3-D frequency-domain CSEM inversion using unconstrained optimization

Eldad Haber(1), Uri Ascher(2), Douglas W. Oldenburg(3)*, Roman Shekhtman(4), & Jiuping Chen(3),
(1) EMI-Schlumberger, Richmond, CA; (2) Dept. of Computer Science, UBC, Vancouver, Canada; (3) UBC-
Geophysical Inversion Facility, Dept. of Earth & Ocean Sciences, UBC, Vancouver, Canada;

Summary
We develop an inversion algorithm for computing frequency domain electromagnetic inversion for conductive
bodies in the low frequency regime. The algorithm is based on an inexact Gauss-Newton method where only
sensitivities times vector are calculated. Because of the anticipated heavy computational load, one has to
answer many practical questions in order to wisely use the generic numerical inverse methodology. In this
presentation, we address some of these questions. A synthetic model stimulating a CSAMT survey has been
carried out as a demonstration of the application of this generic inverse method.

Introduction
In this paper we discuss the solution of electromagnetic inverse problems in the frequency domain using an un-
constrained Inexact Gauss-Newton formulation.

The approach is complementary to our constrained approach, presented in a companion abstract, and it serves
as a demonstration of the application of this generic inverse method.

The Forward Problem

The frequency-dependent Maxwell equations can be written as

\( \nabla \times \mathbf{E} - \omega \mu \mathbf{H} = 0 \)  \hspace{1cm} (1a)

\( \nabla \times \mathbf{H} - (\sigma - \omega \epsilon) \mathbf{E} = s_{s}(\omega) \)  \hspace{1cm} (1b)

where \( \omega \) is the frequency, \( \mathbf{E} \) and \( \mathbf{H} \) are the electric and magnetic fields, \( \mu \) is the permeability, \( \sigma \) is the conductivity, \( \epsilon \) is the permittivity and \( s_{s} \) is a source. The boundary
conditions over the entire boundary of the spatial domain, \( \partial \Omega \), are

\( \mathbf{n} \times \mathbf{H} = 0. \)  \hspace{1cm} (2)

We let \( \hat{\sigma} = \sigma - i \omega \epsilon \). As discussed in Haber and Ascher(2001a) and Haber et al.(2000a), this form is not fa-
vorable for iterative solvers, especially when \( |\omega \hat{\sigma}| \) is small (for example in the air). We therefore reformulated the
problem prior to discretizing it further such that it is more amenable to applying standard iterative solvers.

A Helmholtz decomposition with Coulomb gauge is applied, decoupling the \( \text{curl} \) operator:

\( \mathbf{E} = \mathbf{A} + \nabla \phi, \quad \nabla \cdot \mathbf{A} = 0 \quad \text{in} \ \Omega \)

\( \mathbf{A} \cdot \mathbf{n} = 0 \quad \text{on} \ \partial \Omega. \)

After adding a stabilization term and differentiating (Haber and Ascher, 2001a), this leads to the system

\( \nabla \times \mu^{-1} \nabla \times \mathbf{A} - \nabla \mu^{-1} \nabla \cdot \mathbf{A} + i \omega \hat{\sigma} (\mathbf{A} + \nabla \phi) = i \omega \mathbf{s} \)  \hspace{1cm} (3a)

\( \nabla \cdot (\hat{\sigma} (\mathbf{A} + \nabla \phi)) = \nabla \cdot \mathbf{s} \)  \hspace{1cm} (3b)

in \( \Omega \), subject to

\( \mathbf{n} \times \nabla \times \mathbf{A} = 0, \quad \mathbf{n} \cdot \mathbf{A} = 0, \)  \hspace{1cm} (3c)

\( \mathbf{n} \cdot \nabla \phi = 0, \)  \hspace{1cm} (3d)

on the boundary \( \partial \Omega \).

Following Haber and Ascher(2001a) and Haber et al.(2000a), we use a finite volume approach for the dis-
cretization of (3) on an orthogonal, staggered grid. We choose to discretize \( \mathbf{A} \) on cell faces and \( \phi \) at cell cen-
ters. Note that the modified conductivity \( \hat{\sigma} \) is averaged harmonically at cell faces, whereas the permeability is
averaged arithmetically at edges.

