Mathematical Models and Methods in Applied Sciences

Coulomb friction and other sliding laws in a higher order glacier flow model

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We consider a widely used higher order glacier flow model with a variety of parameterizations of wall slip, including Coulomb friction, regularized Coulomb friction laws and a power law. Mathematically, the Coulomb friction problem is found to be analogous to a classical friction problem in elasticity theory. We specifically analyze the case in which slip is possible everywhere at the boundary, in which case the weak formulation becomes a semi-coercive convex minimization problem which has a solution only if a solvability condition representing force and torque balance is satisfied. Going beyond previous work, we study the uniqueness of solutions in depth, finding that non-unique solutions are possible under very specialized circumstances. Further, in an extension of work by Campos, Oden and Kikuchi, we show that solutions to the regularized Coulomb friction and power law problems converge to the Coulomb friction problem in appropriate parametric limits, provided the latter is unique, and briefly discuss the implications of possible non-unique solutions for a priori error estimation in numerical approximations.

Keywords: Wall slip; convex minimization; elasticity; glacier flow.

AMS Subject Classification: 5J85,76D07

1. Introduction

Wall slip is an important phenomenon in many glacial flows as well as in many other situations in mechanics. In glaciology, sliding accounts for many interesting dynamical features such as glacier surges (oscillatory behaviour due to periodic activation of slip at the boundary, see e.g. Ref. 13) and ice stream flow (channelized fast ice flow, see e.g. Ref. 31). More recently, observations in Greenland have indicated that percolation of meltwater to the ice sheet bed can lead to reduced friction at the bed and hence to ice sheets responding much more rapidly to changed climatic conditions than would otherwise be possible,[33] making the study of wall slip a topic of pressing concern in glaciology. In this paper, we present an analysis of a model for glacier flow with a number of different friction laws describing different modes of wall slip: a Coulomb friction law, a regularized Coulomb friction law and a power law, all of which can be motivated theoretically and empirically.

Glaciers flow viscously, but the results obtained here can also be applied to analogous friction problems in elasticity theory. Generically, the problem we obtain from the glaciological application and study in this paper is the minimization of functionals of the form

$$J(\boldsymbol{v}) = J_0(\boldsymbol{v}) - l(\boldsymbol{v}) + j(\boldsymbol{v}), \tag{1.1}$$

defined on a suitable Sobolev space X, where J_0 is a lower semi-continuous, semi-coercive convex functional, l is a continuous linear functional and j is a lower semi-continuous convex functional defined in terms of an integral over the boundary of the domain. Physically, J_0 represents energy dissipation in the flow (or energy stored due to deformation in elastic analogue problems), while l represents external forces and j represents wall friction.

In our application, J_0 is not coercive because rigid body motions do not induce stresses or viscous dissipation, and hence $J_0(\boldsymbol{v} + \boldsymbol{r}) = J_0(\boldsymbol{v})$ for any rigid body motion $\boldsymbol{r}$. Rigid body motions represent the only subspace of X over which dissipation J_0 is invariant, and it is possible to show using Korn's inequality that J_0 is semi-coercive. By this, we mean that there exists a $C > 0$ and finite dimensional subspace $X_0 \subset X$ with an associated projection operator $\pi : X \mapsto X_0$ such that $J_0(\boldsymbol{v}) \geq C \|\boldsymbol{v} - \pi(\boldsymbol{v})\|$ for all $\boldsymbol{v} \in X$; in our case, we can identify X_0 with the set of rigid body motions..

Because J_0 is invariant under addition of rigid body motions, it also follows that J_0 is not strictly convex. The fact that J_0 is neither coercive nor strictly convex then implies that the same may be true of J, and this leads to a number of complications, such as the possibility of non-existence and non-uniqueness of minimizers of J. These have already been identified elsewhere for elastic problems with frictional slip by Duvaut and Lions,[8] Kikuchi and Oden[21] and Panagiotopoulos.[25] On an abstract level, the present paper aims to extend their existence and uniqueness results, and also to study the relationship between different friction laws. For instance, a Coulomb friction assumes shear stress $\boldsymbol{\sigma}_t$ at the boundary to be parallel to the direction of motion $\boldsymbol{u}$ but independent of its magnitude, so $\boldsymbol{\sigma}_t \propto \boldsymbol{u}/|\boldsymbol{u}|$. A power law puts $\boldsymbol{\sigma}_t \propto |\boldsymbol{u}|^{\epsilon-1}\boldsymbol{u}$ for some constant ϵ. This mimics Coulomb friction as $\epsilon \to 0$, but does it follow that the solution of a power-law problem converges to the solution of the corresponding Coulomb friction problem?

On a practical level, we choose a glacier flow model to illustrate our approach to these questions because glacier flow provides a natural and scientifically relevant setting in which the different slip scenarios that we consider can all occur in practice. A word on the model is therefore in order before we start. The model, due to Blatter[3], is an approximation of Stokes' equations in which the component of normal stress in the direction of the vertical (z-) axis σ_{zz} is treated as hydrostatic. Using this as a closure relation for pressure, a simplified model is obtained that solves only for the component of velocity in the xy-plane, and the associated stress components σ_{ij}, where $i = x$ or y. The justification for Blatter's approximation can be found in the shallow geometry of most glaciers; we therefore have a thin-film model, but by contrast with most other thin-film models, the present one is not depth-integrated. There is a strong rationale for this, as Blatter's model can describe both slow and fast sliding, while depth-integrated thin film models (such as the standard lubrication approximation[14] or membrane-type models[23]) can usually only describe one or the other. Our aim in this paper is, however, not to dwell on the construction of Blatter's model (see a separate paper with R. Hindmarsh, submitted to *Quart. J. Mech. Appl. Math.*), but to present a mathematical analysis of the model. Previous analyses of Blatter's model, which is becoming increasingly widely adopted in numerical glacier flow simulations, are due Colinge and Rappaz,[7] Glowinski and Rappaz, [16] Chow *et al.*,[6] Rappaz and Reist[28] and Reist[29]. The main feature that sets ours apart is a more sophisticated treatment of slip at the boundary. A related analysis for a depth-integrated thin film model may be found in Ref. 31.

