the true field at the pole shown in Figure 1b is good. This is a good result, considering that the projection operator is singular at the equator and the RTP operator is undefined in the traditional sense. Displayed in Figure 4a is the predicted total field data at the equator, which is calculated by applying the projection operator to the recovered RTP field. The predicted data misfit the noisy data by the expected amount, and they are a good representation of the true field at the equator as shown in Figure 1a.

The performance of the algorithm can be understood by examining power spectra of recovered RTP fields. We first invert the above data set using only the weighted smallest model objective function by setting $\alpha_p = \alpha_q = 0$ in equation (5). This is equivalent to using a traditional regularized RTP operator. (We provide details about this in a generalized comment at the end of this paper.) Here, for brevity, we have not produced the RTP field but, instead, only its power spectrum, which is shown in Figure 5a. A notch, in the direction of the magnetic declination, is clearly evident, and this leads to the elongation of the RTP field in the declination direction. The notch arises because the objective function can be minimized by having

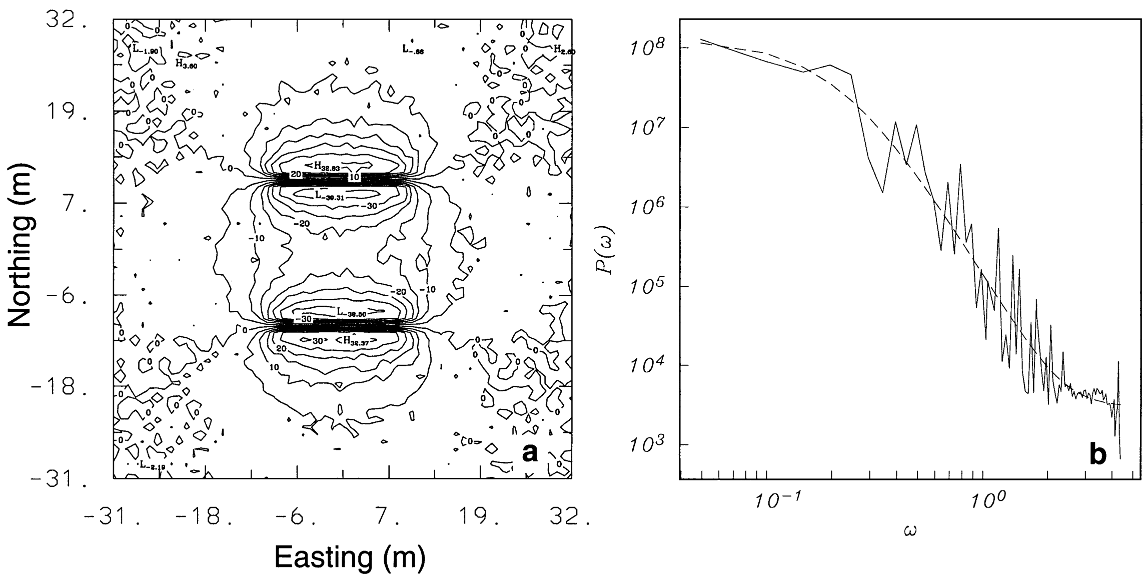

FIG. 3. Panel (a) shows the noisy total field data at the equator. The contaminating noise is uncorrelated Gaussian noise and has a zero mean and a standard deviation of 1 nT. Panel (b) illustrates the smooth representation of the radially averaged power spectrum. The solid line is the radially averaged power spectrum of the total field data in (a). The dashed line is the representing function in equation (13). The resulting parameters are $\beta = 2.26$ and $\omega_0 = 0.228$, which are used in the inversion to recover the RTP field in Figure 4b.

