Transient electromagnetic inversion for multiple targets
Lin-Ping Song\textsuperscript{a}, Douglas W. Oldenburg\textsuperscript{a}, Leonard R. Pasion\textsuperscript{a,b} and Stephen D. Billings\textsuperscript{b}

\textsuperscript{a}Department of Earth and Ocean Sciences, University of British Columbia
6339 Stores Road, Vancouver, BC, Canada, V6T 1Z4
\textsuperscript{b}Sky Research Inc, Suite 112A, 2386 Est Mall, Vancouver, BC, Canada, V6T 1Z3

ABSTRACT
In UXO contaminated sites, there are often cases in which two or more targets are likely close together and the electromagnetic induction sensors record overlapping signals contributed from each individual target. It is important to develop inversion techniques that have the ability to recover parameters for each object so that effective discrimination can be performed. The multi-object inversion problem is numerically challenging because of the increased number of parameters to be found and because of the additional nonlinearity and non-uniqueness. An inversion algorithm is easily trapped in a local minimum of the objective function that is being minimized. To tackle these problems we exploit the fact that, based on an equivalent magnetic dipole model, the measured electromagnetic induction signals are nonlinear functions of locations and orientations of equivalent dipoles and linear functions of their polarizations. Based on these conditions, we separate model parameters into nonlinear parts (source locations and orientations) and linear parts (source polarizations) and proceed sequentially. We propose a selected multi-start nonlinear procedure to first localize multiple sources and then get the estimated polarization tensor matrix for each item through a subsequent or a nested linear inverse problem. It follows that the orientations of the objects are estimated from the computed tensor matrix. The resultant parameter set is input to a complete nonlinear inversion where all of the dipole parameters are estimated. The overall process can be automated and thus efficiently carried out both in terms of human interaction and numerical computation time. We validate the technique using synthetic and field data.

Keywords: Transient electromagnetic induction, Unexploded ordnance, Multiple Objects, Nonlinear inversion

1. INTRODUCTION
Buried unexploded ordnance (UXO) has been remaining a serious environmental problem caused by the previous wars, conflicts, and military training in a number of the places of the world. Its cleanup is an urgent task but the cost can be prohibitively high due to unnecessary excavation of harmless materials that can be left safely in the ground. To achieve the cost-effective remediation of UXO contaminated sites, a number of techniques or tools have been developed to detect, characterize, and distinguish between UXO and clutter. Among them, electromagnetic induction (EMI) sensing has proven to be one of the most promising methods for discriminating UXO from metallic debris and is increasingly being studied and applied in the UXO community.\textsuperscript{1–10}

This paper is concerned with the inversion of multiple targets using transient electromagnetic (TEM) data. The problem has arisen from the real scenarios in which two or more targets are closely spaced and the EMI sensors can record overlapping signals contributed from each individual target. Current EMI processing methodologies, which are based on the assumption that measured EMI responses are due only to a single object, are unsuitable for the overlapping case and yield inaccurate or misleading information in a subsequent process of UXO discrimination or classification. Therefore, it is important to develop inversion techniques that have the ability to recover parameters for each object so that effective discrimination can be performed.
There are a few studies on the processing of overlapping EMI signals. Hu and Collins\textsuperscript{11} adopted the well-known blind source separation (e.g., independent component analysis, ICA\textsuperscript{12}) technique and attempted to separate unknown dipolar signatures of each object from the mixed EMI signals and identify the objects by comparing the extracted source signatures with those in a library. This technique requires that the dipolar sources are statistically independent and follow a non-Gaussian distribution, but that noise is Gaussian. The researchers in Ref. 13 attempted to invert overlapping signals for locations and orientation parameters of equivalent dipoles and their polarizations by solving a nonlinear optimization problem. Their numerical study demonstrated some feasibility when inverting multi-objects from overlapping single time channel EMI data. Nevertheless, the study and practical use of solutions for this class of EMI inverse problem is still at an early stage and more efforts are needed to develop techniques and applications for general practice.

The inversion of a data set that includes multiple objects is conceptually straightforward, but has numerical challenges. This problem has more parameters to be solved for and thus the nonlinearity and non-uniqueness of the inverse problem has greatly increased. Also, the choice of a good initial point becomes more crucial for the global convergence of this high-dimensional nonlinear problem. To address these difficult issues in our development, we categorize the model parameters into nonlinear parts (source locations and orientations) and linear parts (equivalent dipole polarizations) and then solve them separately and/or sequentially. To a certain extent this approach can avoid the tradeoff existing between the different classes of the parameters if solved simultaneously and hence help mitigate the solution ambiguity. In the separation procedure, we propose to implement a multi-start nonlinear search about the optimal source locations over an active region of interest. This region of interest can be defined on the basis of the spatial distribution of the EMI responses. Details of the algorithm are described in section 3.

