

- [14] G. GOLUB AND C. V. LOAN, *Matrix Computations*, Johns Hopkins University Press, 1983.
- [15] E. HABER AND U. ASCHER, *Fast finite volume modeling of 3d electromagnetic problems with highly discontinuous coefficients*, submitted to SISC, (1999).
- [16] R. HIPTMAIR, *Discrete hodge operators*, tech. report, Report No. 126, SFB 382, Universitt Tbingen, October 1999.
- [17] J. JIN, *The Finite Element Method in Electromagnetics*, John Wiley and Sons, 1993.
- [18] P. R. MCGILLIVRAY, *Forward Modelling and Inversion of DC Resistivity and MMR Data*, PhD thesis, University of British Columbia, 1992.
- [19] J. D. MOULTON, *Numerical implementation of nodal methods*, PhD thesis, Institute of Applied Mathematics, University of British Columbia, 1996.
- [20] T. NATAKA AND K. FUJIWARA, *Summary on the results of benchmark problem 13 (3d nonlinear magnetostatic model)*, COMPREL, 11 (1992), pp. 345–369.
- [21] I. PERUGIA, V. SIMONCINI, AND M. ARIOLI, *Linear algebra methods in mixed approximation of magnetostatic problems*, SISC, 21 (1999), pp. 1085–1101.
- [22] P. RAVIART AND J. THOMAS, *A mixed finite element method for second order elliptic problems*, vol. 606, Springer-Verlag, Berlin and New York, 1977.
- [23] Y. SAAD, *Iterative Methods for Sparse Linear Systems*, PWS Publishing Company, 1996.
- [24] J. SANZ-SERNA AND M. CALVO, *Numerical Hamiltonian Problems*, Chapman and Hall, 1994.
- [25] S. TUREK, *Efficient Solvers For Incompressible Flow Problems*, New York: Macmillan, 1999.

- rithms for constrained multibody dynamics*, J. Nonlinear Dynamics, (to appear).
- [2] O. BÍRÓ AND K. PREIS, *On the use of the magnetic vector potential in the finite element analysis of three-dimensional eddy currents*, IEEE Transactions on Magnetics, 25 (1989), pp. 3145–3159.
  - [3] A. BOSSAVIT, *Whitney forms: a class of finite elements for three-dimensional computations in electromagnetism*, IEEE Proc. A, 135 (1988), pp. 493–500.
  - [4] ———, *Computational Electromagnetism*, Academic Press, 1998.
  - [5] R. J. BOWERS AND B. A. BIDWELL, *Geophysics and uxo detection*, The leading edge, 18 (1999). n. 12.
  - [6] F. BREZZI AND M. FORTIN, *Mixed and Hybrid Finite Element Methods*, Springer-Verlag, New York, 1991.
  - [7] G. CAREY AND T. ODEN, *Finite Elements, a second course*, Prentice-Hall, 1983.
  - [8] N. DEMERDASH AND R. WANG, *Theoretical and numerical difficulties in 3d vector potential methods in finite element magnetostatic computations*, IEEE Trans on Magnetics, MAG-26 (1990), pp. 1656–1658.
  - [9] H. ELMAN, *Iterative methods for problems in computational fluid dynamics*, Published in Iterative Methods in Scientific Computing, R. Chan, T. Chan and G. Golub, Eds. Springer-Verlag, (1997), pp. 271–327.
  - [10] ———, *Fast stokes solvers*, (1998). submitted.
  - [11] M. FORTIN AND R. GLOWINSKI, *Augmented lagrangian methods: Applications in the numerical solution of boundary-value problems*, Studies in Mathematics and Its Applications, 15 (1983).
  - [12] P. E. GILL, W. MURRAY, AND M. H. WRIGHT, *Practical Optimization*, Academic Press, New York, 1981.
  - [13] V. GIRAULT AND P. RAVIART, *Finite Element Methods for Navier-Stokes Equations*, Springer, Berlin-Heidelberg, 1986.

$k$	grid size	CG iterations	Flops
2	8	25	$4.1 \times 10^6$
2	16	31	$5.1 \times 10^7$
2	32	45	$7.3 \times 10^8$
2	64	78	$9.0 \times 10^9$
10	8	26	$4.2 \times 10^6$
10	16	37	$6.0 \times 10^7$
10	32	42	$6.9 \times 10^8$
10	64	79	$9.1 \times 10^9$
100	8	46	$7.6 \times 10^6$
100	16	63	$1.0 \times 10^8$
100	32	91	$1.4 \times 10^9$
100	64	155	$1.8 \times 10^{10}$
1000	8	83	$1.4 \times 10^7$
1000	16	132	$2.2 \times 10^8$
1000	32	253	$3.6 \times 10^9$
1000	64	389	$5.0 \times 10^{10}$

Table 2: Summary of the numerical experiments for the edge elements.

roughly comparable and for this small extra effort, we obtain  $\mathbf{H}$  with second order accuracy vs the first order accuracy that the edge elements give.

