Funnel Functions in Linear and Nonlinear Appraisal

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This paper presents new methods for appraising the nonuniqueness inherent in linear and nonlinear inversion problems. The usual approach of constructing unique averages of the model is abandoned in favor of computing upper and lower bounds to assure products of the model with exact boundary functions of variable width. The bounds are found by extremizing a suitable objective function and using a linear programming problem. Information about the average value of the model is presented in terms of funnel functions and uncertainty functions. In an alternate approach, information about the model averages is summarized by global bound curves. These functions are upper and lower bounds to the individual integral of the model, and they provide a plausible insight about the resolving power of the data. Once computed, they can also be used to generate approximate funnel functions. The methods presented here are also applicable to the nonlinear problem, but a new reformulation is necessary so that a global area of the model can be extremized. This has been done, but its global extremum is achieved only through iteration. Consequently, the potential difficulties arising from nonconvergence of a linearized algorithm are avoidable consequence to a local, rather than global, extremum are ever present. As an example of the nonuniqueness problem, I have analyzed the magnetotelluric responses taken near an ocean spreading center in the Pacific. The data are shown to process very poor resolving power, and hence the large variability in acceptable models is now understood.

INTRODUCTION

The inversion of geophysical data sets has received a great deal of attention over the past two decades. The literature abounds with different approaches, but most of the papers can be classified into one of two groups. Papers within the first group are concerned with constructing a model that will reproduce the observations. Papers within the second group are concerned with nonuniqueness, because it is well appreciated that for any given set of observations there exist infinitely many models that adequately reproduce those data. The purpose of this paper is ts present a new technique by which this nonuniqueness can be quantified.

Before proceeding, let us first define some notation. We shall assume that the models m under investigation are members of a Hilbert space M and that the data for the present are linear functionals of m, written as

\[ \mathbf{e} = \sum_{j=1}^{N} \mathbf{g}_j \mathbf{m}_j \]

(1)

The element \( \mathbf{e}_j \), also a member of H, is the kernel function for the datum \( \mathbf{e}_j \). The notation (1) will be used throughout this paper to indicate the inner product.

Our goal is to quantify the nonuniqueness inherent in the inversion of (1). One possible approach is to compute

\[ m(B_0, \Delta) = \langle m, R(B_0, \Delta) \rangle \]

(2)

where

\[ R(B_0, \Delta, z_0) = \begin{cases} \frac{\Delta}{\sqrt{2}} & |z - z_0| \leq \Delta/2 \\ 0 & \text{otherwise} \end{cases} \]

(3)

is a unimodular boxcar of width \( \Delta \) centered on \( z_0 \). Clearly, if the averages in (2) could be determined for arbitrary \( z_0 \) and \( \Delta \), then our goal is achieved. Unfortunately, this can never be done for realistic data sets, otherwise \( m(B_0, \Delta) \) could be recovered uniquely by letting \( \Delta \to 0 \).

Nevertheless, estimating the averages in (2) remains a worthwhile goal, and we shall pursue that end by casting the problem into its proper framework, that of reference theory.

The mathematical foundation for reference theory has been presented by Bucha in 1977.

A major goal of that work can be stated as follows: given a set of inadequate observations (e) that are linear or nonlinear functions of a model m, what are the predicted values of other functionals of m? Let \( p \) denote another element in H that generates a predicted datum \( \mathbf{d} = \langle p, m \rangle \). The initial formulation by Bucha considered the possibility that many data would be predicted, but that extension is not necessary for the ideas developed here. Given \( \langle p, \mathbf{e} \rangle \), and \( \mathbf{p} \), the goal of inversion theory is to compute an estimate for \( \mathbf{m} \), i.e., \( \mathbf{m} = \langle \mathbf{m}, p \rangle \). The space \( \mathbf{m} \) is a subspace of the vector space of \( \mathbf{p} \) such that \( \mathbf{m} = \langle \mathbf{m}, p \rangle \). Let \( \mathbf{B} \) denote that portion of the subspace generated by \( \mathbf{m} \) that is perpendicular to \( \mathbf{M} \). Breaking \( \mathbf{p} \) into a part \( \mathbf{p} \), which lies in \( \mathbf{B} \), and a part \( \mathbf{p} \), which lies in \( \mathbf{M} \), produces the prediction

\[ \mathbf{e} = \langle \mathbf{e}, \mathbf{m} \rangle = \langle \mathbf{e}, \mathbf{m} \rangle + \langle \mathbf{e}, \mathbf{m} \rangle \]

\[ \langle \mathbf{e}, \mathbf{m} \rangle + \langle \mathbf{e}, \mathbf{m} \rangle = \mathbf{d} + \mathbf{d} \]

(4)

where \( \mathbf{m} \) and \( \mathbf{m} \) correspond to those portions of the model that lie in \( \mathbf{B} \) and \( \mathbf{M} \), respectively.

With regard to the evaluation of (4) there are two cases to be considered, depending upon whether or not \( \mathbf{m} \neq 0 \). We shall consider each in some detail.