The technique is evaluated using synthetic data and is then applied to processing field data. All of these results presented in section 4 enhance our understanding of the overlapping signal problem and show that, for data of sufficient quality, our multi-target inversion methodology is efficient in terms of computational accuracy and speed, and is useful in supporting practical clearance operations.

2. THE PHYSICAL MODEL AND THE PROBLEM FORMULATION

Consider a standard EMI system consisting of a transmitting coil and a receiving coil which may be co-located or not. In active sensing, a primary field emitted from a transmitter illuminates a nearby object and its abrupt changes with time induces eddy currents in the metal object. These induced currents produce a transient secondary magnetic field that is measured by a receiver. In practice, an array of sensors is generally positioned above the surface to interrogate an object. For a UXO survey where a target dimension is often small relative to the target-sensor distance, we can adequately describe low frequency EMI responses to a metal target by an equivalent induced dipole,\textsuperscript{2,3,5,8} which is characterized by a 3 × 3 magnetic polarizability tensor (MPT)

\[
P(t) = \begin{bmatrix}
    p_{11}(t) & p_{12}(t) & p_{13}(t) \\
p_{21}(t) & p_{22}(t) & p_{23}(t) \\
p_{31}(t) & p_{32}(t) & p_{33}(t)
\end{bmatrix},
\]  

(1)

where the elements of the tensor \(p_{ij}(t)\) represents a dipole component in the \(i\)th Cartesian direction due to a primary field in \(j\)th Cartesian direction and \(p_{ij} = p_{ji}\) when \(i \neq j\). The equivalent dipole polarizability tensor \(P(t)\) can have an eigen-decomposition,

\[
P(t) = E \begin{bmatrix}
    L_1(t) & 0 & 0 \\
    0 & L_2(t) & 0 \\
    0 & 0 & L_3(t)
\end{bmatrix} E^T,
\]  

(2)

where \(E = [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]\) is an Euler rotational matrix where \(\mathbf{e}_i, (i = 1, 2, 3)\) is the orthonormal eigenvector representing the \(i\)th principal direction of dipolar polarization with respect to a reference system, and \(L_i(t)\) is the associated eigenvalue, i.e., the principal polarization strength that is a function of the geometry and material of a target.

For the \(i\)th measurement \((i = 1, \cdots, M)\), the secondary response \(d_i\) at time \(t\), due to an equivalent dipole source \(P(t)\) located at \(\mathbf{r}\), is written as\textsuperscript{1,2,5,8}

\[
d_i(\mathbf{r}_{Rx,i}, t) = \mathbf{B}_R^T(\mathbf{r}, \mathbf{r}_{Rx,i})P(t)\mathbf{B}_T(\mathbf{r}, \mathbf{r}_{Tx,i}),
\]  

(3)
where \( \mathbf{B}_R(\mathbf{r}, \mathbf{r}_{Rx}) = [B^R_x \ B^R_y \ B^R_z]^T \) and \( \mathbf{B}_T(\mathbf{r}, \mathbf{r}_{Tx}) = [B^T_x \ B^T_y \ B^T_z]^T \) are the magnetic induction fields at the object location generated by the receiver and transmitter coils with their centers at \( \mathbf{r}_{Rx} \) and \( \mathbf{r}_{Tx} \), and the superscripts \( x, y, \) and \( z \) denote the Cartesian components of a field and the superscript \( T \) denotes a transpose. The magnetic fields are computed via a line integral along each of coils.\(^1\)\(^,\)\(^10\) Eq. (3) describes the basic EMI process of illuminating, scattering, sensing.

For the inversion development, we can re-arrange (3) as

\[
d_i(\mathbf{r}_{Rx}, t) = \mathbf{a}_i^T(\mathbf{r}, \mathbf{r}_{Rx}, \mathbf{r}_{Tx}, \mathbf{r}) \mathbf{q}(t),
\]

where \( \mathbf{a}_i(\mathbf{r}, \mathbf{r}_{Rx}, \mathbf{r}_{Tx}) \) is a \( 6 \times 1 \) column vector representing spatial sensitivities of the \( i \)th sensor to the object located at \( \mathbf{r} \), and \( \mathbf{q}(t) \) a \( 6 \times 1 \) column vector whose components are the elements of the polarizability tensor \( P(t) \) of an object. They are given by

\[
\begin{align*}
\mathbf{a}_i^T(\mathbf{r}, \mathbf{r}_{Rx}, \mathbf{r}_{Tx}) &= \begin{bmatrix}
B^R_x B^T_x + B^R_y B^T_y + B^R_z B^T_z & 
B^R_x B^T_y - B^R_y B^T_x & 
B^R_x B^T_z + B^R_y B^T_y - B^R_z B^T_x \\
B^R_x B^T_x - B^R_y B^T_y + B^R_z B^T_z & 
B^R_x B^T_y + B^R_y B^T_x & 
B^R_x B^T_z - B^R_y B^T_y + B^R_z B^T_x \\
B^R_x B^T_x + B^R_y B^T_y - B^R_z B^T_z & 
B^R_x B^T_y - B^R_y B^T_x & 
B^R_x B^T_z + B^R_y B^T_y + B^R_z B^T_x
\end{bmatrix} \\
\mathbf{q}(t) &= \begin{bmatrix}
p_{11}(t) & p_{12}(t) & p_{13}(t) & p_{22}(t) & p_{23}(t) & p_{33}(t)
\end{bmatrix}
\end{align*}
\]