There are two important consequences of the construction of Blatter's model. First, we have a two-dimensional mechanical problem posed on a three-dimensional domain, in the sense that there are only two velocity components that we solve for, and similarly we only concern ourselves with the associated six stress components σ_{ij}, where $i = x$ or y while $j = x$, y or z. This leads to a somewhat odd-looking formulation, but nonetheless we are able to cast it in a succinct form that allows the application of standard analytical tools in mechanics, such as Korn's inequality. Secondly, the realization that stress in the vertical can be approximated as hydrostatic also implies that normal stress at the base of the glacier can be approximated as hydrostatic. In practice, this makes it unnecessary to include normal stress effects in our description of friction, as normal stresses can be taken as prescribed. This greatly simplifies the mathematical treatment of friction, as explained in e.g. Campos *et al.*[4]

Blatter's model ultimately feeds into simulations of the evolution of the ice surface, and hence into simulations of future ice sheet evolution in the context of climate change.[27,18] Even though the model incorporates only the two horizontal velocity components, the vertical velocity component can be computed *a posteriori* as the three-dimensional velocity field is divergence-free,[3] and this allows the ice surface evolution to be computed.

Given the motivation for studying both, Blatter's glacier flow model and the abstract minimization problem for functionals of the form (1.1), the specific goals of our paper are the following: First, we formulate Blatter's model with a number of different friction laws motivated by classical solid mechanics as well as by theoretical and empirical work in glaciology, focusing on the invariance of this model under rigid body motions in our formulation (§2). Next cast the resulting set of partial differential equations in their weak form. This extends previous work by Rappaz and Reist.[28] Subsequently, we show that this weak form corresponds to a convex minimization problem of the form (1.1), in which the semi-coercivity of the functional J_0 arises through the invariance of stresses under rigid body motions (§3). We demonstrate the existence of minimizers of J under a force and moment balance

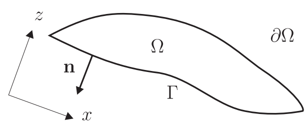

Fig. 1. Geometry of the problem. Note that the y-direction is suppressed here, and the z axis is oriented perpendicularly from the mean slope of the bed.

previously identified by e.g. Duvaut and Lions[8] (Theorem 4.2). In simple terms, we demonstrate that solutions exist provided the maximum force due to friction at the base of a glacier exceeds the net gravitational force on the glacier, and provided the maximum torque due to friction forces exceeds the torque to gravitational forces. Then we derive necessary and sufficient conditions for these solutions to be unique, showing that non-uniqueness only occurs under very specific conditions that allow a rigid body motion to be added to the velocity field of the glacier while leaving friction forces at the boundary unchanged (Theorem 4.3). We also show that these conditions cannot be satisfied if there is non-zero friction everywhere at the base of the glacier (Corollary 4.1) . Next, we demonstrate that the solutions corresponding to the different friction laws that we consider (classical and regularized Coulomb friction as well as power-law friction laws) converge to the Coulomb friction solution in certain parametric limits, and hence that Coulomb friction can be approximated numerically using a variety of regularizations (§5). For instance, we demonstrate in Theorem 5.3. Finally, we consider the convergence of finite element approximations for the friction laws we consider (§6). We conclude with some numerical examples in §7.

2. The model

Having specified a Cartesian coordinate system $Oxyz$ with the z-axis oriented perpendicularly upwards from the mean slope of the bed (figure 1), let the glacier occupy an open, bounded domain $\Omega \subset \mathbb{R}^3$ with Lipschitz boundary $\partial\Omega$. Over time, Ω will naturally evolve as a result of snowfall, surface melting and the flow of the glacier. Here, we only concern ourselves with determining the velocity field for a given glacier geometry and treat Ω as fixed. Blatter's model then requires that the x- and y-components of velocity, u and v, satisfy

$$\frac{\partial}{\partial x}\left[2\mu\left(2\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right)+f_1=0, \quad (2.1a)$$

$$\frac{\partial}{\partial y}\left[2\mu\left(2\frac{\partial v}{\partial y}+\frac{\partial u}{\partial x}\right)\right]+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\frac{\partial}{\partial z}\left(\mu\frac{\partial v}{\partial z}\right)+f_2=0, \quad (2.1b)$$

where f_1 and f_2 are gravitational driving terms associated with the surface inclination of the glacier. These equations are equivalent to equations (2.37) and (2.38) in Blatter's paper.[3] μ denotes ice viscosity, which may depend on position and on the strain rate scalar s defined through

$$s = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \left(\frac{\partial v}{\partial y}\right)^2 + \frac{1}{4}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \frac{1}{4}\left(\frac{\partial u}{\partial z}\right)^2 + \frac{1}{4}\left(\frac{\partial v}{\partial z}\right)^2}. \tag{2.2}$$

A widely used rheological model for ice is *Glen's law*[26], a shear-thinning power law that sets

$$\mu(s; x, y, z) = \frac{1}{2}B(x, y, z)s^{p-2} \tag{2.3}$$

with $p \in (1, 2]$ and $B > 0$ a 'rate factor' that depends on local temperature, liquid content and impurities in the ice. A more common notation in glaciology[26] puts $p = 1 + 1/n$ where $n = 3$ is frequently used, so $p = 4/3$.