Case 1: \( \mathbf{m} = 0 \) and Bucha's Gilbert Approximation

In this case, \( \mathbf{e} = \mathbf{0} \), that i \( \mathbf{p} = \mathbf{B} \), where \( \mathbf{B} \) are real constants. Equation (4) reduces to

\[ \mathbf{e} = \langle \mathbf{e}, \mathbf{m} \rangle = \sum_{j} \mathbf{p}_j \]

and thus \( \mathbf{e} \) is determined uniquely because it is a linear
combination of the data [\(c\)]. This result plays an essential role in the appraised analysis of Backus and Gilbert [1968, 1970]. They consider only prediction kernels contained wholly in \(\mathcal{S}\), but rather than specifying \(p\) a priori, they provide a formalism for finding the constants \(\beta\) = \(\beta(c_d)\), for which

\[
\rho_{\mathcal{E}}(z, z') = \mathcal{A}(z, \Delta) + \sum_{\mathcal{J}} \beta_{\mathcal{J}}(z, \Delta) \delta(z, z')
\]

is close to a Dirac delta function centered on \(z_d\). The importance of their approach cannot be overemphasized. Not only does their work show that unique information about the model can be determined from a finite set of data, but the nature of the constructed averaging windows \(\mathcal{A}(z, \Delta)\) quantified, for the first time, the resolving power of the data.

Let the unique Backus-Gilbert average be represented

\[
\mathcal{M}(\Delta) = (\mu, \mathcal{A}(\Delta))
\]  

(5)

We note that this representation is closely related to our problem of computing averages in the form of (2). In particular, if \(\mathcal{A}(z, \Delta)\) is narrow, without significant sidelobes, and centered on \(z_d\), then \(\mathcal{M}(\Delta)\) is a good representation of \(\mathcal{M}(\Delta)\) when \(\Delta\) corresponds to the width of \(\mathcal{A}(z, \Delta)\). Moreover, if \(\mathcal{A}(z, \Delta)\) is sufficiently narrow, no further knowledge about \(\mathcal{M}(\Delta)\) in a neighborhood of \(z_d\) might be required. The need to compute \(\mathcal{M}(\Delta)\) as per (2) would be obviated.

The difficulty arises when \(\mathcal{A}(z, \Delta)\) does not have these characteristics. Practical interpretations about the numerical value of the model can be greatly humpered if the averaging function resembles a boxcar, has significant sidelobes, or is not centered near \(z_d\). As an example, Oldenburg and Samson [1979] attempted to estimate the plasma density near the center \((r = 0)\) of a plasma cylinder by inverting laser interferometric data. Averaging functions for \(r > 0\) were shown to possess significant sidelobes, and consequently, attempts to infer numerical estimates for the average value of the plasma density near the origin were not successful. They were able to conclude that the density decreased toward the axis, but the lack of numerical bounds was disappointing.

The laser interferometry problem introduces another difficulty. All mathematically acceptable models are included in the Backus-Gilbert averages, and yet the plasma density is a positive function. Since this knowledge cannot be included into the Backus-Gilbert appraisal, it is not known to what extent such a priori information could alter the interpretation.

Case 2: \(\rho^2 \neq 0\)

In this case the prediction kernel \(\rho\) has a projection onto \(\mathcal{S}\). But the observations provide no constraints on \(\rho^2\), and hence \(\rho = \rho^1 + \rho^2\) is unbounded unless additional constraints can be found to bound \(\rho^2\). Backus chooses to do this by imposing a global bound on \(\rho\). However, for many investigations this form of constraint does not satisfy our needs; a particular example of this occurs when the model is known to be positive.

From the foregoing discussion it is apparent that the problem of computing averages of the model in the form of (2) is but a special case in a more general formalism: the inference theory prediction kernel of interest here is simply the boxcar function in (3). In this paper I shall illustrate the usefulness of linear programing methods in solving this inference problem. The introduction of physical constraints will permit upper and lower bounds on \(\mathcal{M}(\Delta)\) to be computed. Moreover, the formalism is applicable to linear and nonlinear problems with accurate or imprecise data. The only difficulties presented in applying the technique to the nonlinear problems are the possibility of nonconvergence of an iterative algorithm and the chance of becoming trapped in a local extremum rather than finding the true global maximum or minimum.

Information about the model is presented in terms of funnel functions which show, in a quantitative manner, the tradeoff between uncertainty in \(\mathcal{M}(\Delta)\) and the width of the boxcar averaging function. The funnel functions can be computed directly, and therefore exactly (at least for the linear problem) or they can be generated approximately from two global bound curves which maximize and minimize the uncertainty integral of the model. These approximate funnel functions are guaranteed to lie on or outside the exact funnel functions, and hence the bounds on \(\mathcal{M}(\Delta)\) that are derived from global bound curves are not necessarily optimum. Nevertheless, the linear and nonlinear examples presented in this paper indicate that the approximation can be good.

The advantages of computing the global bound curves are twofold: (1) they provide immediate insight about the resolving power of the data, and (2) once generated, they can be used to compute approximate funnel functions at any depth.

The paper proceeds by applying the exact and approxi-
mate methods to successively more complicated problems. A linear model with accurate data is considered first. Followed by the effects of inaccurate observations. The final part of the paper is concerned with a nonlinear problem. I have inverted synthetic and observed magnetoelectric data and shown that bounds on the average conductivity can be reliably obtained but that the true resolving power of the data is much poorer than previously supposed.