On the other hand, rewriting the decomposition (2) of the magnetic polarizability tensor as

\[
P(t) = \sum_{i=1}^{3} L_i(t) \mathbf{e}_i \mathbf{e}_i^T,
\]

and substituting (6) into (3), we obtain another scalar-product form of the secondary response

\[
d_i(\mathbf{r}_{Rx}, t) = \mathbf{g}_i^T(\mathbf{r}, \xi, \mathbf{r}_{Rx}, \mathbf{r}_{Tx}) \mathbf{m}(t),
\]

where \( \xi \) denotes one set of Euler angles \((\phi, \theta, \psi)\) that describes the orientation of an dipolar object and hence determines the Euler vectors \( \mathbf{e}_i \), and

\[
\begin{align*}
\mathbf{g}_i^T(\mathbf{r}, \xi, \mathbf{r}_{Rx}, \mathbf{r}_{Tx}) &= \begin{bmatrix}
B^R_x \mathbf{e}_1 \mathbf{e}_1^T B_T & B^R_y \mathbf{e}_2 \mathbf{e}_2^T B_T & B^R_z \mathbf{e}_3 \mathbf{e}_3^T B_T
\end{bmatrix} \\
\mathbf{m}(t) &= \begin{bmatrix}
L_1(t) & L_2(t) & L_3(t)
\end{bmatrix}
\end{align*}
\]

In the development, we assume that dipolar location and orientation is independent of time.

Consider \( \eta \) multiple objects within the field of sensor view at locations of \( \mathbf{r}_1, \cdots, \mathbf{r}_\eta \) with orientations \( \xi_1, \cdots, \xi_\eta \) and characterized with polarizations \( \mathbf{q}_1(t), \cdots, \mathbf{q}_\eta(t) \) or \( \mathbf{m}_1(t), \cdots, \mathbf{m}_\eta(t) \), each of which is defined in (5) and (8). Assume that the EMI interactions between the objects are negligible. By linear superposition of individual contributions, we have

\[
d_i(\mathbf{r}_{Rx}, t) = \sum_{k=1}^{\eta} \mathbf{a}_i^T(\mathbf{r}_k, \mathbf{r}_{Rx}, \mathbf{r}_{Rx}) \mathbf{q}_k(t) = \sum_{k=1}^{\eta} \mathbf{g}_i^T(\mathbf{r}_k, \xi_k, \mathbf{r}_{Rx}, \mathbf{r}_{Rx}) \mathbf{m}_k(t)
\]

where \( \mathbf{a}_i(\mathbf{r}_k, \mathbf{r}_{Rx}, \mathbf{r}_{Rx}) \) and \( \mathbf{g}_i(\mathbf{r}_k, \xi_k, \mathbf{r}_{Rx}, \mathbf{r}_{Rx}) \) defined in (5) and (8) are the spatial sensitivities of the \( i \)th sensor to the \( k \)th object located at \( \mathbf{r}_k \) with orientation \( \xi_k \).

To simplify the notations we shall now suppress the position vectors of the sensor coils. It is understood that the sensor information is a subscript-indexed in the recordings and the sensitivity vectors. The measurements made at \( M \) sensing locations, in the presence of noise, can be conveniently expressed in a vector-matrix notation

\[
\mathbf{d}(t) = \sum_{k=1}^{\eta} A(\mathbf{r}_k) \mathbf{q}_k(t) + \mathbf{n}(t),
\]

or

\[
\mathbf{d}(t) = \sum_{k=1}^{\eta} G(\mathbf{r}_k, \xi_k) \mathbf{m}_k(t) + \mathbf{n}(t),
\]

where \( \mathbf{d}(t) = [d_1(t), \cdots, d_M(t)]^T \) is \( M \times 1 \) measured data vector at time \( t \), \( \mathbf{n}(t) \) is the additive noise vector, and \( A(\mathbf{r}_k) \) is a \( M \times 6 \) matrix denoting the sensitivities of the \( M \) sensors to the \( k \)th object located at \( \mathbf{r}_k \). Its transpose is given by

\[
A^T(\mathbf{r}_k) = \begin{bmatrix}
\mathbf{a}_1(\mathbf{r}_k) & \cdots & \mathbf{a}_M(\mathbf{r}_k)
\end{bmatrix}.
\]
$G(r_k, \xi_k)$ is a $M \times 3$ matrix denoting the sensitivities of the $M$ sensors to the $k$th object located at $r_k$ and oriented at $\xi_k$. Its transpose is given by

$$G^T(r_k, \xi_k) = \left[ g_1(r_k, \xi_k) \ldots g_M(r_k, \xi_k) \right].$$

The observed EMI response, as illustrated in Eq. (9) and (10), is a linear combination of the finite dipole polarizations of multiple objects with spatial weighting coefficients as a function of the locations and/or orientations of objects for a fixed sensor array. This is a generic dipole-based expression for modeling and inversion of EMI anomalies. Both formulations will be interchangeably used in the following development.