In terms of comparison to other codes. It was reported in [8] that nodal methods break when  $\mu/\mu_0 = 1000$  and thus our method is obviously superior to the nodal methods. Our method is also better than [21] because it requires roughly a third of the storage and it is simpler to program while giving similar results in terms of accuracy.

## 7 Acknowledgment

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## References

- [1] E. BAYO AND A. AVELLO, *Singularity-free augmented lagrangian algo-*

$k$	grid size	CG iterations	Flops
2	8	15	$4.5 \times 10^6$
2	16	22	$5.5 \times 10^7$
2	32	32	$7.5 \times 10^8$
2	64	61	$9.1 \times 10^9$
10	8	16	$4.9 \times 10^6$
10	16	25	$6.2 \times 10^7$
10	32	39	$7.7 \times 10^8$
10	64	66	$9.5 \times 10^9$
100	8	32	$9.6 \times 10^6$
100	16	50	$1.2 \times 10^8$
100	32	82	$1.5 \times 10^9$
100	64	131	$1.9 \times 10^{10}$
1000	8	63	$1.8 \times 10^7$
1000	16	111	$2.6 \times 10^8$
1000	32	203	$3.7 \times 10^9$
1000	64	347	$5.3 \times 10^{10}$

Table 1: Summary of the numerical experiments.

In order to check the accuracy of our method and compare its efficiency, we use edge elements as suggested in [2]. The edge elements have been generated on the same grid. In this case the stiffness matrix has size of the number of edges, that is 1944, 13872, 104544 and 811200. We see that the size of the stiffness matrices which evolve from edge elements and the size of the stiffness matrix in our method is roughly equivalent (this is because the number of unknowns we have is equivalent to the number of faces while in [2] the number of unknowns is equivalent to the number of edges). In terms of fill-in, the storage of all our matrices  $M, B, \hat{L}$  requires roughly 1.5 more disk space.

In order to solve the edge element system we have used CG with SSOR preconditioner which makes for a good comparison with our new method. In terms of accuracy, the results of the two methods agree within the discretization error. The computational effort of the edge element discretization is presented in Table 2. We can see that the edge elements take more iterations to converge but require less floating point operations. This is due to the inversion of the mass matrix in our method. However, the amount of work is

to overcome this problem. It is possible to approximate  $M$  by a diagonal mass matrix. This procedure is often called lumping [16]. One such lumping strategy is to use the diagonal finite volume mass matrix [15] as an approximation to  $M_H$ . Let us define this matrix as  $\hat{M}_H$ . In this case  $\hat{M}^{-1}$  is a diagonal matrix and we can calculate the matrix

$$\hat{L} = B^T \hat{M}^{-1} B$$

with moderate expense. The matrix  $\hat{L}$  is a good approximation to the matrix  $L$ . Therefore, we can use it in order to precondition the system [24]. In this work we have used the SSOR [14, 23] of  $\hat{L}$ . The SSOR requires the choice of a parameter  $0 < \omega < 1$  which we have set to be 0.5.

Thus below we apply the CG method to (37) and precondition with the SSOR of  $\hat{L}$ .

## 6 Computational examples

In order to test and compare the results we use a model problem similar to [21]. We assume a uniform cube made of permeable material with permeability  $\mu = k\mu_0$  in the ground where  $\mu = \mu_0$ . The grid is uniform spanning the space  $[-0.8m, 0.8m]^3$ . The cube is of size  $0.2m^3$  and it is located in the middle of the grid from  $-0.1$  to  $0.1$ . The boundary conditions are Dirichlet with  $(\mathbf{n} \times \mathbf{H})_{\partial\Omega} = \mathbf{n} \times [0, 0, 1]^T$ . Our goal is to test the accuracy of the method as well as its numerical properties. Since our method results in large but manageable systems we are able to refine our grid and solve the system with  $8^3, 16^3, 32^3$  and  $64^3$  cells. This corresponds to problems of size 1728, 13056, 101376 and 798720 unknowns respectively. We also change the discontinuity ratio  $k$  from 2 to  $10^3$ . This allows us to check the method and its robustness. The results are summarized in Table 1. In the experiments we stop the iterations when the relative residual is less than  $10^{-6}$ . From the above we see that the iteration count as well as the floating point operations grows which is what we expect from a well conditioned system when using preconditioned CG. We also see that the problem is harder to solve as the discontinuity is increased. This is no surprise and our method shares this property with the **div-grad** system [19]. However although the number of iterations grows with the discontinuity and with the grid, the solution is still manageable even for the  $64^3$  grid with large discontinuity where the number of unknowns is very large.

## 5 Solution of the resulting linear system

In this section we discuss the solution of the discrete system (35). Although, it is possible to design preconditioners directly for the system (35) (see for example [9, 25, 21]) we use a Schur complement approach to reduce the system into a smaller dimensional system and precondition the Schur complement. Similar approach is taken in mixed methods for the electrostatic case where the (1,1) block is easily invertible [7]. We note that such approach is not possible in the formulation [21] because the (1,1) block there is not invertible.