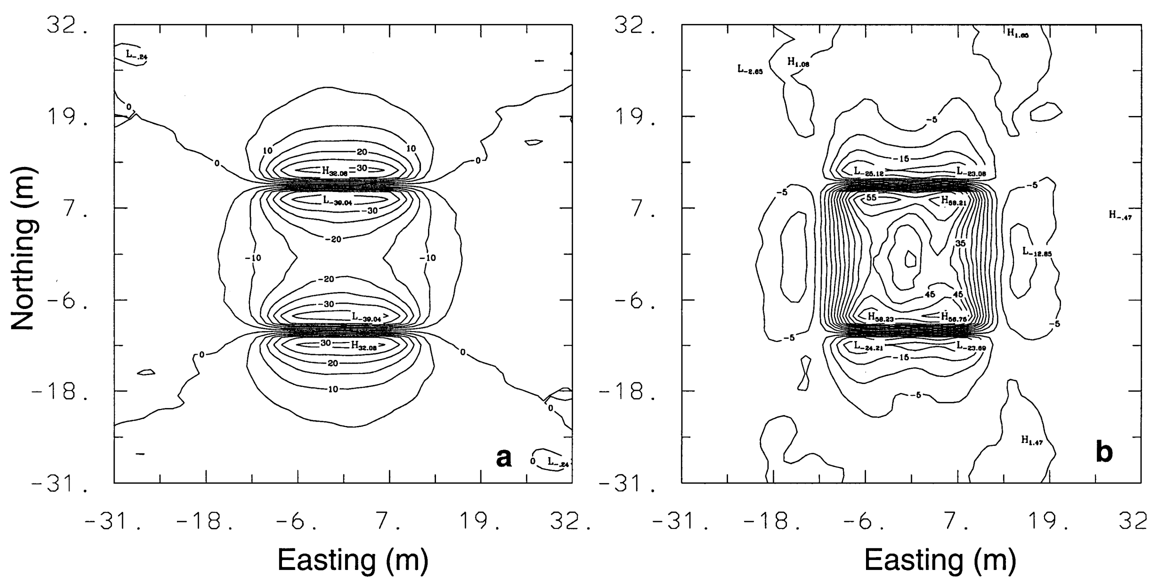

FIG. 4. Panel (a) shows predicted total field data that are produced when the noisy data in Figure 3a are reduced to the pole using the inversion algorithm. They can be compared with the true data in Figure 1a. Panel (b) shows the RTP field. This result can be compared with the true field at the pole shown in Figure 1b; it is a good representation.

very small values of the model (the RTP field), and the resultant model can still reproduce the near-zero field data there because the projection operator $G(p, q)$ is essentially zero. In comparison, the RTP power spectrum from the current algorithm in Figure 5b does not show the notch. This is because the derivative terms in the model objective function require the weighted power spectrum to be as flat as possible. Therefore, although the data do not require a high power spectrum along the direction of declination, the model objective function causes the notch region to be smoothly filled and hence a more realistic spectrum of the RTP field is produced. This is what makes the current algorithm perform better at low magnetic latitudes.

Finally, as a comparison with the published results in the literature, we have also applied this algorithm to the total field data at a magnetic latitude of 15°. Figure 6a shows the noise-contaminated data. The standard deviation of the Gaussian noise is again 1 nT. Figure 6b shows the RTP result. This can be compared with the RTP results under the same condition in Hansen and Pawlowski (1989), Mendonca and Silva (1993), and Keating and Zerbo (1996). We feel that our result has greater symmetry and fewer artifacts.

FIELD EXAMPLE

The total field magnetic data set from Dixon Seamount has become a classic example in illustrating RTP algorithms, and