Before we specify boundary conditions, we condense our notation somewhat by exploiting the symmetry properties of (2.1). In particular, due to the approximations behind the model, it exhibits invariance only under translations in the horizontal and rotations about vertical axes, and we can capture this in a more succinct notation. Let $\boldsymbol{u} = (u_1, u_2, u_3) = (u, v, 0)$, where we include a dummy third component of velocity because it leads to a simpler notation when dealing with the three-dimensional domain. In the same vein, define $\boldsymbol{x} = (x_1, x_2, x_3) = (x, y, z)$, and let strain rate $\boldsymbol{D}(\boldsymbol{u})$ be defined in the usual way through

$$D_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{2.4}$$

We also define the following inner product $(\cdot, \cdot)$ and associated norm $|\cdot|$ on $\mathbb{R}^{3\times 3}_{\mathrm{symm}}$, the space of real symmetric 3-by-3 matrices:

$$(\boldsymbol{A}, \boldsymbol{B}) = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\left(A_{ij}B_{ij} + A_{ii}B_{jj}\right), \qquad |\boldsymbol{A}| = \sqrt{(\boldsymbol{A}, \boldsymbol{A})}. \tag{2.5}$$

It is easy to show that the strain rate scalar s in (2.2) is simply $s = |\boldsymbol{D}(\boldsymbol{u})|$. We can then define a stress 'tensor' Σ_{ij} as

$$\Sigma_{ij} = 2\mu(|\boldsymbol{D}(\boldsymbol{u})|; \boldsymbol{x})\left[D_{ij}(\boldsymbol{u}) + \sum_{k=1}^{3} D_{kk}(\boldsymbol{u})\delta_{ij}\right] \tag{2.6}$$

where δ_{ij} is the usual Kronecker delta. In terms of Σ_{ij}, (2.1) can be written succinctly as

$$\sum_{j=1}^{3}\frac{\partial \Sigma_{ij}}{\partial x_j} + f_i = 0 \tag{2.7}$$

for $i = 1, 2$ (note that we deliberately do not apply the summation convention here in order to avoid confusion over the range of the indices i and j). Rewriting the strain rate components $D_{ij}(\boldsymbol{u})$ in terms of derivatives of u and v will confirm that the formulations (2.1) and (2.7) are equivalent.

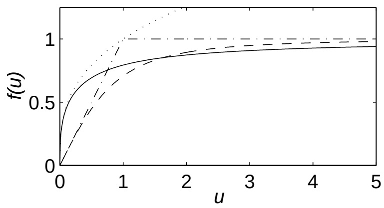

Fig. 2. An illustration of different friction laws, with dependence on $\boldsymbol{x}$ suppressed. Shown are (2.9) (solid line) with $\Lambda = 1$, $p = 4/3$, (3.7) with $\Lambda = 1$ (dashed line), (3.8) with $\tau_c\ C = 1$ (dot-dashed line) and the power law (2.8c) with $\epsilon = p - 1 = 1/3$ (dotted line).

As indicated, analyses of (2.1) with suitable prescriptions for μ and mixed Dirichlet-Neumann conditions have appeared in the literature,[6,7,16,28] and our main focus in this paper is on more sophisticated boundary conditions that are motivated by empirical and theoretical work on glacier sliding.[11,12,15,17,30] We consider three distinct cases: Coulomb friction, a velocity-dependent monotone friction law that behaves as Coulomb friction at large velocities, and a power law.

Let $\Gamma \subset \partial\Omega$ be the base of the glacier (figure 1), and let $\boldsymbol{n}$ denote the outward-pointing unit normal to the boundary $\partial\Omega$. Define a 'tangential' shear stress vector on Γ through $\Sigma_i^t = -\sum_{j=1}^{3} \Sigma_{ij} n_j$, where $i = 1, 2$. This agrees with the definition of bed shear stress in Blatter's original paper[3] up to a positive geometrical factor that is, in Blatter's approximation scheme, close to unity (and can in any case be absorbed into the bed strength parameter τ_c below, which also absorbs any normal stress effects as indicated at the end of the introduction). Coulomb friction[17] with a prescribed bed yield stress $\tau_c \geq 0$ then gives rise to the either-or statement

$$\begin{aligned} \Sigma_i^t &= \tau_c u_i/|\boldsymbol{u}| \qquad \text{when } |\boldsymbol{u}| > 0, \\ |\boldsymbol{\Sigma}^t| &\leq \tau_c \qquad \text{when } \boldsymbol{u} = \boldsymbol{0} \end{aligned} \tag{2.8a}$$

on Γ, where $|\boldsymbol{\Sigma}^t| = \sqrt{\Sigma_1^{t2} + \Sigma_2^{t2}}$ and $|\boldsymbol{u}| = \sqrt{u_1^2 + u_2^2}$ denote the usual Euclidian norms. Typically, τ_c is assumed to be proportional to *effective pressure*, defined as the difference between normal stress at Γ and water pressure in a thin sediment layer at the bed (cf. Ref. 17). In Blatter's model, normal stress is treated as being hydrostatic, and if the water pressure is known, then τ_c can be treated as a prescribed function. The development below assumes this to be the case.

As an alternative, we consider velocity-dependent friction laws in the form

$$\Sigma_i^t = \tau_c f(|\boldsymbol{u}|; \boldsymbol{x}) u_i/|\boldsymbol{u}|, \tag{2.8b}$$

where the function $f : [0, \infty) \times \Gamma \mapsto [0, \infty)$ is a continuous, non-decreasing function in its first argument with $f(0; \boldsymbol{x}) = 0$ and $\lim_{u\to\infty} f(u; \boldsymbol{x}) = 1$ for $\boldsymbol{x} \in \Gamma$. Friction laws of this form behave as Coulomb friction at high velocities, but avoid the either-or nature of the Coulomb friction law. Lastly, we also consider power laws of the form[11,19,20]

$$\Sigma_i^t = \tau_c |\boldsymbol{u}|^{\epsilon-1} u_i, \tag{2.8c}$$

where $\epsilon > 0$. As a particular example of (2.8b) motivated by the work on subglacial cavity formation in Ref. 30 is

$$f(u;\boldsymbol{x}) = \left(\frac{u}{u+\Lambda(\boldsymbol{x})}\right)^{p-1}, \tag{2.9}$$

where $p > 1$ is the same coefficient as in (2.3), and $\Lambda(\boldsymbol{x}) > 0$ is determined by local glacier bed roughness.

Regardless of the friction law specified at the bed, the upper glacier surface is always traction-free (see also Ref. 28, equations (1.20) and (1.21)):

$$\Sigma_{ij} n_j = 0 \tag{2.10}$$

on $\partial\Omega \setminus \Gamma$.