**Linear Problem (Accurate Data)**

**Method A: Funnel Functions**

We begin with defining equation for a linear process

\[
e_{i} = (m, q_{i}), \quad j = 1, \ldots, N
\]

(6)

where \(e\) are the data and are assumed to be accurate. The model \(m(d)\) is defined over the interval \((a, b)\), and \(q_{i}\) is the kernel function for the \(j\)th datum. Our goal is to find the average value of \(m(d)\) around an arbitrary point \(z_d\). That is, we wish to find

\[
m(\mathcal{M}(\Delta)) = \mathcal{B}(\mathcal{M}(\Delta)), (m)
\]

(7)

where \(\mathcal{B}(\mathcal{M}(\Delta))\) is defined by (3). The average \(\mathcal{M}(\Delta)\) is equal to the true model viewed through a unimodular boxcar averaging function centered on \(\Delta\). Clearly, \(\mathcal{M}(\Delta)\) cannot be found uniquely from the data unless coefficients \(\{\delta_{j}(\Delta)\}\) can be found, such that

\[
\sum_{j} \delta_{j}(\Delta)q_{i}(\Delta) = \mathcal{B}(\mathcal{M}(\Delta)), (2)
\]

In general this cannot be done. But if physical constraints are imposed upon the model, then it is possible to bound \(\mathcal{M}(\Delta)\). That is, upper and lower bounds \(\mathcal{M}^+(\Delta)\) and \(\mathcal{M}^-(\Delta)\) can be constructed such that

\[
\mathcal{M}^+(\Delta) \leq \mathcal{M}(\Delta) \leq \mathcal{M}^-(\Delta)
\]
Intuitively, these bounds will depend upon \( z_0 \), the data, the nature and number of kernel functions, and \( \Delta \). In particular, for any given set of \( g(z) \) and \( z_0 \) we expect the upper and lower bounds to be very different when \( \Delta \) is sufficiently small, but anticipate that they become more confining as \( \Delta \) increases; \( m^* \) and \( m^* \), when plotted as a function of \( \Delta \), will (at least initially) converge for increasing \( \Delta \). I have therefore chosen to call such a plot the funnel diagram for \( \zeta_0 \).

The funnel functions \( m^* \) and \( m^* \) can be computed directly by using linear programming methods. To do so, we first create a partitions on the region \( \{0, a\} \) with boundaries at \( \{0, e_0, e_1, \ldots, e_{L-1}, e_L = a\} \). We suppose that the depth of interest \( \zeta_0 \) lies at the center of the jth partition and that the partitioning is symmetric about \( z_0 \).

\[
\gamma_j = \int_{e_{j-1}}^{e_j} g(z) dz = i = 1, a
\]

then the data equations may be written as

\[
e_j = \sum_{i=1}^{N} \gamma_i m_i \quad j = 1, N
\]

For purposes of illustration we shall assume that \( m(z) \) is a nonnegative function of \( z \). The decision-making offered here also allows for models that are both positive and negative, but in that case, each of the model parameters \( m_i \) must be written as the difference between two positive variables, that is \( m_i = r_i - q_i \), where \( r_i, q_i \geq 0 \).

Linear programming allows us to the maximize or minimize an objective function

\[
d = \sum_{i=1}^{N} w_i m_i
\]

subject to equality or inequality constraints on linear combinations of the positive parameters \( m_i \). The \( w_i \) in (10) are a set of positive weights which may be adjusted to suit our needs. It is this flexibility that is evoked to make (10) look like a discretized form of (7). We do this by attempting to find a set of weights \( w_i \) such that

\[
\delta_k = \sum_{j=1}^{N} w_j g_j(z_0) \quad \Delta_k
\]

is thus the average value of \( m(z) \) over a width \( \Delta_k \) centered on \( z_0 \). If

\[
\delta_k = \sum_{j=1}^{N} w_j g_j(z_0) \quad \Delta_k
\]

is the width of the jth partition element, then, for our symmetric partition

\[
\Delta_k = \frac{\Delta_k}{2} + \sum_{j=1}^{N} h_j \Delta_i \quad k = 0, 1, \ldots
\]

and the desired weights in (11) are given by

\[
w_k = \frac{h_k \Delta_k}{\Delta_0} \quad p - k \leq i \leq p + k
\]

### Functions

In the context of the model, the value of \( \delta(z) \) in the distance between the upper and lower funnel functions. Its value indicates the uncertainty in the average value of \( m \) over a width \( \Delta \) centered on \( z_0 \), and therefore I shall refer to \( \delta(z) \) as the uncertainty function.

Enhancement of our knowledge of \( \delta(z) \), obtained by incorporating additional constraints or data, can be shown quantitatively by computing the uncertainty function. This is carried out in Figure 1 (c, j) for the linear example presented.