The Schur complement system of (36) is obtained by block elimination. First, we can write that

$$\begin{pmatrix} h \\ \psi \end{pmatrix} = -M^{-1} B a$$

and therefore

$$L a = B^T M^{-1} B a = f \tag{37}$$

This system is symmetric positive definite and therefore standard techniques such as CG are effective for its solution. The size of  $L$  is equivalent to the number of faces we have in our discretization and it is therefore a large matrix.

Note that in order to apply CG one need not calculate the dense matrix  $M^{-1}$  explicitly but rather calculate its action on a vector  $M^{-1}v$ . This is easily done by solving the system

$$M u = v \tag{38}$$

with other iterative methods (here we use CG again). The solution of this system is fast because  $M$  is a mass matrix and as discussed in the previous section, its condition number is grid independent. In our numerical experiments we have found that roughly 5-6 preconditioned iterations are needed in order to solve the mass matrix system to accuracy of  $10^{-6}$ . The precondition is done by simple scaling [14] and require very little effort.

However, while it is very easy to calculate the product of the system (37) with a vector, there remain the problem of preconditioning this system. As stated before, although  $M$  is easily invertible  $M^{-1}$  is not sparse and therefore  $L = B^T M^{-1} B$  is not sparse as well. However, there is a simple way

and

$$M_\psi^{(e)} = \mu_e h_x h_y h_z \quad (33)$$

Using the element matrices we can rewrite the variational form as

$$\mathcal{L}_d = \frac{1}{2}(h^T M_H h + \psi^T M_\psi \psi) - a^T C h + \psi^T D a + f^T a \quad (34)$$

where we now have the stiffness matrices  $M_H, M_\psi, C$  and  $D$ . The matrices  $M_H$  and  $M_\psi$  are  $\mu$  weighted mass matrices. The matrices  $M_H$  has six diagonals and the matrix  $M_\psi$  is diagonal. The matrix  $C$  is a representation of the **curl** operator and as such, it contains a null space which is spanned by the divergence matrix represented by  $D$ .

Differentiating the discrete Lagrangian with respect to  $h, a, \psi$  yields the following discrete system.

$$\begin{pmatrix} M_H & 0 & -C^T \\ 0 & M_\psi & D \\ -C & D^T & 0 \end{pmatrix} \begin{pmatrix} h \\ \psi \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -f \end{pmatrix} \quad (35)$$

The system (35) is a KKT system of the form

$$\begin{pmatrix} M & B \\ B^T & 0 \end{pmatrix} \quad \text{where} \quad M = \begin{pmatrix} M_H & 0 \\ 0 & M_\psi \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} C^T \\ D \end{pmatrix} \quad (36)$$

The matrix  $B$  represents the discretization of the operator  $[\nabla \times, \nabla \cdot]^T$  which upon consistent discretization (as we have) is of full-rank [13].

The block  $M$  represent a mass matrix. For constant  $\mu$ , such matrices have condition number which is  $\mathcal{O}(1)$  [10] independent on the mesh, and therefore, are easily invertible. In the case that  $\mu$  is not constant, the matrix has a condition number which is proportional to  $\max(\mu)/\min(\mu)$ . However, the condition number of the matrix can be reduced to  $\mathcal{O}(1)$  by simple scaling [21]. This plays an important role when inverting the matrix as we see in the following section.

With the above characteristics of  $M$  and  $B$ , the KKT matrix (36) is invertible [23] however, it is indefinite. We now discuss the solution of the discrete system.

where  $h_x, h_y$  and  $h_z$  are the cells dimensions. We now rewrite the variational form (25) in a discrete form:

$$\begin{aligned} \mathcal{L}_2 = & \sum_e \frac{1}{2} \left[ h_{(e)}^T \left( \int_e \mu \mathbf{N} \mathbf{N}^T dV \right) h_{(e)} + \psi_{(e)} \left( \int_e \mu dV \right) \psi_{(e)} \right] - \quad (29) \\ & h_{(e)}^T \left( \int_e (\nabla \times \mathbf{N} \cdot \mathbf{W}^T) dV \right) a_{(e)} - a_{(e)}^T \left( \int_e (\nabla \cdot \mathbf{W}) dV \right) \psi_{(e)} + \\ & a_{(e)}^T \left( \int_e (\mathbf{W} \cdot \mathbf{f}) dV \right) \end{aligned}$$

Calculating the integrals in (29) we obtain the element matrices and (29) is rewritten as

$$\mathcal{L}_2 = \sum_e \frac{1}{2} \left[ h_{(e)}^T M_H^{(e)} h_{(e)} + \psi_{(e)} M_\psi^{(e)} \psi_{(e)} \right] - a_{(e)}^T C^{(e)} h_{(e)} - \psi_{(e)}^T D^{(e)} a_{(e)} + a_{(e)}^T f$$