3. Weak formulation

The weak formulation of problem (2.7) subject boundary conditions (2.8) and (2.10) follows standard procedure,[8] with only minor alterations to account for the fact that $i \in (1,2)$ in (2.7), while j ranges from 1 to 3. Let $\boldsymbol{v} = (v_1, v_2, 0)$ be a smooth vector-valued test function defined on $\bar{\Omega}$. Multiply (2.7) by $v_i - u_i$ and integrate over Ω. Assuming Σ_{ij} and u_i to be sufficiently smooth to apply the divergence theorem, we obtain

$$\int_\Omega \sum_{j=1}^{3} \Sigma_{ij} \frac{\partial(v_i - u_i)}{\partial x_j} - f_i(v_i - u_i)\,\mathrm{d}\Omega - \int_{\partial\Omega} \sum_{j=1}^{3} \Sigma_{ij} n_j (v_i - u_i)\,\mathrm{d}\Gamma = 0. \tag{3.1}$$

Defining $f_3 = 0$ and recognizing that $u_3 = v_3 = 0$, we can sum over $i = 1,2,3$. Since $\Sigma_{ij} = \Sigma_{ji}$, it follows that $\sum_{i=1}^3 \sum_{j=1}^3 \Sigma_{ij} \partial(v_i - u_i)/\partial x_j = \sum_{i=1}^3 \sum_{j=1}^3 \Sigma_{ij} D_{ij}(\boldsymbol{v} - \boldsymbol{u}) = 4\mu(|\boldsymbol{D}(\boldsymbol{u})|;\boldsymbol{x})(\boldsymbol{D}(\boldsymbol{u}), \boldsymbol{D}(\boldsymbol{v}-\boldsymbol{u}))$.

For the Coulomb friction law (2.8a), it is straightforward to show (with $\tau_c \geq 0$) that $-\sum_{i=1}^3 \Sigma_{ij} n_j (v_i - u_i) \leq \tau_c(|\boldsymbol{v}| - |\boldsymbol{u}|)$ (see also Duvaut and Lions, Ref. 8 chapter 5.2, or Ref. 4, 31). In that case, we obtain from (3.1) the variational inequality

$$\int_\Omega 4\mu(|\boldsymbol{D}(\boldsymbol{u})|;\boldsymbol{x})\,(\boldsymbol{D}(\boldsymbol{u}), \boldsymbol{D}(\boldsymbol{v}-\boldsymbol{u})) - \boldsymbol{f}\cdot(\boldsymbol{v}-\boldsymbol{u})\,\mathrm{d}\Omega + \int_\Gamma \tau_c(|\boldsymbol{v}| - |\boldsymbol{u}|)\,\mathrm{d}\Gamma \geq 0, \tag{3.2a}$$

where $\boldsymbol{f} = (f_1, f_2, 0)$. Conversely, with Σ_i^t defined by (2.8b), we obtain the variational equation

$$\int_\Omega 4\mu(|\boldsymbol{D}(\boldsymbol{u})|;\boldsymbol{x})\,(\boldsymbol{D}(\boldsymbol{u}), \boldsymbol{D}(\boldsymbol{v}-\boldsymbol{u})) - \boldsymbol{f}\cdot(\boldsymbol{v}-\boldsymbol{u})\,\mathrm{d}\Omega + \int_\Gamma \tau_c f(|\boldsymbol{u}|;\boldsymbol{x}) \frac{\boldsymbol{u}\cdot(\boldsymbol{v}-\boldsymbol{u})}{|\boldsymbol{u}|}\,\mathrm{d}\Gamma = 0, \tag{3.2b}$$

while for the power law (2.8c), we find

$$\int_\Omega 4\mu(|\boldsymbol{D}(\boldsymbol{u})|;\boldsymbol{x})\,(\boldsymbol{D}(\boldsymbol{u}), \boldsymbol{D}(\boldsymbol{v}-\boldsymbol{u})) - \boldsymbol{f}\cdot(\boldsymbol{v}-\boldsymbol{u})\,\mathrm{d}\Omega + \int_\Gamma \tau_c |\boldsymbol{u}|^{\epsilon-1} \boldsymbol{u}\cdot(\boldsymbol{v}-\boldsymbol{u})\,\mathrm{d}\Gamma = 0. \tag{3.2c}$$

When augmented with appropriate prescriptions for ice viscosity μ and sliding friction f, and cast in suitable functions spaces, a well-developed framework for analyzing elliptic variational problems of the form (3.2) is available.[8,21,25] We cast our ice flow problem in this framework first, and then describe the novelties that set the present paper apart from previous work.

With a power law rheology of the form (2.3), our elliptic problem is effectively of p-Laplacian form. We allow for slight generalizations by using a viscosity function which is required to behave as a power law only at high strain rates, as indicated by definition 3.1 below. Note that we consider generalizations of the p-Laplacian only with $p < 2$, implying that viscosity $\mu(s;\boldsymbol{x})$ decreases with strain rate s. This is motivated by the fact that ice is usually observed to be a shear-thinning material, and leads to $W^{1,p}$ spaces with $p < 2$ as the natural setting for our problem. The extension of the results in this paper to $p \geq 2$ is straightforward but requires slightly different stability estimates in the later parts of the paper, where we deal with the convergence of regularized Coulomb friction laws. (As with the results for $p < 2$ presented in this paper, the necessary stability estimates for $p > 2$ also follow from the work of Liu and Barrett.[22])

Definition 3.1. Let $g(t;\boldsymbol{x}) := 4t\mu(t;\boldsymbol{x})$ for $t > 0$, $\boldsymbol{x} \in \Omega$ and define $g(0;\boldsymbol{x}) = 0$. Assume that $g : [0,\infty) \times \Omega \mapsto [0,\infty)$ is continuous and strictly increasing with respect to its first argument for all $\boldsymbol{x} \in \Omega$ except possibly on a subset of measure zero. Let

$$G(t;\boldsymbol{x}) := \int_0^t g(s;\boldsymbol{x})\,\mathrm{d}s, \tag{3.3}$$

and assume that there exist constants $C_1 > C_2 > 0$ and $p \in (1,2]$ and a function $s_0 \in L^p(\Omega)$ such that for all $t > 0$

$$G(t;\boldsymbol{x}) < C_1(1+t^p) \qquad \forall \boldsymbol{x} \in \Omega, \tag{3.4}$$

$$G(t;\boldsymbol{x}) > C_2 t^p \qquad \text{wherever } t > s_0(\boldsymbol{x}), \tag{3.5}$$

except possibly on a subset of Ω of measure zero. Assume also that $G(|v|;\boldsymbol{x})$ is measureable for all $v \in L^p(\Omega)$, and that $\boldsymbol{f} \in [L^{p/(p-1)}(\Omega)]^3$. For the different cases in (3.2), we assume the following:

(i) For (3.2a), let $\tau_c \in L^{p/(p-1)}(\Gamma)$ and $\tau_c \geq 0$ almost everywhere (a.e.) on Γ.