The uncertainty in the average value of the model at any \( z_0 \) depends upon the width of the boxcar averaging function, the data, and the imposed physical constants \( m^*(z) \) and \( m^*(z) \). As \( \Delta \to 0 \), \( \delta(z) \) will become equal to \( m^*(z) - m^*(z) \). As \( \Delta \) increases, the data will be forced to gather inferences, and will decrease. For some problems it may even be that the kernel functions can generate a boxcar, and hence for the appropriate \( \Delta \), and the upper and lower funnel functions meet. Then \( \delta(z) \) is known uniquely, and \( \delta(z) = 0 \). In a qualitative sense the value of \( \delta(z) \) in an indication of the width \( \Delta \) of the kernel function into a unimolecular boxcar function. When \( \delta(z) \) is small, we expect that the kernel functions can nearly reproduce \( m^*(z) \), but their ability must worsen as \( \delta(z) \) increases.
Method B: Global Bound Curves

The funnel diagrams and uncertainty functions summarize our knowledge about the average value of the model around $z_0$ and if such information is desired for only one or two points, then the direct computation of these curves is an efficient way to proceed. Often though, we wish to determine bounds on $\hat{m}(z_0, \Delta)$ for all $z_0$ in the range $[0, a]$. In addition we seek a representation of information which yields insight into the resolving power of the data at all depths. These objectives can be met at least approximately, by computing global bound curves. Let

$$n(z) = \int_0^z \frac{m(\xi) d\xi}{\sigma(\xi)}$$

be the indefinite integral of the model. Our goal is to find upper and lower bounds for $\hat{m}(z_0, \Delta)$. Written in terms of $n(z)$ we have

$$\hat{m}(z_0, \Delta) = \frac{1}{\Delta} (n(z_0 + \Delta/2) - n(z_0 - \Delta/2))$$

I wish now to define two global bound curves $r^*(z)$ and $r^-(z)$, which have the following properties:

$$r^*(z) = \int_0^z m(\xi) d\xi \quad 0 < z < a$$

(18)

$$r^-(z) = \int_0^z m(\xi) d\xi \quad 0 < z < a$$

In (18), $m$ is any model that reproduces the data and simultaneously satisfies all other imposed physical constraints. The global bound curves may be computed by using linear programming in exactly the same manner as the funnel functions were computed. The only difference is that the objective function is now unity for $u < q_0$ and zero for $u > q_0$ in (18). A typical plot of $r^*(z)$ and $r^-(z)$ is given in Figure 2. It is possible to generate approximate funnel functions from the global bound curves. The desired bounds on $m$ are
and

\[ m(z, \Delta) = \frac{1}{2} \int_{z}^{z+\Delta} r(z) - r(z-\Delta/2) \, dz \]

These inequalities can be improved somewhat by incorporating bounds imposed only by the applied physical constraints. Let \( m^-(z) \) and \( m^+(z) \) be the upper and lower bounds on \( m(z) \) and define

\[ m_- = \int_{z}^{z+\Delta} m^-(z) \, dz \]

\[ m_+ = \int_{z}^{z+\Delta} m^+(z) \, dz \]

The desired inequalities are

\[ m(z, \Delta) \leq \max(m^-(z+\Delta/2), m^+(z+\Delta/2)) - m_- \]

\[ -r(z-\Delta/2), m_+ \]

\[ m(z, \Delta) \leq \min(m^-(z+\Delta/2), m^+(z+\Delta/2)) - m_+ \]

The right sides of these equations represent, respectively, approximations \( m_+ \) and \( m_- \) to the true funnel functions \( m_+ \) and \( m_- \). The main concern with this approach is that the approximate bounds established in (19) are guaranteed to lie on or outside those from the exact solution, that is, \( m_+ \leq m(z, \Delta) \leq m_- \). From a practical viewpoint the usefulness of the global bound curves depends upon how closely \( m_+ \) and \( m_- \) approximate the exact values. The work carried out here suggests that these approximations may be adequate.

The global bound curves for the numerical example are shown in Figure 3. These generated in the absence of model constraints are shown in Figure 3a. The highest resolving power occurs near zero, where the upper and lower bound curves have a small vertical separation. The resolving power decreases monotonically toward greater depths, even though the upper and lower bound curves meet at \( z = 1.0 \). Such convergence is unusual and occurs (in this problem only) because the first data kernel has a test value for all \( z \), and hence the corresponding datum is itself an average of the model over the range \( 0.31 \).

The global bound curves generated after restricting the model to lie within the range \( 0.5 \leq m(z) \leq 2.0 \) are shown in Figure 3b. The enhanced resolution, compared to that in Figure 3a, is quite evident. In particular, the average value of the model for small \( z \) is now very well determined.

The approximate funnel functions for the constrained problem are compared with the exact functions in Figure 4. For \( z = 0.5 \) the worst discrepancy—approximately 10%—occurs in an estimate of the lower bound value near \( \Delta = 0.35 \). The exact and approximate funnel functions for \( z = 0.75 \) are nearly identical at all values of \( \lambda \).

**Linear Problem: Inaccurate Data**

Geophysical data are always inaccurate, and consequently any practical algorithm must accommodate such errors. Real data equations have the form

\[ z = \left( c + \epsilon \right) \eta \]

where \( \epsilon \) is additive noise. It must be assumed here that \( \eta \) is Gaussian noise with zero mean and standard deviation \( \sigma \). It
is also assumed that the noise for different data is uncorrelated, that is, \( \text{Cov}[\epsilon_i, \epsilon_j] = 0 \) for \( i \neq j \). However, if correlated errors are important, they can be handled by taking appropriate linear combinations of the initial data to arrive at new data whose errors are statistically independent [Gilbert, 1971].

