The electromagnetic response of a sphere in the moment domain: a building block for the construction of arbitrary 3D shapes

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Summary

We present a fast approximate forward modelling approach whose aim is to model confined conductors in a resistive background. The method consists of two novel elements, the use of the moment transform, and the use of superposition of the electromagnetic response of small spheres to construct that of arbitrary shapes. Transformation of the time-derivative of the magnetic field measurements to moments of the magnetic field is obtained via the moment transform. The moment-domain electromagnetic response of a small sphere excited by a plane wave is used as a building block to construct arbitrary shapes and simulate their response to a dipole excitation. The electromagnetic response generated by the superposition of these small spheres compares favourably with the analytic response of a large sphere and a plate excited by both plane wave and dipole sources. In addition to proposing a fast approximate forward modelling approach, our aim has been to lay the ground work for a fast imaging algorithm.

Introduction

We describe an approximate forward modelling approach which is designed to produce the response of confined conductors as described by Kaufman (1978). He explained that regardless of shape and conductivity, the response of a confined conductor in the time-domain could be expressed as the sum of decaying exponentials. Under his formulation, the late-time response is expressed as a single exponential time-decaying function. The time-constant (τ) of this late-time function is related to the shape, size and conductivity of the conductor. Although we explicitly use the τ of a small sphere as a building block, we show examples of successfully matching the τ of both a large sphere and a plate. Based on these results, we suggest that the superposition of the electromagnetic response of many spheres of time-constant τ may be used to construct the electromagnetic response of an arbitrary shape also of time-constant τ.

In addition, our use of the moment transform as described by Smith and Lee (2002a) and (2002b), holds several advantages over the time-domain formulation. The principal benefit of the moment formulation is speed. A second benefit as advocated by Smith and Lee (2002a), is the ability of higher order moments to image either deep or very conductive targets.

Although not described here, our ultimate goal is to pose this approximate forward problem as an inverse problem. Our ability to model arbitrary forward shapes in a general fashion where we subdivide our model space in a mesh of cubic cells is key to the success of this technique. In its present state, and because it is based upon the free-space response of a sphere, the imaging algorithm is expected to perform well under the condition that the response is due to confined conductors in a resistive host. In this scenario, vortex currents and not galvanic currents dominate the response. Because our building block is a sphere in free-space, it is only the vortex currents that are taken into account. The approach is to transform the time-domain data to the moment domain and from these data to produce an image in terms of τ, the product of conductivity σ, magnetic permeability μ and an arbitrary dimension squared. Then, depending on the shape of the objects that are recovered in the image, a conversion from τ to σ is possible.

The moment transform

The moment transform has the following form

\[ \tilde{b}(n) = \int_0^\infty t^n \frac{\partial \tilde{b}(t)}{\partial t} \, dt \]  \hspace{1cm} (1)

where t is time and \( \frac{\partial \tilde{b}(t)}{\partial t} \) is the time-derivative of the magnetic field due to a step-off current. Smith and Lee (2002a) show that the zero-order moment corresponds to the inductive limit and that the first-order moment corresponds to the resistive limit. In practice, field data are transformed with an incomplete moment transform where the integration is over the measurement time. However, we still apply the full moment transform in our modelling with the understanding that for confined conductors, the difference between the two is minimal.

The moment transform of a sphere in a uniform field

The use of the moments of a sphere in a uniform field has been advocated by Smith and Lee (2002a) as a simple model to simulate a sphere excited by a dipolar field. They further show that for the zero-order moment only, the sphere excited by a dipolar field may be simulated with multiple small-radii uniform-field spheres. The small spheres are placed to fill the volume of the larger sphere. In this paper, we extend their formulation to arbitrary moments by applying a superposition of the small spheres in terms not of conductivity σ, but of the time-constant τ. In addition, the concept of superposition is generalized with the goal of constructing arbitrary 3D shapes.

The time-domain response of a sphere excited by a uni-
form field is

\[
\frac{\partial \tilde{b}(t)}{\partial t} = \frac{b_0}{2} \left( \frac{2a^3 \cos \theta}{r^3} \hat{r} + \frac{a^3 \sin \theta}{r^3} \hat{\theta} \right) \Xi(t)
\]  

(2)

where

- \( \hat{r} \): unit vector in the radial direction.
- \( \hat{\theta} \): unit vector in the azimuthal direction.
- \( \Xi(t) \): time-domain response.
- \( b_0 \): primary magnetic flux density at the sphere (T).
- \( a \): radius of the sphere (m).
- \( r \): distance from the sphere to the measurement point (m).
- \( \theta \): latitude angle of measurement point (rad).
- \( x_s \): zeros of spherical bessel function of order 0 with index \( s \).