(ii) For (3.2b), assume that $\tau_c \in L^{p/(p-1)}(\Gamma)$, $\tau_c \geq 0$ a.e. on Γ. Let $f : [0,\infty) \times \Gamma \mapsto [0,\infty)$ be continuous and non-decreasing in its first argument with $f(0;\boldsymbol{x}) \equiv 0$ and let $\lim_{u\to\infty} f(u;\boldsymbol{x}) = 1$ a.e. on Γ. Defining

$$\psi(v,\boldsymbol{x}) := \int_0^v f(u;\boldsymbol{x})\,\mathrm{d}u, \tag{3.6}$$

let f be such that $\psi(|u|;\boldsymbol{x})$ is measurable for every $u \in L^p(\Gamma)$.

(iii) For (3.2c), assume that $0 < \epsilon \leq p-1$ and $\tau_c \in L^{p_\epsilon/(p_\epsilon-1)}$ for some $p_\epsilon \geq 1+\epsilon$, and $\tau_c \geq 0$ a.e. on Γ.

With p related to the viscosity function μ through the conditions above, define $X = \{(v_1, v_2, v_3) \in [W^{1,p}(\Omega)]^3 : v_3 = 0\}$, endowed with the usual norm on $[W^{1,p}(\Omega)]^3$:

$$\|\boldsymbol{v}\|_X := \left(\int_\Omega |\boldsymbol{v}|^p + \left(\sum_{i=1}^3 \sum_{j=1}^3 \frac{\partial v_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \right)^{p/2} \mathrm{d}\Omega \right)^{1/p},$$

and let X' be the dual of X. Any $\boldsymbol{u} \in X$ that satisfies (3.2a), (3.2b) or (3.2c) for all $\boldsymbol{v} \in X$ will be considered as a weak solution of the system (2.7) with boundary conditions (2.8a), (2.8b) and (2.8c) respectively.

There are many choices of functions μ and f that satisfy the conditions above: f defined in (2.9) with $\Lambda \in L^p(\Gamma)$, $\Lambda > 0$ a.e. is an example of a suitable friction function, as are

$$f(u;\boldsymbol{x}) = \frac{u}{\sqrt{u^2 + \Lambda(\boldsymbol{x})^2}}, \tag{3.7}$$

$$f(u;\boldsymbol{x}) = \begin{cases} C(\boldsymbol{x})u/\tau_c(\boldsymbol{x}) & \text{if } u \leq \tau_c(\boldsymbol{x})/C(\boldsymbol{x}), \\ 1 & \text{if } u \geq \tau_c(\boldsymbol{x})/C(\boldsymbol{x}), \end{cases} \tag{3.8}$$

where $C^{-1} \in L^{2p/(p+1)}(\Gamma)$, $C^{-1} \geq 0$ a.e. Equation (3.8), though awkward-looking, can be motivated glaciologically. Specifically, it states that friction increases linearly with $|\boldsymbol{u}|$ as $\boldsymbol{\Sigma}^t = C\boldsymbol{u}$ until $|\boldsymbol{\Sigma}^t|$ reaches the critical value τ_c, after which it remains at that value for higher velocities (see figure 2). A friction law of this form is applicable if two sliding processes act in parallel, one leading to a linear friction law (e.g. Refs. 11, 24) and the other to a Coulomb friction law. For μ, Glen's law (2.3) with $B \in L^\infty(\Omega)$, $B > B_0 > 0$ a.e. will do (where B_0 is a constant), as will

$$\mu(t;\boldsymbol{x}) = B\left[t^2 + D_0(\boldsymbol{x})^2\right]^{(p-2)/2} \tag{3.9}$$

with $D_0 \in L^p(\Omega)$, and the implicitly defined viscosity functions in Refs. 6, 7, 16.

Associated with the variational problems (3.2) are the problems of minimizing the functionals

$$J_i(\boldsymbol{v}) := \int_\Omega G(|\boldsymbol{D}(\boldsymbol{v})|;\boldsymbol{x}) - \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}\Omega + j_i(\boldsymbol{v}), \tag{3.10a}$$

over X, where

$$j_1(\boldsymbol{v}) := \int_\Gamma \tau_c |\boldsymbol{v}| \,\mathrm{d}\Gamma \tag{3.10b}$$

$$j_2(\boldsymbol{v}) := \int_\Gamma \tau_c \psi(|\boldsymbol{v}|;\boldsymbol{x}) \,\mathrm{d}\Gamma \tag{3.10c}$$

$$j_3(\boldsymbol{v}) := \int_\Omega \tau_c (1+\epsilon)^{-1} |\boldsymbol{v}|^{1+\epsilon} \,\mathrm{d}\Gamma \tag{3.10d}$$

We show next that the variational problems (3.2) are in fact equivalent to these minimization problems. Moreover, we show that the functionals J_i defined above have certain properties that help ensure the existence of minimizers. Recall that a convex functional on a reflexive Banach space is guaranteed to have a minimizer over a closed and bounded convex set K if it is proper and weakly lower semi-continuous.[5,9] Essentially, this is the case because closed and bounded sets in reflexive Banach spaces are weakly compact. Let $\inf_{\boldsymbol{v}\in K} J(\boldsymbol{v})$ be the greatest lower bound of J over the set K, which exists if J is proper, and take a sequence of points $\boldsymbol{v}_n \in K$ such that $\lim_{n\to\infty} J(\boldsymbol{v}_n) = \inf_{\boldsymbol{v}\in K} J(\boldsymbol{v})$. By compactness, the sequence $\{\boldsymbol{v}_n\}$ then has a weakly convergent subsequence with limit $\boldsymbol{v}_\infty \in K$, and the lower semi-continuity of J ensures that $J(\boldsymbol{v}_\infty) \leq \inf_{\boldsymbol{v}\in K} J(\boldsymbol{v})$, whence $J(\boldsymbol{v}_\infty) = \inf_{\boldsymbol{v}\in K} J(\boldsymbol{v})$, and $\boldsymbol{v}_\infty$ minimizes J over K. In our example, it is straightforward to show that the J_i are indeed convex, proper and lower semi-continuous.