3. INVERSION TECHNIQUE

In a transient electromagnetic (TEM) system, we usually have measurements of $D = [d(t_1), \ldots, d(t_N)]$ at a series of times, say, $t_1, \ldots, t_N$. Given space-time data $D$, and provided that the number $\eta$ of buried objects within the field view of sensors is known or assumed, the goal of our inverse problem is to determine the set of model parameters of multi-source locations and orientations and their principal transient polarizations, i.e., $[r_1, \xi_1, \ldots, r_\eta, \xi_\eta, m_1(t_j), \ldots, m_\eta(t_j)], j = 1, \ldots, N$, as described in the above, that best explain the spatial-temporal data.

The inversion of a data set that includes multiple objects is conceptually simple, however it is numerically non-trivial. It involves more parameters (multiple source locations, orientations, and polarizations) to be solved for and thus the nonlinearity and non-uniqueness of the inverse problem is greatly increased. As a result, there might be a number of different source configuration that can trade for each other for fitting the data equally well. Alternatively, a minimizing optimization algorithm in a high-dimensional model space becomes quite sensitive to the initial guess of a model and is easily trapped in a false solution that is a local minimum of a functional of noisy data and the unknowns of the problem.

To tackle these aforementioned numerical challenges and exploit the fact that the EMI response is linear with respect to dipolar polarization and nonlinear with respect to the locations and/or orientations of objects, we employ a solution strategy that decomposes the inverse problem into several steps, each of which seeks to resolve one major set of model parameters. The procedure is first to solve for non-linear location parameters and subsequently solve for linear polarization parameters. With an optimal estimate of locations and dipolar polarizations, the orientations of each object can be extracted from estimated dipolar polarizability tensors through an eigen-decomposition in Eq. (2) and then further optimized. For a time-domain systems that have sufficient time channels to characterize the decay behavior of the polarizability, we further seek the set of parameters in a parametric model of dipolar polarizations by either fixing the locations and orientations of multiple objects derived in the previous two-step procedure, or incorporating these nonlinear parameters into the inversion for an update. These steps are elaborated in the following.

3.1 Searching for Optimal Locations

In this inversion step, based on Eq. (9), model parameters are separated into into two sets: a nonlinear part consisting of multi-source locations $r = \text{vec}[r_1, \ldots, r_\eta]$ and a linear part consisting of source polarizations $f(t) = \text{vec}[q_1(t), \ldots, q_\eta(t)]$ at time $t$, where $\text{vec}[\cdot]$ represents a vectorization operation, i.e., stacking matrix columns into a single column. Note that notation $r$ is used to represent a vector of the multi-source location parameter in the following discussion. The primary concern here is to find source locations, while the polarization estimate can be followed for a specific set of the location parameters.

To start the inversions, we need an initial guess for the locations of the $\eta$ objects. This might be obtained by examining the EMI response distribution assuming that each source contributes to the field distribution with an associated peak. Then the centroids of these peaks can provide initial horizontal locations of objects. Given that set of horizontal locations, we attempt to assign a depth combinations of the objects in range (reasonably say 0 - 1.2 m) and select, by a forward search, initial depths that are associated with a minimum of an objective function. However, the centroid estimate is biased toward the peak position for a tilted object and the overlapping signals impose additional complications on this estimate. Thus this initial guess for a multi-object scenario cannot be guaranteed to be good. In many cases it is challenging to assign the starting point as there is often no
knowledgeable way to connect spatial anomalies to each individual unknown object because the field pattern is a complicated function of object locations and orientations as well as transmitter-receiver configuration specific for an instrument and sensor distribution. This is described in the results section.