We express it as

\[
\Xi(t) = -\frac{\tau}{\pi} \sum_{s=1}^{\infty} e^{-x_s^2 \tau/r^2}
\]

This equation can be written as

\[
\tilde{b}(n) = -\frac{\mu_0 b_0}{2} \left( \frac{2a^3 \cos \theta}{r^3} \hat{r} + \frac{a^3 \sin \theta}{r^3} \hat{\theta} \right) \sum_{s=1}^{\infty} \frac{1}{j^{2n+2}} 6\tau^n \Gamma(n+1) \frac{1}{\pi^{2n+2}} \sum_{s=1}^{\infty} 1
\]

(4)

This equation can be written as

\[
\tilde{b}(n) = \Xi(n) g(\tau) a^3 \tau^n
\]  

(5)

where \( \Xi \) is a function of the moment order, \( g \) is a function of geometry, \( a \) is the radius of the sphere and \( \tau = \sigma \mu_0 a^2 \). We see that under the moment formulation, the moment-dependent and the \( \tau \) dependent elements are decoupled. In contrast, under the time-domain formulation, the time-dependent and the \( \tau \) dependent elements are coupled. It is in part this decoupling that leads to faster calculations.

**Superposition**

Superposition relates the moments of the magnetic field at a particular location to a distribution of conductive spheres. We express it as

\[
\tilde{b}_i(n) = f(n) \sum_{j=1}^{M} \alpha_j g(\tilde{r}_{ij}) \alpha_j \tau^n
\]  

(6)

where the \( i \) subscripts refer to a measurement location, the \( j \) subscripts refer to a small-radius sphere location and \( \alpha_j \) are the radii of the small spheres. Consider a model space which we subdivide into cubic cells. In each cell we place a small sphere. We compare the response of a large sphere to that due to the superposition of many small spheres. We consider the case where the large sphere is far away from the measurement point such that \( \tilde{r}_{ij} \approx \tilde{r} \). As the discretization in terms of the small spheres becomes finer, the number of small spheres which fill the large sphere will converge to

\[
M = \frac{4\pi a^3}{3c}
\]

(7)

where \( c \) is the length of the side of a small cubic cell and the above expression is the ratio of the volume of a large sphere \( 4\pi a^3/3 \) over the volume of the small cubic cells \( c^3 \).

We equate the moment of the magnetic field due to a large sphere to that due to the superposition of \( M \) small spheres and observe that

\[
f(n) g(\tau) \tau^n a^3 \approx f(n) M g(\tau) \tau^n a^3
\]

(8)

By reorganizing the two equations above, we obtain

\[
a^3 = \sqrt{\frac{6 \epsilon_c \pi}{\tau}}
\]

(9)

This corresponds to placing a small sphere of volume equal to the volume of the small cubic cell in which it is placed.

Note that our ability to use superposition of small spheres to reproduce the response of a large sphere is a function of posing the problem in terms of \( \tau \), not a function of using the moment formulation. If we choose, we can apply the superposition principle in the time-domain.

**First Test: The plane wave source**

In the above discussion, we assumed that \( r_{ij} \approx r \) which is true only if the sphere is far away from the measurement point. In this section, we generate synthetic examples to demonstrate the validity of superposition as a function of moment order and distance from the sphere.

Synthetic data, which are moments of the vertical magnetic field, are generated on a plane 150m (Fig. 1a and 1b) and 50m (Fig. 1c and 1d) above the sphere. In all cases, \( \tau = 2s, a = 50m \) and \( c = 5m \). In the first comparison (Fig. 1a and 1b), \( n = 1 \) and in the second comparison, \( n = 2 \).

The numerical tests confirm that superposition holds for all moments and its accuracy is independent of distance from the sphere to the measurement point. The accuracy of superposition is mostly determined by the level of discretization. For the 50m radius sphere and \( c = 5m \), differences are on the order of 1%.

**Second Test: The dipole source**

The expression for the fields and their moments due to a
Here we show his solution for the magnetic potential due to a pole source. For spheres the full sphere response is sometimes referred to as the dipole approximation of the full sphere response. Conversely, the sphere excited by a dipole source is more complicated. Nabighian (1970) solved this problem in the time domain. Because the infinite sum is independent of the sphere parameters, it can be pre-calculated and therefore, the moments of the magnetic field due to this sphere are calculated much more quickly than their time-dependent counterparts.