The linear programming solutions developed here can treat (20) in either of two ways. In both, the objective function is unchanged from the analysis with accurate data, but in the first method the data constraints are satisfied only to within the limits

\[
e_j^0 - a_{ij} \leq m_j \leq e_j^0 + a_{ij}
\]

where \( a \) is some number (usually 1 or 2) that determines the goodness of fit. In the second method the data constraints are written in equality form [Lewy and Fullagar, 1981]

\[
e_j = e_j^0 = \sum a_{ij} x_i + \gamma_j, \quad j = 1, N
\]

The \( \gamma \) are a set of numbers to be determined from the linear programming solution. With the normalization used, each \( \gamma \) is a particular realization of a Gaussian random variable with zero mean and unit standard deviation. Let \( x^1 \) be the 1-norm of these variables, that is

\[
x^1 = \sum_{j=1}^N |\gamma_j|
\]

The statistics of \( x^1 \) have been given by Parker and Mckean [1980]. The expected value is

\[
E[x^1] = \sqrt{2 \pi} N
\]

and the standard deviation is

\[
s_d(x^1) = \left(1 - \frac{2}{\pi} N\right)^{1/2}
\]

Based upon these statistics, we introduce the criterion

\[
\sqrt{2 \pi} N - a \left(1 - \frac{2}{\pi} N\right)^{1/2} - B \leq x^1
\]

(24)

Equation (24) provides a global constraint upon the parameters \( q_j \). The constant \( a \) in (24) is a goodness-of-fit parameter which usually acquires a value of 1. The quantity \( B \) is a bias value which can be used to produce better fits to the observations than the \( L_1 \) norm statistics would dictate. This is sometimes preferable because a solution acceptable with the \( L_1 \) norm misfit criterion can be unacceptable with a chi-squared criterion. If the ultimate acceptability of a model is characterized by its chi-squared value, it may be necessary to require a slightly smaller value of \( x^1 \). This can be done by setting \( B > 0 \) in (24).

When data with statistical errors are to be inverted, it is often preferable to use (22) and (24) rather than (21) [Fullagar, 1981]. To see this, suppose that \( a \) is specified to be unity in (21). The choice that all \( N \) data lie within \( a = 0.05 \), and this becomes vanishingly small with increased \( N \). It is increased until there is a good probability that all the data lie within \( a \). At this point, the chi-squared misfit may be unacceptable. On the other hand, the use of equality constraints (equations (22) and (24)) allows one or two data to have large misfits, while at the same time the global misfit is kept to within acceptable limits.

To show the effects of errors on the global bound curves and funnel functions, we shall use the inequality constraints (21). In Figure 5c the global bound curves are computed for misfits corresponding to \( a = 0.05 \) and \( a = 0.01 \). The model was also restricted to lie between 0.5 and 2.0. As expected, for \( a > 0 \), the difference between the upper and lower global bound curves increases with allowable misfit errors.

Approximate funnel functions may be computed directly from the global bound curves, and those for \( \omega = 0.5 \) are displayed in Figure 5c. The importance of the physical constraints imposed upon the model are again evident. For instance, when data are fit only to within a value of 0.05, the box averaging function centered at \( \omega = 0.5 \) must have a width greater than \( 0.05 \) before the data affect the value of the lower bound. The uncertainty functions plotted for different misfits are shown in Figure 5c.

Nonlinear Appraisal: Background

In the linear problem we saw that the computation of the funnel functions might be particularly valuable when the Backus-Gilbert resolving power was poor and when physical constraints on the model were available. In the nonlinear problem, there is an additional reason for computing the
funnel functions. A deficiency of the Backus-Gilbert approach is that the unique averages refer only to those models that are linearly close to the constructed model. Yet for many geophysical experiments, especially electromagnetic sounding, the data do not have sufficient resolving power to force the constructed model to lie linearly close to the earth model. This occurs even in those circumstances where it has been proven that a unique solution to the problem exists if an infinite number of accurate data have been provided. Thus, globally distinct solutions are possible, and linearized approaches cannot illuminate the true variability of model space.

I would now like to apply the techniques developed for linear appraisal to the nonlinear problem. A formalism will be developed that is applicable to any nonlinear problem, but the specific applications here will be directed toward the magnetotelluric (MT) problem.

In a one-dimensional nonlinear problem the model is related to the observations by

$$e^f(m_0) = E[m_0(z)] + n_i$$

where $F$ is a nonlinear operator, $m_0(z)$ is the model, and $n_i$ is additive noise. A particularly successful method for constructing and acceptable model is to linearize the problem [Backus and Gilbert, 1968]. In general, a starting model $m_0(z)$ will be a sufficiently poor guess that the predicted responses

$$e^f(m_0) = E[m_0(z)]$$

will not be in acceptable agreement with the observations. A perturbation $\delta m_0(z)$ is sought such that a new model $m_j(z) = m_0(z) + \delta m_0(z)$ produces responses that are in closer agreement with $e^f$ than those produced from $m_0(z)$. Expanding $e^f(m_0 + \delta m_0)$ produces