Theorem 3.1. *Under the assumptions in definition 3.1, the functionals J_1, J_2 and J_3 in (3.10) are convex, proper and weakly lower semi-continuous on X. Furthermore, solutions of (3.2a), (3.2b) and (3.2c) are minimizers of J_1, J_2 and J_3, respectively, and vice-versa.*

Proof. From definition 3.1, we find that $\partial G(t;\boldsymbol{x})/\partial t = g(t;\boldsymbol{x})$ is non-negative and strictly increasing in t for $t \geq 0$ and all $\boldsymbol{x} \in \Omega$ except possibly on a subset of measure zero. By redefining G on that set as necessary, $G : [0,\infty) \times \Omega \mapsto [0,\infty)$ is strictly convex and strictly increasing in its first argument. Moreover, the matrix norm $|\cdot|$ is a convex functional on $\mathbb{R}^{3\times3}_{\text{symm}}$. Hence the mapping $\boldsymbol{A} \mapsto G(|\boldsymbol{A}|;\boldsymbol{x})$ is strictly convex on $\mathbb{R}^{3\times3}_{\text{symm}}$ for any fixed $\boldsymbol{x} \in \Omega$ except possibly on a set of measure zero. Moreover, from (3.4) and the definition of $|\boldsymbol{D}(\boldsymbol{v})|$, one can derive the growth bound

$$J_0(\boldsymbol{v}) := \int_\Omega G(|\boldsymbol{D}(\boldsymbol{v})|;\boldsymbol{x})\,\mathrm{d}\Omega \leq \frac{3C_1}{2}\,\|\boldsymbol{v}\|^p + C_1\mathrm{mes}\,(\Omega). \tag{3.11}$$

Hence J_0 is convex and bounded on bounded subsets of X, whence J_0 is strongly continuous and therefore weakly lower semi-continuous on X.[9] Similarly, it follows from definition 3.1 that ψ is convex in its first argument with $\psi(u;\boldsymbol{x}) \leq u$ for all $\boldsymbol{x} \in \Gamma$, and hence that the functionals j_1, j_2 and j_3 are convex, proper and weakly lower semi-continuous on X, whence the same properties follow for the functionals J_1, J_2 and J_3.

Further, direct differentiation and application of theorem 4 in Evans[10], chapter 8.2.3, then shows that the operator $A : X \mapsto X'$ defined by

$$\langle A(\boldsymbol{v}),\boldsymbol{w}\rangle := \int_\Omega 4\mu(|\boldsymbol{D}(\boldsymbol{v})|;\boldsymbol{x})\,(\boldsymbol{D}(\boldsymbol{v}),\boldsymbol{D}(\boldsymbol{w}))\;\mathrm{d}\Omega \tag{3.12}$$

is the Gâteaux-derivative of J_0. Likewise, it can be shown that the Gâteaux derivatives of j_2 and j_3 are given by $\langle j_2'(\boldsymbol{v}),\boldsymbol{w}\rangle = \int_\Gamma \tau_c f(|\boldsymbol{v}|;\boldsymbol{x})\boldsymbol{v}\cdot\boldsymbol{w}/|\boldsymbol{v}|\,\mathrm{d}\Gamma$ and $\langle j_3'(\boldsymbol{v}),\boldsymbol{w}\rangle = \int_\Gamma \tau_c|\boldsymbol{v}|^{\epsilon-1}\boldsymbol{v}\cdot\boldsymbol{w}\,\mathrm{d}\Gamma$. The equivalence between solving the variational problems in (3.2) and minimizing the functionals in (3.10) follows by standard methods in convex analysis.[9] □

The remaining complication in our problem is that we need to minimize the J_i not over some closed and bounded convex set K, but over the entire, unbounded

Banach space X. The usual way to deal with this is to invoke coercivity. That is, one shows that the $J(\boldsymbol{v}) > J(\boldsymbol{0})$ if $\boldsymbol{v}$ lies outside some closed and bounded ball $B \subset X$, whence the a $\boldsymbol{v}$ that minimizes J over the closed and bounded convex set B also minimizes J over the entire space X. Coercivity is therefore the missing ingredient in demonstrating the existence of a minimizer, while strict convexity is usually invoked in problems of this type to ensure the uniqueness of the minimizer.

For the basic Coulomb friction case, problem (3.2a) is mathematically a variant of the classical elastic Coulomb friction problem with prescribed normal stress at the boundary.[4,8,21,25] The analysis of problem (3.2a) in these references however proceeds almost entirely (except for an existence proof similar to theorem 4.2 below) under the additional restriction that $\boldsymbol{u} = \boldsymbol{v} = \boldsymbol{0}$ on a prescribed part of $\partial\Omega$ with positive measure. This device corresponds physically in our case to the glacier ice being firmly 'stuck' to some part the bed. Mathematically, it simplifies the analysis of the problem considerably because it ensures that the space of admissible functions in the variational problem does not include rigid body motions (i.e., velocity fields that leave the stress tensor invariant), and consequently ensures that the functional J_0 defined in (3.11) is coercive by Korn's inequality, as well as being strictly convex. This avoids such issues as the possibility of non-existence or non-uniqueness of solutions studied in §4 below.