Where the matrices  $M_H^{(e)}, M_\psi^{(e)}, C^{(e)}$  and  $D^{(e)}$  are the element matrices. The **curl** matrix over an element is given by

$$C^{(e)} = \begin{pmatrix} 0 & -\frac{h_x h_y}{6} C_1 & \frac{h_x h_z}{6} C_2 \\ \frac{h_x h_y}{6} C_2 & 0 & -\frac{h_y h_z}{6} C_1 \\ -\frac{h_x h_z}{6} C_1 & \frac{h_y h_z}{6} C_2 & 0 \end{pmatrix} \quad (30)$$

where

$$C_1 = \begin{pmatrix} -2 & 2 & -1 & 1 \\ -1 & 1 & -2 & 2 \end{pmatrix} \quad \text{and} \quad C_2 = \begin{pmatrix} -2 & -1 & 2 & 1 \\ -1 & -2 & 1 & 2 \end{pmatrix}$$

The **div** matrix over an element is given by

$$D^{(e)} = \begin{pmatrix} -h_z h_y & h_z h_y & -h_z h_x & h_z h_x & -h_x h_y & h_x h_y \end{pmatrix} \quad (31)$$

Finally, the mass matrices are given by

$$M_H^{(e)} = \begin{pmatrix} \frac{1}{36} \mu_e h_x h_y h_z M_e & 0 & 0 \\ 0 & \frac{1}{36} \mu_e h_x h_y h_z M_e & 0 \\ 0 & 0 & \frac{1}{36} \mu_e h_x h_y h_z M_e \end{pmatrix} \quad (32)$$

where

$$M_e = \begin{pmatrix} 4 & 2 & 2 & 1 \\ 2 & 4 & 1 & 2 \\ 2 & 1 & 4 & 2 \\ 1 & 2 & 2 & 4 \end{pmatrix}$$



Thus in each element, the unknowns  $\mathbf{A}, \mathbf{H}, \psi$  can be written as

$$\mathbf{A}^{(e)} = \sum_{i=1}^6 \mathbf{W}_i^{(e)} a_i^{(e)} = \mathbf{W}^T a_{(e)} \quad (26a)$$

$$\mathbf{H}^{(e)} = \sum_{i=1}^1 2 \mathbf{N}_i^{(e)} h_i^{(e)} = \mathbf{N}^T h_{(e)} \quad (26b)$$

$$\psi^{(e)} = \xi_i \psi_{(e)} \quad (26c)$$

where

$$\mathbf{W}_i^{(e)} = W_{xi}^{(e)} \hat{x}; \quad \mathbf{W}_{i+2}^{(e)} = W_{yi}^{(e)} \hat{y}; \quad \mathbf{W}_{i+4}^{(e)} = W_{zi}^{(e)} \hat{z}; \quad i = 1, 2,$$

and

$$\mathbf{N}_i^{(e)} = N_{xi}^{(e)} \hat{x}; \quad \mathbf{N}_{i+4}^{(e)} = N_{yi}^{(e)} \hat{y}; \quad \mathbf{N}_{i+8}^{(e)} = N_{zi}^{(e)} \hat{z}; \quad i = 1, 2, 3, 4,$$

and  $\xi = 1$  are a constant function over the element. The functions  $\mathbf{W}$  are given by

$$\begin{aligned} W_{x1}^{(e)} &= \frac{1}{h_x} \left( \frac{h_x}{2} - x \right); & W_{x2}^{(e)} &= \frac{1}{h_x} \left( \frac{h_x}{2} + x \right) \\ W_{y1}^{(e)} &= \frac{1}{h_y} \left( \frac{h_y}{2} - y \right); & W_{y2}^{(e)} &= \frac{1}{h_y} \left( \frac{h_y}{2} + y \right) \\ W_{z1}^{(e)} &= \frac{1}{h_z} \left( \frac{h_z}{2} - z \right); & W_{z2}^{(e)} &= \frac{1}{h_z} \left( \frac{h_z}{2} + z \right) \end{aligned}$$