$$e_f(m_0 + \delta m_0) = e_f(m_0) + \int_0^\alpha \delta m_0(z) \frac{dz}{dx}$$

plus higher-order terms which can be neglected in a linearized formulation. At each iteration, $e_f(m_0 + \delta m_0)$ is equated to $e_f$ to yield the linearized equations

$$\delta m_j = e_f(m_j) - \int_0^\alpha \delta m_j(z) \frac{dz}{dx}$$

$$j = 1, N$$

This is a set of $N$ equations to be solved for a function $\delta m_0(z)$, and a suitable $\delta m_0(z)$ is easily found by minimizing a particular function of $\delta m_0(z)$. Subject to using (25) as constraints. This can be accomplished by minimizing

$$\phi = \int_0^\alpha [\delta m_0(z)]^2 \frac{dz}{dx}$$

and using the spectral expansion method [Parker, 1977]. The procedure is terminated when the $\delta m_j$ becomes sufficiently small.

A disadvantage of this method of construction is that the norm is always applied to the perturbation and never to the model itself. It would be more desirable to search for models that minimized or maximized a global norm, but this requires that the nonlinear problem be written in the form of (26). For the problem attempted here it is crucial that this be done because the objective function to be extremized must pertain to the model and not to a perturbation.

Fortunately, our goal is easily achieved by writing $\delta m_0(z) = m_0(z) - m_j(z)$ and substituting $\delta m_0(z)$ into (25) to get

$$\delta m_0(z) + \int_0^\alpha m_0(z) \delta m_0(z) \frac{dz}{dx} = \int_0^\alpha m_0(z) \delta m_0(z) \frac{dz}{dx}$$

The left side of (26) is recognized as modified data. It includes a small error as well as a new inner product of the starting model with the kernel function. An updated model $m(z)$ can be found by minimizing any norm of $m_j(z)$. That is, norms such as $|m_j(z)|$, $|m_j(z)|^2$, $|m_j(z)|^4$, etc., could be minimized to compute globally 'smallest' or 'flattest' models. Of course there is still a need to iterate toward a solution, since the kernel functions in (26) depend upon the starting model. Moreover, there is the question of whether the iterative algorithm will converge to a local maximum or minimum rather than to the desired global extremum. This is an important point, but I shall defer discussion of it until later.
Having cast the data constraints into the linearized form of (26), we can proceed directly to the computation of the funnel functions and global bond curves. However, before carrying out a practical application of this procedure, I shall first present some required background for the magnetotelluric problem.

One-Dimensional MT Problem

The goal of magnetotellurics is to recover the conductivity structure of the earth by inverting electromagnetic responses observed on the earth’s surface. The responses are the ratio of the magnetic to electric fields and are known only at discrete frequencies ω₀. A plethora of algorithms exist that will invert these observations to find a one-dimensional conductivity structure 𝛿(𝑧). Many of these methods are linearized versions and therefore iterative in nature, but some solve the fully nonlinear problem. The surprising result that has come to light is that when only limited numbers of data are inverted, the individual algorithms can produce very different solutions. Moreover, any attempt to explore model space by varying the parameters of a particular construction technique can, at best, illuminate only a fraction of the acceptable model types. This point is crystallized by considering the application of different algorithms to MT data sets taken on the Pacific floor [Oldenburg et al., 1983].

The existence of globally distinct models is disturbing because it reduces the chance that the averages generated from the linearized appraisal of Backus and Gilbert will be valid for the earth. The following analysis, however, will show that the upper and lower bounds on the average value of the model represent a substantial improvement in our quest for determining information about the conductivity structure.

We begin with the linearized version of the MT problem as formulated by Oldenburg [1979]. The differential equation for the electric field 𝑃 at a frequency ω₀ is

$$E'(ω₀, 2) + iω₀σ(2)E(ω₀, 2) = 0$$  \hspace{1cm} (27)$$

where the primes denote differentiation with respect to 2. The responses are

$$r_2 = B(ω₀, 0) = \frac{E(ω₀, 0)}{E(ω₀, 0)}$$  \hspace{1cm} (28)$$

where B is the magnetic field. The change in the response caused by a perturbation 𝜒(2) in the conductivity is derived to be

$$δr_2 = -\mu_0 \int \frac{E(ω₀, 2) + δ(ω₀, 2) E(ω₀, 0)}{E(ω₀, 0)} \, d2$$  \hspace{1cm} (29)$$

Finally, if the data are considered to be the amplitude and phase of 𝜒(2) rather than its real and imaginary parts, we may write 𝜒(2) = 𝑓(2) exp(𝑖θ(2)), where

$$f(2) = \int G(ω₀, 2) \, d2$$  \hspace{1cm} (30)$$

and

$$G(2) = -\mu_0 f(2) \frac{E(ω₀, 2)}{E(ω₀, 0)}$$

Clearly, equations (30) are in the form of (25) and hence can be changed into their global form (26). That is

$$b_f = \int b_2 G(ω₀, 2) \, d2$$  \hspace{1cm} (31)$$

where 𝑏_2 (2) is a starting model, and 𝑏_f and 𝑓 are the kernel functions and data, respectively, for the amplitudes or phases.