We write the fully discretized system as

\( \begin{pmatrix}
L_{\mu} + i \omega M_{\hat{\sigma}} & i \omega M_{\hat{\sigma}} \nabla_{h} \\
\nabla_{h} \cdot M_{\hat{\sigma}} & \nabla_{h} \cdot M_{\hat{\sigma}} \nabla_{h} 
\end{pmatrix}
\begin{pmatrix}
\mathbf{A} \\
\phi
\end{pmatrix} =
\begin{pmatrix}
i \omega \mathbf{s} \\
\nabla_{h} \cdot \mathbf{s}
\end{pmatrix} \hspace{1cm} (4)

where \( \nabla_{h}, \nabla_{h} \times \) and \( \nabla_{h} \) are matrices arising from the discretization of the corresponding continuous operators, \( M_{\hat{\sigma}} \) arises from the operator \( \hat{\sigma}(\cdot) \) and \( L_{\mu} \) is the discretization of the operator \( \nabla \times \mu^{-1} \nabla \times - \nabla \mu^{-1} \nabla \). This linear system can be solved using standard iterative
methods (Saad, 1996) and effective preconditioners can be designed for it (Haber et al., 2000a; Aruliah and As-
cher, 2002). Briefly, for small enough \( \omega \), the system is dominated by its diagonal blocks and therefore a good
preconditioner can be obtained by using an approximation of the matrix

\( \begin{pmatrix}
L_{\mu} & 0 \\
0 & \nabla_{h} \cdot M_{\hat{\sigma}} \nabla_{h}
\end{pmatrix} \hspace{1cm} (5) \)
3-D frequency-domain CSEM inversion

It is possible to use one multigrid cycle or an Incomplete LU factorization (ILU) of (5) to obtain an effective preconditioner. In order to treat the forward problem in a generic way we rewrite the problem as

$$A(m)u - q = 0.$$  \hspace{1cm} (6)

Given the model \(m\) we can solve for the fields \(u\) by “simply” inverting the forward operator

$$u = A(m)^{-1}q$$

Note however that one never calculates \(A^{-1}\) but rather computes \(A^{-1}q\) using an iterative solver.

The inverse problem

In the inverse problem we try to fit the data and our a-priori information given the forward problem as a constraint. This can be written as

$$\begin{align*}
\min & \quad \frac{1}{2} || W_d (Qu - b) ||^2 + \frac{1}{2} \beta || W (m - m_{\text{ref}}) ||^2 \\
\text{s.t} & \quad A(m)u - q = 0
\end{align*}$$

where \(Q\) is a projection matrix, which projects the potentials \(A, \phi\) to a field component; \((W_d^T W_d)^{-1}\) is the noise covariance matrix and \(W\) is a model weighting matrix which is a discretization of the operator

$$W = (\alpha_s w, \quad \alpha_s \partial_s (w \cdot), \quad \alpha_d \partial_d (w \cdot), \quad \alpha_e \partial_e (w \cdot))^T$$

where \(w\) is a spatial weighting, and the \(\alpha\)'s are constants that control the relative weights in the regularization functional.

In this work we consider the unconstrained formulation to the inverse problem and thus, we use (6) and generate an unconstrained optimization problem of the form

$$\min \left\{ \frac{1}{2} || W_d (QA(m)^{-1}q - b) ||^2 + \frac{1}{2} \beta || W (m - m_{\text{ref}}) ||^2 \right\}. \hspace{1cm} (8)$$

Differentiating (8) with respect to \(m\) we obtain the nonlinear gradient system

$$g(m) = -G(m,u)^T A(m)^{-T} Q^T W_d^T (QA(m)^{-1}q - b) + \beta W^T W (m - m_{\text{ref}}) = 0,$$

where we introduce the sparse matrix \(G(m,u) = \partial A(m)u / \partial m\). Note that in order to evaluate the gradient one needs to solve the forward and the adjoint problem. Also, comparing (9) with the usual unconstrained formulation we see that we can express the sensitivity matrix as

$$J = -QA(m)^{-1}G(m,u).$$

This is a key observation in the solution of the inverse problem as we need not calculate the sensitivity matrix but only evaluate the sensitivity times vector for a Gauss-Newton iteration (Haber et al., 2000b; Haber and Ascher, 2001b)

The Gauss-Newton iteration for the system can be written as

$$(J^T J + \beta W^T W)dm = -g(m)$$  \hspace{1cm} (11)

In order to solve the system we use a preconditioned conjugate gradient method and thus only products of the form \(Jv\) and \(J^T w\) need to be calculated. Using equation (10) we can do that without calculating \(J\) explicitly. One iteration of the conjugate gradient solution therefore requires solving a forward and an adjoint problem. As a preconditioner for our system we use \(W^T W\) which works reasonably well as long as \(\beta\) is large enough. In order to further save computational time, we use an inexact Gauss-Newton formulation (Kelley, 1999) and solve (11) to a rough tolerance (typically \(10^{-1} - 10^{-2}\)), which usually requires only few conjugate gradient iterations. After we solve the system we update our model by

$$m_{k+1} = m_k + \alpha \delta m$$

The parameter \(\alpha\) is initially set to 1 and we use a quadratic line search if the objective function (8) is not decreased at each iteration. We use the discrepancy principle in order to choose the regularization parameter, that is, we choose \(\beta\) such that

$$||W_d (QA(m)^{-1}q - b)|| \leq \text{tol}$$

where tol is determined by the \(\chi^2\) misfit.