To avoid the difficulty and pitfalls arising from selecting one initial point based on field distribution, we propose a multi-start algorithm as follows. Define a region of interest (ROI) that is sufficiently large to cover the active objects. Within the ROI, a number of points are uniformly or randomly created. Each point consists of the locations for η objects and is a potential solution to the object function

\[ \Phi_d(r) = \sum_{j=1}^{N} \left\| W_j(d_{obs}(t_j) - \tilde{d}(t_j)) \right\|^2, \]  

(11)

where \( W_j \) is a data weighting matrix at time \( t_j \) and is generally chosen as the inverse of estimated standard error of the data, \( d(t_j) \) are the predicted data, and \( \tilde{q}_k(t_j) \) are computed according to (13) given below. Tilde \( \sim \) means an assumed/estimated quantity. After a forward evaluation using (11), these points are sorted, e.g., in ascending order according to the values of \( \Phi_d(r) \). A small population of points with smaller function values \( \Phi_d(r) \) is selected as multi-start points. Then we conduct a nonlinear search to determine optimal source locations by updating those selected starting points using a well-developed minimizer like the Levenberg-Marquardt approach or trust region interior point method.\(^{15,16}\) That is, given a starting or current location point \( r_c \), the location is optimally updated as

\[ \tilde{r} = \arg \min_r \sum_{j=1}^{N} \left\| W_j(d_{obs}(t_j) - \tilde{d}(r_c, t_j) - J(r_c)(r - r_c)) \right\|^2, \]  

(12)

where \( J(r_c) \) is a Jacobian matrix with respect to the location in a linearized step and evaluated at the current location value, \( d(r_c, t_j) \) the predicted data at \( r_c \), and \( \Delta_r \) is a positive scalar representing the trust region size for the variable parameter \( r \) (for details about the algorithms see Refs. 15,16). At this step, \( q_k(t_j) \) are updated accordingly with locations.

Now we turn to the issue of estimating dipolar polarizations for a given set of locations \( \tilde{r} \) in the above processes in (11) and (12). This intermediate estimate of polarizations may be maintained or removed completely using a variable projection (VP) approach.\(^{17}\) The VP approach, through a least-squares projection operator, might be used to transform the objective function in (11) or (12) into a function that is only dependent on source locations \( r \) and the corresponding data and on longer contains polarization quantities anymore. However, minimization of the transformed, location-dependent function involves a derivative computation of the pseudo inverse that could have a numerical instability problem. We chose the way to compute the polarizations explicitly so that not only the potential complications in the VP approach are avoided but also any available constraints on the polarizations can be easily incorporated into the source localization process. For estimating the polarizations, we solve the constrained linear least-squares problem as follows

\[ \tilde{f}(t_j) = \arg \min_{f(t_j)} \left\| d_{obs}(t_j) - \sum_{k=1}^{\eta} A(\tilde{r}_k)q_k(t_j) \right\|^2 \]  

s.t.

\[ p_{k,ii}(t) \geq 0 \]

\[ |p_{k,ij}(t)| \leq \frac{1}{2} \left[ p_{k,ii}(t) + p_{k,jj}(t) \right]. \]  

(13)
The constraints in (13) based on the EMI physics, i.e., the principal polarizations $L_i(t)$ must be positive in Eq. (2). This means that the magnetic polarizability tensor $P(t)$ should be physically sought as a (semi)-definite positive one. However, these constraints are necessary but not sufficient. To use a complete positive definite constraint on the polarization tensor, we may consider employing (semi)-definite programming (SDP) techniques later.

In this step, it is important to determine a proper number of sampling points. Its choice is a compromise among computational speed, convergence, and available computing power. Of course, the greater the number of points are sampled in a ROI, the higher the probability that the starting points are close to the global solution we desire. In that context however, a multi-start algorithm has high computing costs and could be very time-consuming for a very large number of sampling points. The selected multi-start nonlinear optimization we proposed has the advantage of reducing the computational cost of the algorithm and while maintaining its interesting capability of global optimization. In our study, it appears reasonable that the number of sampling points is set up as 300 and the number of selected starting points for nonlinear updates is 10. Our computational experience shows that the multi-star algorithm outperforms the one with an initial point mentioned above. This multi-start algorithm requires little user involvement and can be parallelized.

### 3.2 Determining Optimal Orientations

Given estimated polarization tensors, we can find the orientation angles $\xi = (\phi, \theta, \psi)$ using the decomposition in Eq. (6). The components of the Euler vectors $e_j$ can be expanded as

$$e_{11} = \cos \phi \cos \theta \cos \psi - \sin \phi \sin \psi, \quad e_{12} = \cos \phi \cos \theta \sin \psi + \sin \phi \cos \psi, \quad e_{13} = -\cos \phi \sin \theta,$$

$$e_{21} = -\sin \phi \cos \theta \cos \psi - \cos \phi \sin \psi, \quad e_{22} = -\sin \phi \cos \theta \sin \psi + \cos \phi \cos \psi, \quad e_{23} = \sin \phi \sin \theta.$$  \hspace{1cm} (14)

Thus the angles $\xi = (\phi, \theta, \psi)$ might be obtained as

$$\theta = \arccos(e_{33}), \quad \phi = \arccos\left(-\frac{e_{13}}{\sin \theta}\right), \quad \psi = \arccos\left(\frac{e_{31}}{\sin \theta}\right).$$  \hspace{1cm} (15)

With multi-time channel data, we could have a series of estimated $\tilde{\xi}_j$ for each object, which may not be close to each other at all due to noise and model approximation. For the dipole model assumed here (i.e., its principal polarization directions are not varying with time), they might be simply approximated with the orientations obtained at an early or intermediate time channel by consideration of the signal-to-noise ratio.