and the functions  $\mathbf{N}$  and are given by

$$\begin{aligned} N_{x1}^{(e)} &= \frac{1}{h_y h_z} \left( \frac{h_y}{2} - y \right) \left( \frac{h_z}{2} - z \right) & N_{x2}^{(e)} &= \frac{1}{h_y h_z} \left( \frac{h_y}{2} + y \right) \left( \frac{h_z}{2} - z \right) \\ N_{x3}^{(e)} &= \frac{1}{h_y h_z} \left( \frac{h_y}{2} - y \right) \left( \frac{h_z}{2} + z \right) & N_{x4}^{(e)} &= \frac{1}{h_y h_z} \left( \frac{h_y}{2} + y \right) \left( \frac{h_z}{2} + z \right) \\ N_{y1}^{(e)} &= \frac{1}{h_x h_z} \left( \frac{h_x}{2} - x \right) \left( \frac{h_z}{2} - z \right) & N_{y2}^{(e)} &= \frac{1}{h_x h_z} \left( \frac{h_x}{2} + x \right) \left( \frac{h_z}{2} - z \right) \\ N_{y3}^{(e)} &= \frac{1}{h_x h_z} \left( \frac{h_x}{2} - x \right) \left( \frac{h_z}{2} + z \right) & N_{y4}^{(e)} &= \frac{1}{h_x h_z} \left( \frac{h_x}{2} + x \right) \left( \frac{h_z}{2} + z \right) \\ N_{z1}^{(e)} &= \frac{1}{h_x h_y} \left( \frac{h_x}{2} - x \right) \left( \frac{h_y}{2} - y \right) & N_{z2}^{(e)} &= \frac{1}{h_x h_y} \left( \frac{h_x}{2} + x \right) \left( \frac{h_y}{2} - y \right) \\ N_{z3}^{(e)} &= \frac{1}{h_x h_y} \left( \frac{h_x}{2} - x \right) \left( \frac{h_y}{2} + y \right) & N_{z4}^{(e)} &= \frac{1}{h_x h_y} \left( \frac{h_x}{2} + x \right) \left( \frac{h_y}{2} + y \right) \end{aligned}$$

$\psi = \nabla \cdot \mathbf{A}$  at the nodes. However, the fields  $\mathbf{A}$  at the nodes in cases of large discontinuities are not well behaved and may present singularities (see [13]) which may result in poor accuracy of  $\nabla \cdot \mathbf{A}$  at the nodes (where we calculate  $\psi$  in the case of low order interpolation). On the other hand, (23) implies that we have  $\mathbf{A} \in \mathcal{H}(div)$  and  $\psi = \nabla \cdot \mathbf{A}$  needs to be only in  $\mathcal{H}_0$  which is similar to define it at cell centers where  $\mathbf{A}$  and  $\nabla \cdot \mathbf{A}$  are well defined. We therefore use the weak form (25) in order to discretize the system.

## 4 Discretization of the weak formulation

In order to discretize (25) we divide our domain into rectangular elements and pick the variables to be of a low level interpolation functions over the elements. The vector potential  $\mathbf{A}$  is chosen to be approximated by a  $\mathcal{RT}$  element [22]. The magnetic field is chosen to be a lowest order Whitney or edge element [3, 17] and finally, the scalar  $\psi$  is chosen to be piecewise constant. The numbering of the unknowns on an element is in Figure 1.

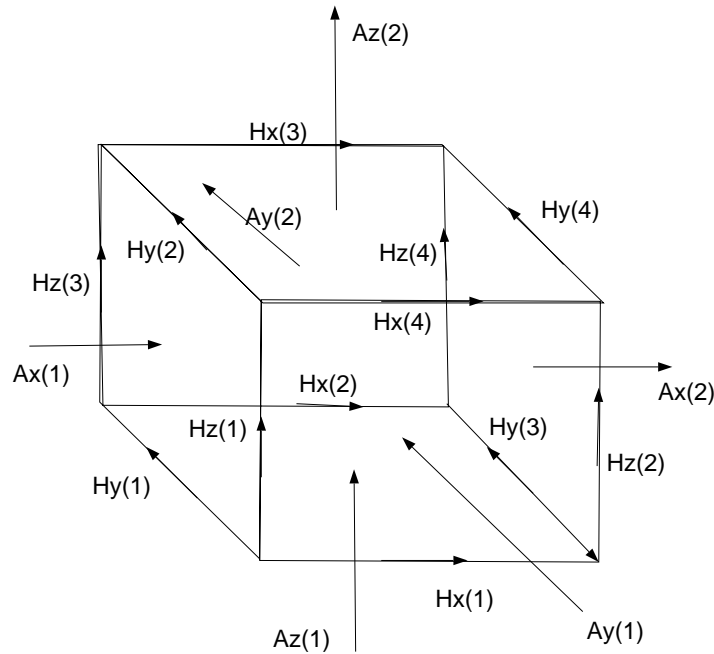


Figure 1: Discretization of the unknowns over an element

### 3 Weak form of the system

In order to discretize the system (21) we first present it in its weak form. It is easy to verify that the system (21) corresponds to the following constraint optimization problem

$$\text{minimize} \quad \frac{1}{2} \int \mu(\mathbf{H}^2 + \psi^2) dV \quad (22a)$$

$$\text{subject to} \quad \nabla \times \mathbf{H} - \nabla\psi - \mathbf{f} = 0 \quad (22b)$$

Note that  $\mathbf{A}$  does not appear in the constrained problem. This is similar to the weak form of the div-grad system (13a) and  $\mathbf{A}$  is introduced as a Lagrange multiplier in the equivalent Lagrangian

$$\mathcal{L}_1 = \int \frac{\mu}{2}(\mathbf{H}^2 + \psi^2) - \mathbf{A} \cdot (\nabla \times \mathbf{H} - \nabla\psi - \mathbf{f}) dV \quad (23)$$

So far in this paper we avoided the question of boundary conditions. However, the above Lagrangian is equivalent to the PDE (21) only for the following *Natural boundary conditions*

$$(\mathbf{n} \times \mathbf{H})_{\partial\Omega} = 0 \quad (24a)$$

$$\psi_{\partial\Omega} = 0 \quad (24b)$$

Note that as discussed, the BC (24b) implies that  $\nabla \cdot \mathbf{A} = 0$ . Here we assume for simplicity only natural BC but other BC can be added [17].