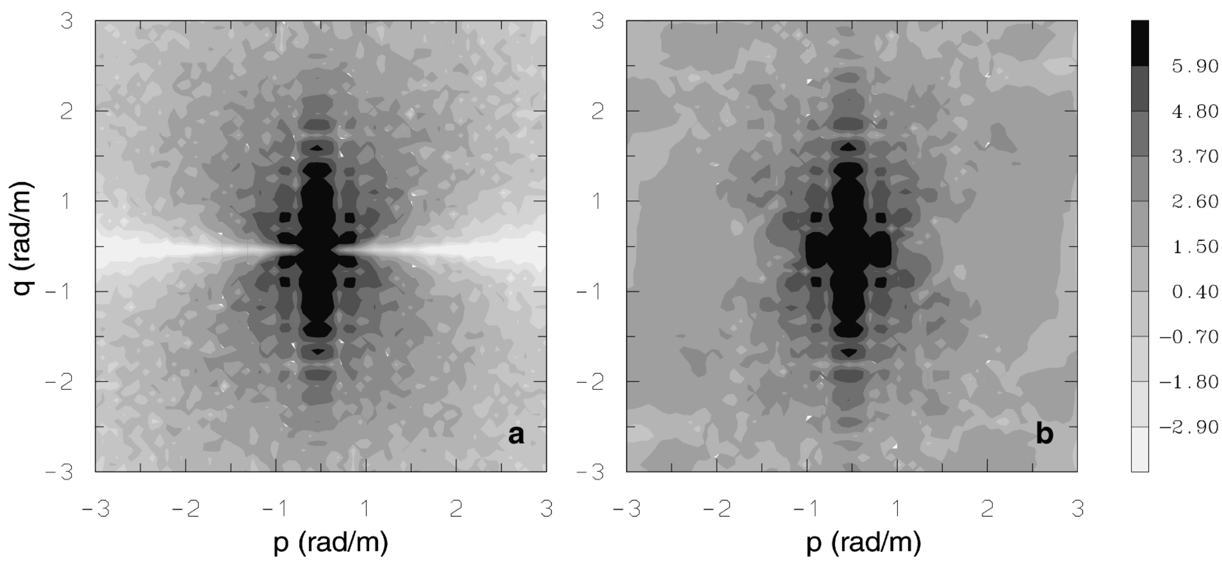

FIG. 5. The power spectra of the RTP fields recovered with different model objective functions. The gray scale indicates the logarithm of the power spectrum. Panel (a) shows the result from the smallest model inversion, which still has the notch in the direction of declination. Panel (b) shows the spectrum recovered with the model objective that includes the derivative terms. The notch in the declination direction is no longer present.