However from the first-order perturbation analysis, we know that the perturbation in the estimated magnetic polarizability tensor can be erroneously propagated into large changes in the determination of the principal directions due to potentially small differences between the principal polarizations. Considering this numerical instability problem, measurement error, and possible tradeoff among the magnetic polarizability tensors of multiple objects, we suggest to use the orientations approximately found using a single time channel as starting one and do a nonlinear update to determine optimal orientations of multi-objects by fixing their locations. The nonlinear optimization algorithm is similar to the one in the location search step but formulation (10) is used. Defining $\xi = \text{vec} [\xi_1, \ldots, \xi_\eta]$ and from a set of starting or current orientations $\xi_c$ of multiple objects, we obtain their optimal estimate as

$$\tilde{\xi} = \arg \min_{\xi} \sum_{j=1}^{N} \left\| W_j (d_{\text{obs}}(t_j) - \tilde{d}(\xi_c, t_j) - J(\xi_c)(\xi - \xi_c)) \right\|^2$$

$$d(\xi_c, t_j) = \sum_{k=1}^{\eta} G(\mathbf{r}_k, \tilde{\xi}_c, k) \mathbf{m}_k(t_j)$$

subject to $\|\xi - \xi_c\| < \Delta_\xi$ \hspace{1cm} (16)

where $J(\xi_c)$ is a Jacobian matrix with respect to the orientations in a linearized step and evaluated at the current value, $\Delta_\xi$ is a positive scalar representing the trust region size, and $d(\xi_c, t_j)$ are the predicted data. Similar to dealing with $\mathbf{q}_k(t)$ in the location search step, $\mathbf{m}_k(t)$, as a nested linear inverse solution, is updated accordingly with the orientations during this nonlinear step.
3.3 Parametric Model Fitting

In this step, we parameterize the principal polarizations as $L_i(t) = k_i t^{-\beta_i} e^{-\gamma_i t}$ described in Refs. 5, 10 and wish to find the set of $(k_i, \beta_i, \gamma_i)$ parameters corresponding to ith polarization for each object. This is also a nonlinear inverse problem. To do that, the sets of $(k_i, \beta_i, \gamma_i)$ parameters can be first initialized by linearly fitting to the log-transformed discrete polarization $\ln L_i(t_j) = \ln k_i - \beta_i \ln t_j - \gamma_i j$, $j = 1, \cdots, N$, that are computed in the last step. These initial parameters are further updated nonlinearly by fitting the measured data using fixed locations and orientations derived from the last steps. Finally, we update these parametric model simultaneously with locations and orientations parameters. The parametric solution obtained using the fixed nonlinear spatial parameter $(r, \xi)$ might be better since in this problem there does not exist a tradeoff between the spatial and polarization parameters sets of $(k, \beta, \gamma)$, e.g., an ambiguity existing between the depth and $k$ parameters. As an option, the final parametric solution is picked with a smaller misfit value from above two nonlinear solutions.

4. RESULTS

We now demonstrate our inversion procedure using simulated and real Geonics-EM63 TEM data. This EMI system has 26 off-time measurement channels ranging from 0.18 ms to 25 ms and is mounted on a cart with the square transmitter (1 m x 1 m) and receiver coils (0.5 m x 0.5 m) positioned along a common vertical axis.

To evaluate convergence and accuracy of the technique, we do not assume that we are dealing with axisymmetric bodies in the following inversions. In a mathematical development, the number of targets can be assumed to be large. In practice, however treatment of overlapping signals due to two targets is a major problem. We focus on that here.

4.1 Simulated data

In the synthetic survey, the EM-63 sensors are assumed to be positioned over 5 m x 5 m area with a line spacing of 25 cm and station spacing of 10 cm. The sensor height is set up at 30 cm. The simulated data were generated from a series of two-object configurations where their locations and orientations were randomly varied. These are attempts to simulate various cases of practical interest that could arise from deep and shallow objects, strong and weak sources, two similar items, or smaller spatial separation for which two objects render single-peak-like anomaly in space. The Gaussian noise, obeying the rule of $0.6/\sqrt{t} \ast \text{Randn}$ that is often observed in the transient EMI signals, was added to the simulated data. Randn denotes a normal random number generator of $\mathcal{N}(0, 1)$.