In this weak form we have  $\mathbf{A} \in \mathcal{H}_0$ ,  $\mathbf{H} \in \mathcal{H}(\text{curl})$  and  $\psi \in \mathcal{H}_1$  and the smoothness of  $\mathbf{A}$  is relaxed while the smoothness of  $\mathbf{H}$  and  $\psi$  is enhanced. This is important as it allows us to find better approximations for the magnetic field  $\mathbf{H}$  than for the magnetic vector potential  $\mathbf{A}$ .

While it is important to have the smoothness of  $\mathbf{H}$  enhanced,  $\psi$  is only a fictitious variable and we can relax its smoothness by integrating by parts, obtaining

$$\mathcal{L}_2 = \int \frac{1}{2}\mu(\mathbf{H}^2 + \psi^2) - \nabla \times \mathbf{H} \cdot \mathbf{A} - (\nabla \cdot \mathbf{A})\psi + \mathbf{A} \cdot \mathbf{f} dV \quad (25)$$

Although it is possible to discretize either (23) or (25) we have found that (25) behaves better for very large discontinuities. We suspect that the main reason is that in (23)  $\psi$  is at least in  $\mathcal{H}_1$  which implies continuity of

which is closely connected to augmented Lagrangian and penalty methods [12, 11, 1]. It is possible to write equations (16a) and (17) in their weak form as

$$\min \int [\mu^{-1}(\nabla \times \mathbf{A})^2 - \mathbf{A} \cdot \mathbf{f}]dV \quad (18)$$

and

$$\min \int [\mu^{-1}((\nabla \times \mathbf{A})^2 + (\nabla \cdot \mathbf{A})^2) - \mathbf{A} \cdot \mathbf{f}]dV \quad (19)$$

The condition (16b) can be applied to (18) by choosing  $\mathbf{A} \in \mathcal{H}(\text{curl})$  (as in [2]) or it can be approximately applied in (19) (as in [17, 8]). However, as stated in the introduction, these methods lead to low accuracy in  $\mathbf{H} = \mu^{-1}\nabla \times \mathbf{A}$  because it is obtained by numerical calculation of the **curl**, however,  $\mathbf{H}$  is exactly the quantity desired from such calculations. Therefore, we take a different approach and similarly to the electrostatic problem, we do not eliminate  $\mathbf{H}$  from the system. Furthermore, we introduce a scalar function  $\psi$  such that

$$\nabla \cdot \mathbf{A} = \mu\psi \quad (20)$$

and the system is rewritten as

$$\nabla \times \mathbf{A} - \mu\mathbf{H} = 0 \quad (21a)$$

$$\nabla \cdot \mathbf{A} - \mu\psi = 0 \quad (21b)$$

$$\nabla \times \mathbf{H} - \nabla\psi = \mathbf{f} \quad (21c)$$

Note that in our formulation, equation (21c) corresponds to the positive definite system (17). Furthermore, since  $\nabla \cdot \mathbf{f} = 0$  equation (21) implies that  $\nabla^2\psi = 0$  and therefore if we demand Dirichlet boundary conditions on  $\psi$  of the form  $\psi|_{\partial\Omega} = 0$ , we obtain  $\nabla \cdot \mathbf{A} = 0$  identically.

The advantage of adding the unknowns  $\mathbf{H}$  is similar to mixed formulations in the electrostatic case and we can design a method that allows higher order interpolation functions for  $\mathbf{H}$ . Also, introducing  $\psi$  allows us to relax the divergence-free condition which is an arbitrary gauge condition. As we show in the next section this formulation implies a different weak forms that allows different smoothness for the different unknowns.

equipped with the usual norms (see, e.g., [13]).

We now note that while in (11)  $\phi \in \mathcal{H}_1$  the weak formulation (13a) requires  $\phi$  to be only in  $\mathcal{H}_0$ . Also note that in (11), the flux  $\mathbf{J}$  can be obtained only by taking the gradient of  $\phi$  and therefore  $\mathbf{J} \in \mathcal{H}(\text{curl})$  rather in  $\mathcal{H}(\text{div})$  as it should be. However, after the decoupling we obtain  $\mathbf{J} \in \mathcal{H}(\text{div})$  and this allows the use the “right” interpolation for the flux, which keeps its continuity across an interface and not across an edge (see [7, 19, 6]). In most practical applications  $\mathbf{J}$  is approximated by Raviart-Thomas elements whereas  $\phi$  is chosen to be approximated by piecewise constant elements. This has been shown to be important especially in the case of large discontinuities in  $\sigma$  [19].