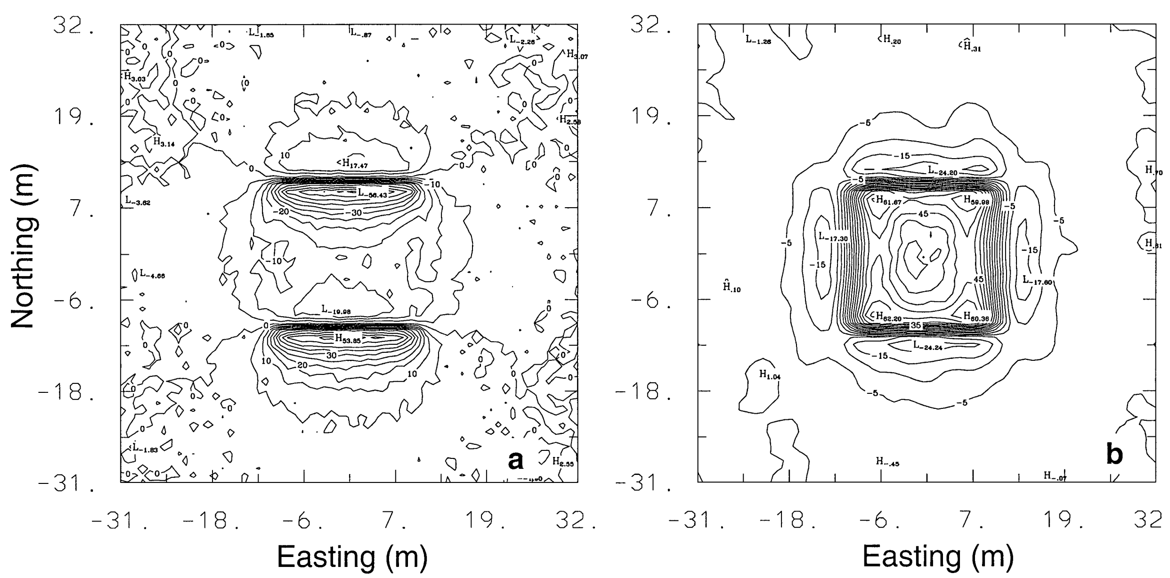

FIG. 6. Panel (a) shows the noisy total field data at the magnetic latitude of 15°. The contaminating noise is uncorrelated Gaussian noise and has a zero mean and a standard deviation of 1 nT. Panel (b) shows the RTP field. This can be compared with different results in the literature.

it serves as a test over which different algorithms can be compared. We therefore also use it as our field example here. The total field data were collected using a towed magnetometer along sparsely distributed tracks (Francheteau et al., 1969). The data were subsequently gridded on a 64 × 64 grid with a spacing of 1 km. Figure 7a displays the data at a contour interval of 100 nT. This data set has been reduced to the pole in several papers (Pearson and Skinner, 1982; Hansen and Pawlowski, 1989; Mendonca and Silva, 1993) using an inclination of 17° and a declination of 9° for both anomaly projection and magnetization. Since gridding was carried out by interpolating and extrapolating widely-spaced observations, the data in Figure 7a are contaminated by noise having long correlation lengths. This is evident in the smooth decay of the radially averaged power spectrum. Hansen and Pawlowski (1989) estimate an rms noise power of 190 nT. We use this value to define the target misfit for our inversion procedure. The estimated power spectral decay is defined by $\omega_0 = 0.4$ and $\beta = 3.35$, and this is incorporated in the inversion through the spectral weighting. The resulting RTP field is displayed in Figure 7b and can be compared with that from Hansen and Pawlowski (1989). There is a general agreement between these two results. Both achieve a similar suppression of high wavenumber noise, but our inversion preserves more details in the main anomaly above the Seamount and does not have apparent structure elongated in the direction of 9° declination. Our RTP field more closely resembles the RTP field computed using the magnetization and geometric parameters of the Seamount that was obtained by Pearson and Skinner (1982, their Figure 3). This may be a validation of the improvement that is produced by a regularized inversion procedure.

DISCUSSIONS

There are three important attributes in our algorithm. The first is the use of the regularization by a model objective function that includes derivative terms. This imposes a global property of smoothness on the recovered $R(p,q)$, and thus it enables the stable recovery of $R(p,q)$ in the neighborhood of $u_x p + u_y q = 0$ (or $v_x p + v_y q = 0$). The second is the degree of regularization determined according to the data error. This ensures that the resultant RTP field is consistent with the data in that $T(p,q)$ are adequately reproduced. The third attribute is the spectral weighting function. This ensures that the recovered $R(p,q)$ has the desired spectral decay. One added benefit is that the noise in the high wavenumber band is automatically suppressed. It is these components that make the current algorithm perform better than existing methods.

Although it is not a realistic radial spectrum, equation (11) is a reasonable representation of the general decay of radially averaged power spectra of magnetic data that are produced by sources with similar distribution in size and depth. In such cases, the power spectra are dominated by a inverse power-law decay followed by a more or less constant noise power. Consequently, the form of spectral weighting given in equation (12) is effective in general. However, there are magnetic data sets whose power spectra are different. For instance, there could be two or more segments that decay as $1/\omega^\beta$ but have different values of β. Then, an appropriate approximation needs to be constructed and the corresponding spectral weighting be used in the inversion. However, exact match between the power spectrum and the representation is not necessary since the spectral weighting only serves to impose a general decay on the recovered RTP field, and the details of the recovered $\mathbf{r}$ are governed by the data constraints and the imposed flatness.