The novelty in the present paper is that we dispense with this device, as is appropriate for glaciers that are not frozen to their beds at any point on their lower boundary. We present uniqueness results in theorem 4.3 and its corollary that are, to our knowledge, new not only to the specific glacier flow problem studied here, but also to generic problems of the form (3.2a). Moreover, we focus in §5 on the use of the alternative friction models in (3.2b) and (3.2c) as regularizations of the Coulomb friction problem (3.2a) in the case where J_0 is non-coercive (or rather, semi-coercive in the sense of §1, the coercive case having been dealt with in part by Ref. 4). The motivation for this is practical: not only is the exact behaviour of subglacial friction a matter of ongoing research, but even allowing that classical Coulomb friction of the form (2.8a) may be the correct friction law, the presence of the non-differentiable term $\int_\Gamma \tau_c |\boldsymbol{v}| \,\mathrm{d}\Gamma$ presents problems for the numerical solution of (3.2a). The purpose of our study is thus in part to demonstrate that, in appropriate parametric limits (such as $\epsilon \to 0$ in (2.8c) or $\|\Lambda\| \to 0$ in (2.9)), the three friction laws in (2.8) produce the same results, and that minimizers of the smoother functionals J_2 and J_3 can be used to approximate minimizers of J_1, thereby extending some of the results in Ref. 4. Lastly, we also touch briefly on the question of numerical approximations of the ice flow problem by finite elements, adapting some of the results in Refs. 6, 16, 21, 28.

4. Existence and uniqueness

As shown in theorem 3.1, the functionals J_1, J_2 and J_3 are convex, proper and weakly lower semi-continuous on the reflexive Banach space X. The remaining ingredient in the standard proof for the existence of a minimizer[9] is coercivity. The first step in establishing coercivity is a simple adaptation of Korn's second inequality from $[W^{1,p}(\Omega)]^3$ to the restriction X.

Theorem 4.1. *Let the subspace of rigid body motions in X be defined through*

$$\mathcal{R} = \{\boldsymbol{v} \in X : \boldsymbol{v}(\boldsymbol{x}) = (v_1^0, v_2^0, 0) + \omega(x_2, -x_1, 0) \text{ with } v_1^0, v_2^0, \omega \in \mathbb{R} \text{ constant}\}$$

Also define a projection $\pi_{\mathcal{R}} : X \mapsto \mathcal{R}$ through

$$\pi_{\mathcal{R}}(\boldsymbol{v})(\boldsymbol{x}) := \bar{\boldsymbol{v}} + \omega(x_2, -x_1, 0) : \qquad \bar{\boldsymbol{v}} = \frac{1}{\mathrm{mes}\,(\Omega)} \int_\Omega \boldsymbol{v}\, \mathrm{d}\Omega,$$

$$\omega = \frac{1}{2\mathrm{mes}\,(\Omega)} \int_\Omega \frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1}\, \mathrm{d}\Omega.$$

Let $X_{\mathrm{rot}} = \{\boldsymbol{v} \in X : \pi_{\mathcal{R}}(\boldsymbol{v}) = \boldsymbol{0}\}$. Then there exists a $C_3 > 0$, depending only on Ω and p, such that

$$\int_\Omega |\boldsymbol{D}(\boldsymbol{v})|^p \, \mathrm{d}\Omega \geq C_3 \, \|\boldsymbol{v}\|^p \tag{4.1}$$

for all $\boldsymbol{v} \in X_{\mathrm{rot}}$.

Proof. This is a straightforward adaptation of the usual proof of Korn's second inequality.[21,32] Trivially,

$$I(\boldsymbol{v}) := \int_\Omega |\boldsymbol{D}(\boldsymbol{v})|^p \, \mathrm{d}\Omega \geq \int_\Omega \left| \sum_{i=1}^3 \sum_{j=1}^3 \frac{1}{2} D_{ij}(\boldsymbol{v}) D_{ij}(\boldsymbol{v}) \right|^p \mathrm{d}\Omega, \tag{4.2}$$

Suppose that (4.1) is not true, and hence that there is a sequence $\boldsymbol{v}_n \in X_{\mathrm{rot}}$ such that $\|\boldsymbol{v}_n\| = 1$, $\forall n$, $I(\boldsymbol{v}_n) \to 0$ for $n \to \infty$. From the Rellich-Kondrachev compactness theorem, it follows that we can find a subsequence $\boldsymbol{v}_{n_j}$, $j = 1, \ldots$ that converges strongly in $[L^p(\Omega)]^3$. From $I(\boldsymbol{v}_{n_j}) \to 0$ and Korn's inequality in $[W^{1,p}(\Omega)]^3$, it follows that $\boldsymbol{v}_{n_j}$ is a Cauchy sequence in X, with strong limit $\boldsymbol{v}_\infty \in X_{\mathrm{rot}}$ (since X_{rot} is closed). Clearly $I : X \mapsto \mathbb{R}$ is continuous, so $I(\boldsymbol{v}_\infty) = 0$. However, from a mollification argument, $I(\boldsymbol{v}) = 0$ if and only if $\boldsymbol{v} \in \mathcal{R}$, so $\boldsymbol{v}_\infty \in \mathcal{R} \cap X_{\mathrm{rot}}$, whence $\boldsymbol{v}_\infty = \boldsymbol{0}$. But this contradicts the limit $\|\boldsymbol{v}_\infty\| = \lim_{j\to\infty} \left\|\boldsymbol{v}_{n_j}\right\| = 1$. □

The physical significance of the set $\mathcal{R}$ of rigid body motions is that strain rate $D_{ij}(\boldsymbol{u})$ and hence the stress tensor Σ_{ij} are invariant under additions of rigid body motions to the velocity field, i.e., $D_{ij}(\boldsymbol{v}) = D_{ij}(\boldsymbol{v} + \boldsymbol{r})$ for any $\boldsymbol{v} \in X$ and $\boldsymbol{r} \in \mathcal{R}$. As we will see later, this invariance underlies the possibility of non-unique solutions for problems (3.2a) and (3.2b).