We are now in a position to compare the accurate multipole expansion of the sphere response to the superposition of many small spheres. As in the previous test, we vary depth of the sphere and the order of the moments. Radius \( a = 50m \), the time-constant \( \tau = 0.0031s \), moment order \( n = 1 \) and the discretization \( c = 10m \) and \( c = 3.33m \) are kept constant throughout the tests (Figs. 2 and 3).

**Fig. 1:** Large sphere (a and c) vs superposition of small spheres (b and d)

**Fig. 2:** Comparison of multipole sphere response to superposition of small spheres. Depth to sphere centre = 200 m. Vertical magnetic dipole source and receiver with 10 m separation.

**Fig. 3:** Comparison of multipole sphere response to superposition of small spheres. Depth to sphere centre = 100 m. Vertical magnetic dipole source and receiver with 10 m separation.

Formally, the magnetic dipole response of the sphere reduces to the uniform field solution discussed earlier. The \( m = 1 \) solution is sometimes referred to as the dipole approximation of the full sphere response. Conversely, the full sphere response is sometimes referred to as the multipole expansion of the sphere solution. For spheres far away from the source and receiver, the dipole approximation is often sufficient to accurately model the sphere response.

The moments are obtained by taking the integral of the time-dependent function

\[
E_m(n) = \int_0^\infty t^n \frac{\partial \tilde{E}_m(t)}{\partial t} dt = -2(2m + 1) \tau^n \sum_{\beta = 1}^{\infty} \frac{1}{x_{s,m}^{2(2m + 1)}}
\]

Because the infinite sum is independent of the sphere parameters, it can be pre-calculated and therefore, the moments of the magnetic field due to this sphere are calculated much more quickly than their time-dependent counterparts.

**Third Test:** The vertical plate

In order to investigate the effectiveness of superposition, consider a plate which is a confined conductor whose shape is substantially different than the sphere. The plate and the sphere are distinguished by their behaviour with respect to the exciting primary field (Dyck and...
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West, 1984). In the case of the plate, the induced dipolar-like moment is perpendicular to the plate independent of primary-field direction whereas for the sphere, the induced dipole moment is aligned in the direction opposite to the primary field. If superposition of small spheres can simulate the response of a plate then the ability of the superposition of small spheres to simulate arbitrary 3D shapes seems promising.

Recalling that our approximate modelling approach is posed in terms of \( \tau \), we determine the relation between the conductivity of the small spheres and the resultant plate. In the case of the small spheres used to construct a large sphere, the relation \( \sigma_{\text{small}} \mu_0 a^2 = \sigma_{\text{big}} \mu_0 a^2 \) is written as

\[
\sigma_{\text{big}} = \frac{b^2}{a^2} \sigma_{\text{small}}
\]

(13)

However since the time constant of a plate (Kaufman, 1994) is \( \tau_{\text{plate}} = \frac{\sigma_{\text{plate}} \mu_0 l}{8.92} \) and that of the sphere is \( \tau_{\text{sphere}} = \frac{\sigma_{\text{sphere}} \mu_0 a^2}{\pi^2} \), then the conductivity-thickness of the plate constructed from small spheres is equivalent to

\[
\sigma l_{\text{plate}} = 1.23 \frac{b^2}{l} \sigma_{\text{sphere}}
\]

(14)

Our plate response is produced using vhplate (Walker and West, 1991) and transformed to the moment domain using a numerical integral evaluation. The plate is vertical, depth to top is 25 m with dimension 100 m x 100 m. The conductivity-thickness of the plate is 0.1S which results in a time-constant for the plate of \( \tau = 1.5 \times 10^{-7} \) s. We choose the radii of the small spheres to be 5m (Fig. 4).

![Fig. 4: Comparison of vertical plate response to superposition of small spheres. Depth to top of plate = 25 m. Vertical magnetic dipole source and receiver with 50 m separation.](image)

The match between the plate response and the approximate model is surprisingly good except at the peak. As with the sphere model, the superposition of small spheres overshoots the true response, in this case by approximately 60%.

Conclusion

The superposition of small spheres seems well suited to produce the inductive response of arbitrary confined conductors. The quality of the match is inversely proportional to the transmitter-target distance, but even when the target is close to transmitter, the true and the approximate profiles are quite similar.

We have presented a method to simulate the inductive response of arbitrary 3D conductive confined targets in a resistive host. The principal advantage of the method is that it is fast and relatively accurate when the target is far from the transmitter. Our motive in developing such an algorithm is that may eventually be used in an imaging capacity. In contrast to rigorous 3D-EM inversion, this imaging approach is relatively independent of the number of transmitters. Given that this imaging approach is relatively accurate when the targets are far from the transmitter and that this method is not penalized by the number of transmitters, it is particularly well suited for the imaging of airborne TEM surveys.

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References