It is interesting to note that many existing methods can be understood as special cases of our algorithm. These methods all seek to form a stable approximation of the RTP operator $G^{-1}(p,q)$ with no consideration about the values of neighboring points. This amounts to replacing $\mathbf{S}^T\mathbf{W}^T\mathbf{W}\mathbf{S}$ in equation (6) by a diagonal weighting matrix. Different choices give rise to

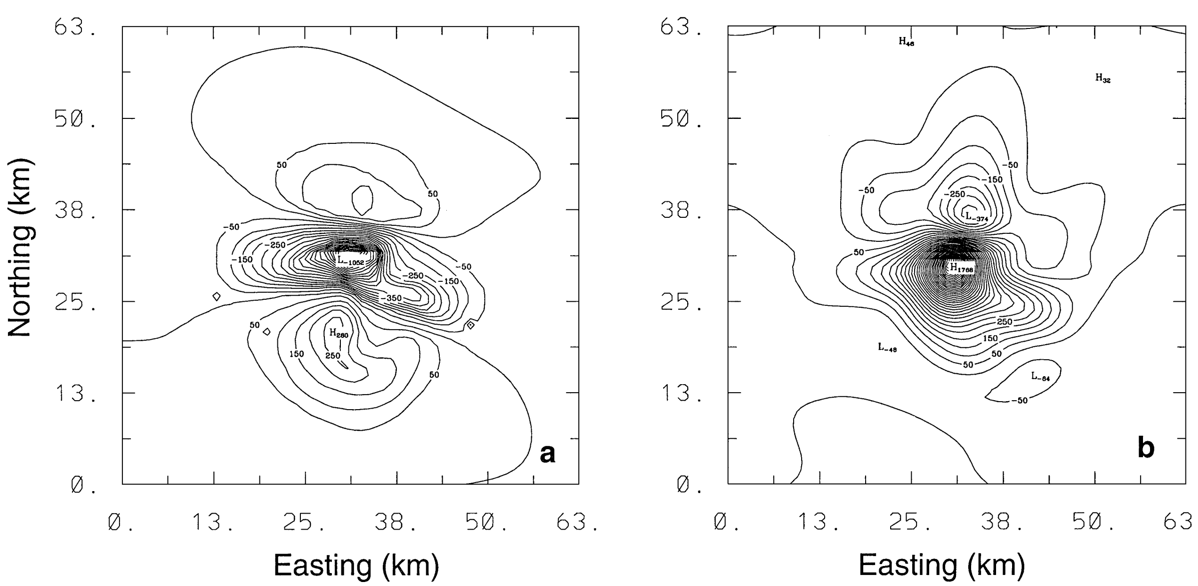

FIG. 7. Panel (a) shows the total field data over Dixon Seamount, which is located at a magnetic latitude of 17° and declination of 9°. The data are located on a 64 × 64 grid. Panel (b) shows the RTP field using the proposed inversion-based procedure. The contour interval is 50 nT in both panels. This RTP field can be compared with different results in the literature; it shows improvement in recovering the details of the main anomaly and suppressing the striping in the direction of declination.

different methods. In particular, if we set $\alpha_s = 1.0$, $\alpha_p = \alpha_q = 0$, and the regularization parameter to be the noise power spectrum (i.e., $\mu = P_n$), the solution of equation (10) has the simple form of

$$R(p,q) = \frac{G^*}{G^*G + P_n\left[1 + (\omega/\omega_0)^2\right]^\beta} T(p,q) = \frac{G^*}{G^*G + \dfrac{P_n}{f(\omega)}} T(p,q), \tag{15}$$

where $G = G(p,q)$ is the projection operator, the superscript $*$ denotes complex conjugate, and $f(\omega)$ is the representation of power spectrum given in equation (11). This is precisely the formula derived by Hansen and Pawlowski (1989) through the Wiener filtering. Thus Hansen and Pawlowski's method is derived as a special case of the inversion algorithm when we choose a model objective function that consists of only the smallest model term, incorporate the spectral weighting, and then set the value of the regularization parameter to being equal to the noise power spectrum.