Selection of regularization parameter

We use a continuation strategy (Haber et al., 2000b; Aruliah and Ascher, 2002) where we cool the regularization parameter slowly to achieve the target misfit. For this method, we need to choose a first guess which is large enough. Here we use the estimate

$$\beta_0 = 100 || Ju||^2 / ||Wv||^2$$

where \(v\) is a random vector. This selection guarantees that at the initial step the model objective function dominates the optimization problem. We then reduce the regularization parameter at each iteration by half.

Practical considerations: A total field formulation

Our formulation works with total field and this can lead to very large grids when the transmitter sources are far away from the survey area. Just like in the CSAMT method, there is no reason to include the transmitters in the model we want to reconstruct. Conventionally, this can be done by solving for the secondary potentials with an equivalent
3-D frequency-domain CSEM inversion

source on the right-hand side. By doing so, the transmitters can be excluded from the discretized model, and the problem reduced to tractable size. However, when formulating the inverse problem in this way, we have to modify the algorithm to include the effects from the primary field, eventually changing the sensitivity matrix and the whole architecture as outlined in the previous section.

Here, we provide a simple total field formulation to deal with this situation, by replacing the source which is outside of the region of interest, with equivalent sources on the boundary of a smaller domain \( \Omega_s \subset \Omega \). In order to do that, we use the primary field \( \mathbf{J}_0 \) and \( \mathbf{H}_0 \) and calculate

\[
\nabla_h \times \mathbf{H}_0 - \mathbf{J}_0 = \delta_r
\]

on \( \Omega_s \). The sources \( \delta_r \) are then used as equivalent sources in solving an inverse problem on the domain \( \Omega_s \). If the primary fields obey Maxwell’s equations, then, physically, the equivalent source is only distributed on \( \partial \Omega_s \), the boundary of \( \Omega_s \). If the primary field is a good approximation to the total field on \( \partial \Omega_s \), this equivalent formulation works fine. In other words, the secondary field must be much smaller than the primary field on the boundary for this to work well. In the next numerical example, we used this approach.

Numerical Example

As a first step to invert a CSAMT data set collected in Penasquito, Mexico, we used the real survey geometry, but a synthetic conductivity model. Fig. 1 shows the survey geometry and the 3-D model which consists of two conductive and one resistive block buried in a uniform halfspace. There are 11 lines with a line spacing of 100 m, running East-West. On each line there are 28 stations at 50 m intervals. Real and imaginary components of \( E_x, E_y, H_x, H_y \), and \( H_z \) for the transmitter \( Tx_5 \), at 3 frequencies (16 Hz, 64 Hz, and 512 Hz) were used for a test. EH3D(Haber et al., 2000a) was used to generate the data, and we then added Gaussian noise that corresponded to 2% for the amplitude and 2 degrees for the phase data. Fig. 2 shows the data without, and with noise, for \( E_x \) and \( H_y \) components at 512 Hz.

The 3-D model for the inversion was \( 3350 \text{m} \times 3000 \text{m} \times 2000 \text{m} \), and was discretized into \( 64 \times 50 \times 30 \) cells. Because the Tx was excluded from this model, we used a 1-D code (Routh and Oldenburg, 2000) to compute the primary fields for generating the sources in the equivalent-source total field formulation. After 6 iterations in the inversion, the final model was 8394, thereby reaching our target misfit 9240. Three slices of the recovered model are shown in Fig. 3. The resistive and two conductive targets are well recovered. For comparison, the predicted data at 512 Hz are also displayed in Fig. 2.

Discussion

We have successfully developed an unconstrained algorithm to invert frequency domain CSEM data. The forward modelling is directly based upon solving a discrete, partial differential operator system in terms of potentials. The inverse problem is formulated as an unconstrained optimization with an inexact Gauss-Newton update at
3-D frequency-domain CSEM inversion

Each iteration. The iteration system is solved using a preconditioned conjugate gradient method. Calculations of the products of sensitivity matrix and a known vector only require solving a forward and an adjoint problem in the CG solution.

The preliminary application of this algorithm to a synthetic CSAMT data set has been promising. The two conductive and one resistive targets are well recovered. With the novel use of an equivalent source total field formulation, we can deal with the situation when the transmitter is far away from the survey area the same way as if the transmitter were included in the discretized model.

Acknowledgement

This work was supported by an NSERC IOR grant and an industry consortium Inversion and Modelling of Applied Geophysical Electromagnetic data (IMAGE) project. Participating companies are Newmont Gold Company, Falconbridge, Placer Dome, Anglo American, INCO Exploration & Technical Services, MIM, Cominco Exploration, AGIP, Muskox Minerals, Billiton, Kennecott Exploration Company.

References