By introducing a partition in the region (0, 2), specifying a starting model 𝑏_2(2), computing the integral of the kernels over the partition elements, and evaluating the inner product of the model with the kernel function, we turn (31) into the form of (10). Funneled functions and global bound curves can then be generated.

Example of Nonlinear Appraisal

I will now illustrate how the appraisal is applied to a nonlinear problem by considering a simple MT example. The true electrical conductivity is a half space with 𝜔₀ = 1.8 S/m in which is embedded a resistive layer having 𝜔₀ = 0.01 S/m. The resistive layer is 1 km thick, and the top is buried 1 km below the surface. Sixteen data were generated at eight periods ranging from 1 to 215 s. The periods were chosen at equally spaced intervals in logarithmic time.

The true conductivity and a linearized model constructed from the algorithm given in Oldenburg [1979] are shown in Figure 6. The starting model for the construction was a half space with 𝜔₀ = 1.0 S/m. The acceptable model shows a region of low conductivity near 1500 m, but the minimum conductivity attained, 𝜔₀ = 0.35 S/m, is significantly greater than the true value of the insulator.

The constructed model indicates a region of decreased conductivity near 1500 m, and in an effort to assess whether all models must have a resistive region near that depth we shall compute the funnel functions for 𝜔₀ = 1500 m. We begin, not by looking at a totally unconstrained problem but by looking at one in which 𝜔₀ = 10 S/m and 𝜔₀ = 0.005 S/m. The funnel functions are shown in Figure 7a. The
bounds are wide, but even so the upper curve shows that the average conductivity in a width of 1400 m (that is from 800 to 2200 m depth) is less than 1.0 S/m. The lower bound curve shows that models exist in which the conductivity can take on its minimum value of $\sigma = 0.005$ S/m in a width ranging up to 1600 m and centered on $z_0 = 1500$ m.

The importance of imposing conductivity constraints is dramatically illustrated by monitoring the funnell functions as $\sigma_{max}$ is decreased. Curves for $\sigma_{max} = 5.0, 2.0, \text{ and } 1.0$ are also shown in Figure 7. The upper and lower bounds for $\sigma_{max} = 1.0$ are quite confining and provide good estimates of the true average conductivity of $\Delta$ greater than about 1200 meters.

Each point of a funnel function is obtained by maximizing or minimizing an objective function subject to data and physical constraints. Thus an acceptable conductivity model is constructed at each extremization. For each such model, each of which maximizes the conductance in the region 1000-2000 m, are shown in Figure 7. When $\sigma_{max}$ is large, the models have concentrations of conductivity only in a few regions and are near zero elsewhere. These models resemble Parker's $D^*$ models (Parker, 1980). As $\sigma_{max}$ decreases, the widths over which the structure achieves this maximum value increases, and in the limit that $\sigma_{max}$ is assigned its smallest value, the conductivity model resembles closely the true model from which the data were generated.

At the outset of this investigation there was some doubt about the practicalities of an iterative algorithm. Of special concern were (1) would the algorithm converge to a solution, and (2) would the final value of the objective function be the

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Fig. 7. Funnell functions for $z_0 = 1500$ m, $\sigma_{min} = 10$ S/m, and $\sigma_{max} = 0.005$ S/m are shown in (a). The conductivity model that maximizes the conductance in the depth range 1000-2000 m is shown in (d). Analogous results for $\sigma_{max} = 5$ S/m, 2 S/m, and 1 S/m are shown in (c)-d).
The global bound curves for a resistive layer embedded in a half space are shown in [10]; α and r are upper and lower bounds functions for the conductance. The exact and approximate funnel functions are given in (6).

desired global extremum or would it be only a local minimum or maximum? Regarding convergence, the concern is the usual one in that the Frechet derivatives are evaluated at the current model rather than at the true model. This introduces the possibility of marching in the wrong direction when using a steepest descent method. There is an added complication in my formulation in that the “data” in (26) include both a Frechet derivative and a starting model. The effective data are thus very dependent upon the initial model, and it would seem that poor initialization could cause the algorithm to fail. Moreover, as with many iterative solutions, convergence of the algorithm depends heavily upon the method of taking the perturbations, and to some extent, this is as much art as science. The method adopted for the MT problem was to initialize the algorithm with the constructed acceptable model obtained from the linearized inversion. The results returned by the linear programming optimization were accepted unless the conductivity in any partition element was changed by more than a factor of 2 from the starting model of that iteration. When this occurred, the change was limited to that factor. Undoubtedly, such a “recipe” does not yield optimum convergence, and further work will have to be done in this area. Nevertheless, this procedure has worked satisfactorily so far.

The second concern is more difficult to contain. A complete theory for nonlinear optimization is not yet available, and so the question of whether a global or global extremum of the objective function has been found cannot be determined directly. However, some confidence may be gained by initializing the algorithm with a different starting model. If the same solution is recovered, it becomes more likely that a true global extremum has been found. This has been the case for the MT example. In the tests carried out so far I obtained the same value of the objective function and the same conductivity model when initialized the algorithm with a half-space model or with the 0.90-acceptable model in Figure 6. In the former case the initial values of the chi-squared merit (estimating that the data had a 1% uncertainty) were approximately 208. In the latter case the initially small chi-squared merit increased rapidly to values of a few hundred and then decreased toward zero. This provides some hope that the linearized analysis may converge to a global extremum.