In the first example, we present a two-object case where the observed spatial data, as shown in Fig. 1 (a) at $t_1 = 0.18$ ms, display a two-anomaly pattern. In this case, we can safely assume the two ROIs in the $x - y$ plane around each peak anomaly to search for the source locations. But to test the algorithm convergence ability, we proceed with the inversion using the circle superimposed in Fig. 1(a) as the ROI and the correspondent data inside it. The use of such a single ROI certainly can make an algorithm difficult since unknown objects have more degrees of freedom to move over a larger space.
Fig. 2 shows the process of the location searching step using the multi-start algorithm. Within the defined ROI in which the depth range is set between 0 and 1.2 m, 300 initial points were randomly generated and evaluated using the misfit function of (11). These points are sorted and plotted (crosses) as Φ_d against the number of sampling points in Fig. 2 (a) and Φ_d versus the depths z_1 and z_2 for the sampled points in Fig. 2 (b). A number of smaller Φ_d points are selected and are shown by the circles in Figs. 2 (a) and (b). These selected multi-start points are updated through the nonlinear optimization and expected to converge the true locations rapidly. Fig. 2 (c) is the convergence curve of the Φ_d as a function of the iteration number starting from one of the selected multi-start point, shown by a pair of circles in Figs. 2 (d). It roughly took the algorithm 10 iterations to reduce the value of Φ_d to 2810 from 57815 and brought that pair of points quickly to the true locations marked as diamonds in Figs. 2 (d).

Given the satisfactorily found source locations, subsequent inversions for orientation angles and a parametric solution are followed and finally all model parameters are determined. Table I lists the ground-truth and the inverted parameters, where subscripts p, s, t represents primary, secondary, and tertiary polarizations. To compare with the observed data in Fig. 1 (a), Fig. 1 (b) and (c) presents the predicted data and the residuals at t_1, where the true (small circles) and inverted (crosses) locations are superimposed. The random pattern in residuals is a sign of a successful inversion. The recovered polarizations (solid curves) in Fig. 1 (d) indicate that the two objects are axis-symmetric. The prediction agrees with the ground-truth. To show how accurate the recovered polarizations are, the true polarizations are plotted as the dashed curves in Fig. 1 (d).

Table 1. Numerical example I. The groundtruth and inverted model parameters.

<table>
<thead>
<tr>
<th></th>
<th>(x, y, z) (m)</th>
<th>(φ, θ, ψ) (deg)</th>
<th>(k_p, β_p, γ_p)</th>
<th>(k_s, β_s, γ_s)</th>
<th>(k_t, β_t, γ_t)</th>
</tr>
</thead>
<tbody>
<tr>
<td>True (Obj 1)</td>
<td>(0.59, -0.26, -0.25)</td>
<td>(239.38, 142.59, x)</td>
<td>(15.77, 0.86, 14.17)</td>
<td>(3.89, 1.17, 10.71)</td>
<td>(3.69, 1.21, 15.50)</td>
</tr>
<tr>
<td>Inverted</td>
<td>(0.59, -0.26, -0.26)</td>
<td>(58.64, 37.62, 3.13)</td>
<td>(16.24, 0.85, 14.08)</td>
<td>(4.04, 1.15, 9.22)</td>
<td>(3.69, 1.21, 15.50)</td>
</tr>
<tr>
<td>True (Obj 2)</td>
<td>(-0.22, 0.32, -0.35)</td>
<td>(60.09, 79.70, x)</td>
<td>(20.87, 0.54, 13.55)</td>
<td>(7.55, 1.13, 10.53)</td>
<td>(7.55, 1.13, 15.51)</td>
</tr>
<tr>
<td>Inverted</td>
<td>(-0.22, 0.32, -0.35)</td>
<td>(60.21, 79.61, 2.72)</td>
<td>(20.07, 0.55, 13.44)</td>
<td>(7.16, 1.14, 10.89)</td>
<td>(7.32, 1.13, 9.82)</td>
</tr>
</tbody>
</table>

Fig. 3 gives the other two-object example when the spatial data at t_1 exhibit a three-peak anomaly. Based on the spatial anomaly distribution, it is hard to guess initial locations of the objects visually and to define a ROI for each possible object. We use a single ROI to cover all suspected anomalies and carry out the inversion algorithm. The predicted data, superimposed with the true (small circles) and inverted (crosses) locations of the objects, are presented in Fig. 3 (b). The correspondent residual maps and the recovered polarizations are given in Figs. 3 (c) and (d). These plots show that the multi-object inversion proceeds very well and the source polarizations, even for a weaker item, are accurately extracted. We omitted presenting the details of the multi-start source search since it was illustrated in the last example.