Our approach to the magnetostatic problem which is driven from the above mixed approach, is stable for very large discontinuities in  $\mu$  and as we shall see in Section 5 it is easy to reduce the discrete system into a positive definite system which allows the use of standard and rapidly converging iterative techniques and preconditioners.

The paper is divided as follows. In Section 2 we reformulate the problem as a first order system similar to the electrostatic case. In Section 3 we derive the weak formulation and show that its multidimensionality allows us to use two different mixed formulations. In Section 4 we use mixed finite elements in order to discretize the system. In Section 5 we discuss the solution of the system by employing sparse matrix techniques. We conclude this paper in Section 6 where we perform numerical experiments on a model problem.

## 2 Reformulation of the magnetostatic problem

Similarly to the electrostatic case we begin with the equations for the magnetic potential

$$\nabla \times \mu^{-1} \nabla \times \mathbf{A} = \mathbf{f} \tag{16a}$$

$$\nabla \cdot \mathbf{A} = 0 \tag{16b}$$

For now, let us ignore the question of BC assuming that they are proper for this problem (see [17, 4]). We shall return to this after we derive our weak form. The system (16) is in principle an over-determined yet consistent. This system is made positive definite by adding a stabilizer [8, 15]

$$\nabla \times \mu^{-1} \nabla \times \mathbf{A} - \nabla \mu^{-1} \nabla \cdot \mathbf{A} = \mathbf{f} \tag{17}$$

In this paper we take a new approach which can be seen as a combination of the two above mentioned approaches namely, we use the magnetic potential but view the constitutive relation (9c) in a weak sense. Our motivation is drawn from the electrostatic problem where mixed type methods are commonly used to solve for large discontinuities (see for example [7, 6, 11, 22, 19] and others). In the electrostatic problem the **div-grad** system

$$\nabla \cdot \sigma \nabla \phi = f \quad (10)$$

can be written in a weak form as a minimization problem

$$\min \int \left[ \frac{1}{2} \sigma (\nabla \phi)^2 + \phi f \right] dV \quad (11)$$

A common discretization of this weak formulation is done by using nodal elements [17, 7, 19].

In mixed methods for electrostatics, the **div-grad** equation is decoupled into two first order systems

$$\nabla \cdot \mathbf{J} = f \quad (12a)$$

$$\sigma^{-1} \mathbf{J} - \nabla \phi = 0 \quad (12b)$$

In its weak form, the problem is a constraint optimization problem of the form

$$\min \int \frac{1}{2\sigma} \mathbf{J}^2 dV \quad (13a)$$

$$\text{subject to } \nabla \cdot \mathbf{J} - f = 0 \quad (13b)$$

Note that  $\phi$  does not appear in the system and it can be viewed as a Lagrange multiplier of the equivalent Lagrangian

$$\mathcal{L} = \int \left[ \frac{1}{2} \sigma^{-1} \mathbf{J}^2 + \phi (\nabla \cdot \mathbf{J} - f) \right] dV \quad (14)$$

In order to see the advantage of the weak form (14) we define the regular Sobolov spaces

$$\mathcal{H}_0 = \{ \mathbf{v} \in \mathbf{L}^2(\Omega) \} \quad (15)$$

$$\mathcal{H}_1 = \{ \mathbf{v} \in \mathbf{L}^2(\Omega); \nabla \mathbf{v} \in \mathbf{L}^2(\Omega) \}$$

$$\mathcal{H}(\text{div}) = \{ \mathbf{v} \in \mathbf{L}^2(\Omega); \nabla \cdot \mathbf{v} \in \mathbf{L}^2(\Omega) \}$$

$$\mathcal{H}(\text{curl}) = \{ \mathbf{v} \in \mathbf{L}^2(\Omega); \nabla \times \mathbf{v} \in \mathbf{L}^2(\Omega) \}$$

Note that (5) differs from (2) only in the source term.

The solution of the system (5) has been addressed in [17, 3, 2, 21, 8] and others. In [2, 17, 8] (and many others) the magnetic potential  $\mathbf{A}$  was introduced such that

$$\mu \mathbf{H} = \nabla \times \mathbf{A} \quad (6)$$

This condition satisfies (5b) and we obtain a system for the vector potential  $\mathbf{A}$  (and appropriate BC for  $\mathbf{A}$  [17]).

$$\nabla \times \mu^{-1} \nabla \times \mathbf{A} = \mathbf{f} \quad (7)$$

This equation does not have a unique solution due to the null space of the curl operator and therefore a gauge condition for  $\mathbf{A}$  is introduced

$$\nabla \cdot \mathbf{A} = 0 \quad (8)$$

In [8, 17] a solution of the problem using nodal elements is derived, employing the gauge condition as penalty. This method has been shown to be unstable when  $\mu$  contains large discontinuities. Better results were obtained using edge elements for  $\mathbf{A}$  [2]. However, this method suffers from low accuracy because  $\mathbf{H}$  is obtained from  $\mathbf{A}$  using numerical differentiation. Furthermore, the resulting discretization of the system (7) is semidefinite and therefore convergence of iterative methods may be slow.