In the linear problem we saw that the approximate funnel functions generated from the global bound curves lay outside the exact curves but that the differences between the solutions could be small. In the nonlinear problem there is a further difficulty with using the global bound curves. This arises because the “data” and kernel functions in (26) are model dependent. Consequently, the data constraints adopt a slightly different form at each point on the global bound curves. Nevertheless, the few examples I have tested suggest that a good approximation to the funnel functions may still be obtained from the global bound curves, and therefore it is worthwhile to construct the global bound curves for the MT example (s = 2.0 km) are shown in Figure 9c.

The notation has been changed slightly to accommodate the usual convention of representing the indefinite integral of the conductivity, that is the conductance by αt. The funnel functions for αt = 1500 m that are generated from these curves are shown in Figure 9d. They can be compared with the true curves, which are also reproduced there.

As a final example I would like to reanalyze a set of MT observations taken on the floor of the Pacific ocean near the Juan de Fuca ridge. The JDF data were acquired by Lovelace and Greenshaw [1981] and analyzed by them and by myself [Oldenburg, 1981]. Using the linearized inversion, I attempted to construct smooth models that were as diverse as possible by initializing the algorithm with different starting models. Four acceptable models were generated, and all had the following character. Proceeding into the earth, they were successively: resistive, conductive, and resistive. These models are reproduced in Figure 9e. On that basis I concluded that the region of low conductivity beneath the conductor was a reversed feature of all models that fit the data. However, counter examples found later [Oldenburg et al., 1983] showed that the low-conductivity region was not dominated by the data. Moreover, the variability observed in the more complete set of acceptable models was discouragingly large.

The nonuniqueness admitted by the JDF data is now understood. In Figure 9f I have plotted the global bound curves after imposing restrictions that the conductivity at all depths lie within the bounds 0.005 ≤ α ≤ 0.2 S/m. For each extremation the iterative algorithm begins with a constructed model, labeled A (in Figure 9c). But even with the realistic model constraints the global bound curves in Figure 9e show that the resolving power of the data is extremely poor. Funnels functions for αt = 60 km are shown in Figure 9f. The bounds are very poor and for the most part reflect only the imposed physical constraints.

The large model variability suggested by the funnel functions is substantiated by examining some of the models that extremized the objective function in the global bound calcu-
Fig. 9. Four smooth acceptable models that fit the JDF responses are shown in (a). These are reproduced from Oldenburg [1981]. The global bound curves for the conductance (in normalized units) are given in (b), and the exact and approximate funneled functions for z = 60 km are shown in (c).

Fig. 10. Shows are four models obtained when computing the global bound curves. The models in (a) and (b) minimize and maximize, respectively, the conductance in the top 95 km. The models in (c) and (d) minimize and maximize, respectively, the conductivity in the top 35 km.
tivity equals the specified lowes hazard. It is surprising that both of these models reproduce the observations to within an acceptable chi-squared value because, within the depth range 40–130 km, the two models are completely at odds with each other. In the other, the other is positive. Analogous results are obtained in Figure 16(c), where the conductance in the top 155 km is maximized then minimized.

DISCUSSION

The funnel functions and global bound curves presented in this paper are new ways of attempting to summarize our knowledge about linear averages of the model. When sufficient explicit knowledge about the model is available, these approaches should enhance the practical interpretation of the data. The work on inference theory presented by Backus [1976, b, c, 1977] provides a solid mathematical foundation for the computation of the funnel functions. All computations in this paper are carried out in a direct manner through linear programming techniques. The only loss of generality in the solution presented here results from parameterization and partitioning, but these effects should be unimportant if the linear-programming partitioning is sufficiently quanitifed.

I have also attempted to extend the computation of funnel functions to nonlinear problems by first linearizing and then iterating. The important step in the process is to rewrite the linearized perturbation equations in such a way that a global norm of the model is sought at each iteration. Because of the linearization there is no guarantee that the algorithm will converge to the global extremum. This is a major disadvantage to all such linearized approaches. Nevertheless, in the cases investigated so far, the algorithm has converged to the same objective function, even when initiated with different starting models. This is encouraging. Also, it is important to point out that useful information is still obtained from this approach, even if the algorithm does get trapped in a local minimum. For the resolving power of the data is at least as poor as that obtained from the algorithm.

A method was also presented from which the funnel functions can be computed approximately. Global bound curves which maximize and minimize the infinite integral of the model were first generated and then used to form inequalities for the average \(V_{\text{avg}}\). These approximate funnel functions are guaranteed to lie outside the true curves, but numerical experiments performed thus far suggest that these are quite close to the exact approximations differ slightly. Moreover, the global bound curves provide immediate insight into the resolving power of the data at all depths, and this alone justifies their computation.

As a final comment it is important to stress that the method presented here is not a paradigm for the nonlinear problem. Much work still remains before nonlinear optimization is fully understood and is computationally feasible. In the meantime, the linearized solution offers great potential for assessing the nonuniqueness inherent in the experiment and provides a firmer foundation from which geophysical inferences can be made.

Conclusions

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