For a fixed sensor array, the EMI spatial anomaly pattern depends on orientation and separation of objects. It is expected that, when the two objects get closer, the anomaly pattern could behave as it was from a single-object. As compared to the previous two examples in which the horizontal separation between the objects are at about 85 ∼ 99 cm, Fig. 4 shows the case when the separation is 60 cm and 30 cm. In both cases, a visual
examination of the observed data of Fig. 4 could most likely lead to a guess that the anomaly pattern is from a single tilted or vertically oriented object. Assuming that we are not certain if these data include one object or two, we conducted inversion experiments with a single-object model and a two-object model. In Figure 4, rows (a) and (b) give the results obtained from a two-object model. The predicted data (the 2nd column), residuals (the 3rd column), and recovered polarizations (the last column) are displayed. Row (c) shows the results for the data in (b) obtained from a single-object inversion. In these difficult cases, the multi-object inversion algorithm performs well and locates the objects correctly and recovers the the polarizations accurately, although the recovered tertiary polarization for that weaker object is different at late times from the ground-truth. On the other hand, the examination of the results from a single-object inversion shows that data cannot be interpreted well. The resultant residuals are large and correlated and thus indicate that remaining signal is not properly accounted for. Consequently, the single-object inversion fails to extract the correct object signature from the data.

Next, we will present the validation of the developed algorithm using field data.

### 4.2 Field data

Here we present the application of multi-object inversion technique to the EM-63 field data that were acquired at Sibert, Alabama. Initially, the data sets were processed using single-object inversions but some difficulties were encountered in fitting the data or comparing the recovered polarizations to the ground-truth. The data associated with these difficult anomalies were then processed with the multi-object inversion. This is a blind test for that we do not have any prior information about buried objects. In the below, two examples are presented.

Results for the first example is in Fig. 5. The observed data at $t_1$, predicted and residuals, and the recovered polarizations after a single-object inversion are shown in Fig. 5 (a). Row (b) gives the results after the two-object inversion. The predicted locations are denoted as the crosses on the two sets of the modelled data. The normalized misfit is 1.10 for the single object and 0.37 for the two-object inversion. The residual distribution illustrates that data are much better explained by the two-object inversion. With further inspection from the recovered polarizations, one infers that the two-object inversion predicts one large, axis-symmetric object and another non-axis symmetrical object. This non-axis symmetrical item was decided to be clutter since it did not seem to have a signature of a regularly shaped ordnance. The result from the single-object inversion fails to provide any useful clues about the object.

When the site for this data set was excavated, a small triangular piece of scrap was found. A photo of the scrap is superimposed on the polarization plot derived from the two-object inversion. The scrap was analyzed by Science Applications International Corporation (SAIC) with TEMTADS to determine the polarization tensor. The TEMTADS derived polarization was convolved to get the polarization for an EM-63 waveform. The ground truth polarizations are plotted as dashed curves in Fig. 5. The two-object inversion recovers the basic features (decay and magnitude) of the scrap polarizations, but their magnitude is slightly off. To identify the second item, a major one found in our two object inversion, we compared the polarizations derived from anomalies of known base plates with the extracted polarizations. Our two-object inversion predicts the presence of a base plate, a kind of circular plate with a normally attached rod, schematically shown on one of the polarization plots.

![Figure 3. Numerical examples II of the multiple object inversion. (a) Observed. (b) Predicted. (c) Residuals. (d) True and recovered polarizations. The notations are the same as those in Fig. 1.](image)
In this instance, scrap was dug up but the remaining object, which we believe is a base plate, was missed and left in the ground.

Fig. 6 gives another example. The single-and two-object inversions have the misfit values of 3.21 and 0.42, respectively. In the inverted results, the two-object inversion predicts an almost identical axis-symmetric object as the recovered in the previous field example. According to the recovered polarizations from the two-object inversion, there is an additional strong object, with an even better fit to the in-air derived polarization, in particular at early times. Again however, the site personnel dug up scrap and left likely a base plate in the ground.

5. CONCLUSIONS

We have considered the problem of inverting multiple objects using TEM data. The problem is of practical interest in UXO contaminated sites where items are often closely spaced and sensed simultaneously within the field view of sensors.

To circumvent the numerical challenges such as increases of non-uniqueness and nonlinearity, becoming trapped into undesirable local minima, being more sensitive to an initial guess because of the increasing number of model parameters to be solved, we have adopted a solution strategy that decomposes a high-dimensional model space into lower-dimensional model spaces. These smaller problems are solved separately and sequentially. Briefly, this is done by

- Seeking nonlinear location parameters and linear polarization tensors, separately.
Optimizing the orientations of the objects.

Finding polarizations of multiple objects in a parametric form for a sufficiently large time window.

By fixing one set of model parameters and optimizing another set, the separated and sequential procedure can have the capability of mitigating the trade-off between the different classes of model parameters. Further more, we have proposed a selected multi-start nonlinear algorithm for source localizations that paves an efficient way to find good initial guess of model parameters, provides a greater robustness for handling the potential local minima, and makes the inversion effectively automated.

We have extensively evaluated the solution technique by using simulated and field data. The results show that...
our technique is robust and importantly, is mostly automated so that little user intervention is required. Blind tests of the technique, using the field data, demonstrated its capability to accurately extract target signatures and show our methodology can be useful in practical clearance operation.

Acknowledgment

The authors are thankful to David Sinex for his help to make the technique run in UXOLab and Joy Rogalla for her some text corrections of the draft.

REFERENCES