CONCLUSION

We have developed an inversion-based reduction-to-the-pole method in the wavenumber domain. The projection operation that calculates the magnetic field at arbitrary latitudes from the vertical field at the pole is treated as a forward modeling process. This allows the RTP to be formulated as an inverse problem and solved through regularization. The final solution field is obtained by minimizing a model objective function that imposes flatness and a desired spectral decay on the Fourier transform of the RTP field. The degree of regularization is chosen so that the observed magnetic data are adequately reproduced. The method can be applied, without restriction, at low latitudes or at the equator.

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REFERENCES

Baranov, V., 1957, A new method for interpretation of aeromagnetic maps: Pseudo-gravimetric anomalies: Geophysics, **22**, 359–383.
Bhattacharyya, B. K., 1965, Two-dimensional harmonic analysis as a tool for magnetic interpretation: Geophysics, **30**, 829–857.
Cordell, L., and Grauch, V. J. S., 1982, Reconciliation of the discrete and integral Fourier transforms: Geophysics, **47**, 237–243.
Francheteau, J., Sclater, J. G., and Craig, H., 1969, Magnetization of a newly discovered seamount in the central Pacific: Geophysics, **34**, 645–651.
Hansen, R. O., and Pawlowski, R. S., 1989, Reduction to the pole at low latitude by Wiener filtering: Geophysics, **54**, 1607–1613.
Keating, P., and Zerbo, L., 1996, An improved technique for reduction to the pole at low latitudes: Geophysics, **61**, 131–137.
Mendonca, C. A., and Silva, J. B. C., 1993, A stable truncated series approximation of the reduction-to-the-pole operator: Geophysics, **58**, 1084–1090.
Pearson, W. C., and Skinner, C. M., 1982, Reduction-to-the-pole of low latitude magnetic anomalies: 52nd Ann. Internat. Mtg., Soc. Expl. Geophys., Expanded Abstracts, 356–358.

APPENDIX A

NUMERICAL PROJECTION OPERATOR

The elements of coefficient matrix **G** are analytically defined by equation (1), which is strictly valid if the area of the data map is infinite. For the finite areas in the practical applications $G(p,q)$, calculated according to that expression, is inconsistent with the data **m** calculated by discrete Fourier transform. The inconsistency is caused by the difference between the integral and discrete Fourier transforms demonstrated by Cordell and Grauch (1982). To overcome this difficulty, we use a more general definition of $G(p,q)$ by rearranging the terms in equation (1):

$$G(p,q) = \frac{T(p,q)}{R(p,q)}, \tag{A-1}$$

that is, the projection operator is the ratio of the Fourier transform of the field with the observing parameter to the Fourier transform of the field at the pole. This provides a means for numerically calculating the projection operator for a finite area. Given a reasonably chosen source, the discrete Fourier transforms of its magnetic fields over the area can be computed, and the ratio then defines the projection operator according to equation (A-1). When the area is that of the observed data, we obtain the projection operator that is consistent with the data **d** calculated by discrete Fourier transform.

We choose the source to be a uniformly magnetized rectangular prism with widths equal to the grid spacings of the observed data in horizontal directions. The depth to the top of the prism is equal to a fraction of the grid spacing and the depth extent is several times of the grid spacing. The magnetic fields are calculated from the prism assuming it is horizontally centered beneath a datum point near the center of the grid. The actual horizontal location of the prism, however, is not crucial. Since the change in the horizontal location only introduces a phase shift in the Fourier transforms of the fields, the effect is cancelled out when the ratio is taken to generate the operator $G(p,q)$.

In general, both the analytical and numerical projection operator produce satisfactory results, but tests have indicated that the RTP field computed using the numerical projection operator is superior. Thus, we prefer to use the numerical projection operator for practical applications.