A second approach to the problem is in [3, 21] where the magnetic flux  $\mathbf{B}$  is introduced and the constitutive relation is added to the system. This yields the system

$$\nabla \times \mathbf{H} = \mathbf{f} \quad (9a)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (9b)$$

$$\mathbf{B} - \mu \mathbf{H} = 0 \quad (9c)$$

and the constitutive relation (9c) is viewed only in a weak sense. This method, although very stable for large discontinuities requires the introduction of three extra Lagrange multipliers and therefore suffers from a large number of unknowns. Furthermore the solution of the resulting system requires some non-trivial linear algebra (see the work of [21]). The main difficulty of this method is that it leads to very large indefinite systems of five vector fields and therefore it requires large memory and sophisticated solution techniques.

of permeability in the ground. The problem is also a corner-stone to the solution of Maxwell's equations in the quasi-static case and therefore efficient techniques for its solution in the case of large discontinuities are important. The problem evolves from Maxwell's equations in the static limit ( $\partial/\partial t(\cdot) = 0$ ) which can be written as

$$\nabla \times \mathbf{E} = 0 \quad (1a)$$

$$\nabla \times \mathbf{H} - \sigma \mathbf{E} = \mathbf{J}^s \quad (1b)$$

$$\nabla \cdot (\mu \mathbf{H}) = 0 \quad (1c)$$

with appropriate boundary conditions on  $\mathbf{H}$  (see [17]). In this paper we assume only Dirichlet type BC i.e.

$$(\mathbf{n} \times \mathbf{H})_{\partial\Omega} = 0.$$

There are two different magnetostatic problems. First, if the source is divergence free (such as the earth's magnetic field) i.e.  $\nabla \cdot \mathbf{J}^s = 0$  then, it is easy to see that  $\mathbf{E} = 0$  and the system reduces to

$$\nabla \times \mathbf{H} = \mathbf{J}^s \quad (2a)$$

$$\nabla \cdot (\mu \mathbf{H}) = 0 \quad (2b)$$

This is an overdetermined yet consistent system for  $\mathbf{H}$ , that is, although there are four equations and three unknowns, it is possible to find  $\mathbf{H}$  which solves the system (2).

Secondly, if  $\nabla \cdot \mathbf{J}^s \neq 0$  (for example in the MMR problem [18]) then there exist an electric potential  $\phi$ , such that  $\mathbf{E} = \nabla\phi$  and we obtain

$$\nabla \times \mathbf{H} - \sigma \nabla\phi = \mathbf{J}^s \quad (3a)$$

$$\nabla \cdot (\mu \mathbf{H}) = 0. \quad (3b)$$

However, this system can be reduced to a similar form as (2) by taking the **div** of (3a), obtaining the well known **div-grad** electrostatic system

$$\nabla \cdot (\sigma \nabla\phi) = -\nabla \cdot \mathbf{J}^s \quad (4)$$

Once this system is solved<sup>1</sup>,  $\phi$  is recovered and one could obtain  $\mathbf{H}$  by solving the system

$$\nabla \times \mathbf{H} = \mathbf{J}^s - \sigma \nabla\phi = \mathbf{f} \quad (5a)$$

$$\nabla \cdot (\mu \mathbf{H}) = 0 \quad (5b)$$

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<sup>1</sup>Note that the solution of the electrostatic equation requires an additional BC on  $\phi$  which is not given in the system (3a). This BC is driven from other physical considerations



# A mixed finite element method for the solution of the magnetostatic problem with highly discontinuous coefficients in 3D

E. Haber\*

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## Abstract

This paper presents a new mixed finite element method for the simulation of magnetostatic problems with highly discontinuous permeability. The method is derived from the well studied mixed formulation for the **div-grad** system that is known to be accurate for very large discontinuities. The method robustness is demonstrated on a test model problem.

**Keywords:** vector potential, Maxwell's equations, solution discontinuities, mixed finite elements, Krylov subspace methods, preconditioning.

## 1 Introduction

Magnetostatic simulation is important and has many applications [2, 18, 20, 17]. In geophysics, this simulation is important in order to simulate the response of unexploded ordnances (UXO's) [5] and for the MMR experiment [18]. In the UXO problem one needs to simulate the response of metal with permeability of roughly  $\mu = 5 \times 10^2 \mu_0$  in a medium of  $\mu = \mu_0$  and therefore it is important to be able to model very large discontinuities. In the MMR problem one needs to simulate the response of any arbitrary distribution

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\*Department of Computer Science and Institute of Applied Mathematics, University of British Columbia, Vancouver, BC, V6T 1Z4, Canada. (haber@cs.ubc.ca).