Moreover, elements of $\mathcal{R}$ (i.e., rotations and translations) appear in the following simple solvability condition.[4,8] Putting $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{r}$ in (3.2a) for any $\boldsymbol{r} \in \mathcal{R}$ and using

$|\boldsymbol{u}+\boldsymbol{r}| \leq |\boldsymbol{u}| + |\boldsymbol{r}|$, a necessary condition for problems (3.2a) to have a solution is obtained:

$$\int_\Gamma \tau_c |\boldsymbol{r}| \, \mathrm{d}\Gamma \geq \int_\Omega \boldsymbol{f} \cdot \boldsymbol{r} \, \mathrm{d}\Omega. \tag{4.3}$$

The same condition can be derived by putting $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{r}$ in (3.2b) and using $f(|\boldsymbol{u}|;\boldsymbol{x}) \leq 1$, $\boldsymbol{u} \cdot \boldsymbol{r}/|\boldsymbol{u}| \leq |\boldsymbol{r}|$.

The physical interpretation of this condition is that solutions cannot exist if the total gravitational force in the xy-plane is greater than the maximum friction force that can be generated at the bed (take $\boldsymbol{r}$ to be a non-zero constant vector), or if the gravitational torque about some axis is greater than the maximum frictional torque (take $\boldsymbol{r}$ to be non-constant, in which case $\boldsymbol{r}(\boldsymbol{x}) = \omega(x_2 - x_2^0, x_1^0 - x_1, 0)$ for some fixed (x_1^0, x_2^0), and $\boldsymbol{r}$ represents a rotation about a vertical axis through (x_1^0, x_2^0)).

The same issues of force and moment balance failure do not arise for the power law case (3.2c) because friction can then be increased without bound purely by dint of increasing velocity. Strengthening the force and torque balance condition (4.3), we the following theorem shows that we also obtain a sufficient condition for solvability. For the Coulomb friction case (2.8a), the following theorem is analogous to theorem 5.2 of Duvaut and Lions,[8] chapter 5, and represents a generalization of theorem 7.4 of chapter 1 of Ref. 8 in the case of the regularized friction law (2.8b).

Theorem 4.2. *Assume that*

$$\int_\Gamma \tau_c |\boldsymbol{r}| \, \mathrm{d}\Gamma > \int_\Omega \boldsymbol{f} \cdot \boldsymbol{r} \, \mathrm{d}\Omega \tag{4.4}$$

for every $\boldsymbol{r} \in \mathcal{R} \setminus \{\boldsymbol{0}\}$. *Then problems (3.2a) and (3.2b) have a weak solution. Problem (3.2c) has a weak solution under the weaker assumption that* $\int_\Gamma \tau_c \, \mathrm{d}\Gamma > 0$.

The proof of this result is rather lengthy. Although there are some significant technical complications, the proof is also conceptually similar to that of theorem 7.4 of chapter 1 in Duvaut and Lions[8], in which the equivalent of our subspace $\mathcal{R} \subset X$ is the set of constant functions. As a consequence, we present the proof in the appendix at the end of this paper.

As we have remarked, the addition to $\boldsymbol{u}$ of a rigid body motion leaves the strain rate tensor $D_{ij}(\boldsymbol{u})$ and hence the stress tensor Σ_{ij} unchanged. Physically, we can expect non-unique solutions if stresses at the boundary are also unchanged under such an addition. This consideration motivates the following result.

Theorem 4.3. *A minimizer of* $\mathbf{u} \in X$ *of* J_1 *or* J_2 *is non-unique if and only if all of the following conditions are met:*

(i) Define $\Gamma_\tau = \{\boldsymbol{x} \in \Gamma : \tau_c(\boldsymbol{x}) > 0\}$. *There must be an* $\boldsymbol{r} \in \mathcal{R} \setminus \{\boldsymbol{0}\}$ *and a function* $\zeta : \Gamma_\tau \mapsto \mathbb{R}$ *such that* $\boldsymbol{u} = \zeta \boldsymbol{r}$ *on* Γ_τ *in the trace sense.*

(ii) ζ *and* $\boldsymbol{r}$ *can be chosen such that* $\Gamma_\tau = \Gamma^+ \cup \Gamma^-$, *where* $\Gamma^+ = \{\boldsymbol{x} \in \Gamma_\tau : \zeta(\boldsymbol{x}) \geq 0\}$ *and* $\Gamma^- = \{\boldsymbol{x} \in \Gamma_\tau : \zeta(\boldsymbol{x}) \leq -1\}$.

(iii) For minimizers of J_2, ψ must further be such that $\psi(|\boldsymbol{r}(\boldsymbol{x})||\zeta(\boldsymbol{x})+\lambda|;\boldsymbol{x})$ is affine in λ for $\lambda \in [0,1]$ a.e. on Γ_τ.
For minimizers of J_1, the relation

$$\int_{\Gamma^+} \tau_c |\boldsymbol{r}| \, \mathrm{d}\Gamma - \int_{\Gamma^-} \tau_c |\boldsymbol{r}| \, \mathrm{d}\Gamma - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{r} \, \mathrm{d}\Omega = 0 \tag{4.5}$$

holds, while for minimizers of J_2, the relation

$$\int_{\Gamma^+} \tau_c \left[\psi(|\boldsymbol{r}|(\zeta+1);\boldsymbol{x}) - \psi(|\boldsymbol{r}|\zeta;\boldsymbol{x})\right] \mathrm{d}\Gamma$$
$$+ \int_{\Gamma^-} \tau_c \left[\psi(-|\boldsymbol{r}|\zeta;\boldsymbol{x}) - \psi(-|\boldsymbol{r}|(\zeta+1);\boldsymbol{x})\right] - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{r} \, \mathrm{d}\Omega = 0 \tag{4.6}$$

holds.

When these conditions are satisfied, $\boldsymbol{u} + \lambda \boldsymbol{r}$ is also a minimizer for any $\lambda \in [0,1]$. Minimizers of J_3 are unique provided $\mathrm{mes}\,(\Gamma_\tau) > 0$*, or equivalently, if $\int_\Gamma \tau_c \, \mathrm{d}\Gamma > 0$.*

Proof. This result is the consequence of simple convexity arguments. Note that each functional is of the form

$$J_i(\boldsymbol{v}) = J_0(\boldsymbol{v}) - l(\boldsymbol{v}) + j_i(\boldsymbol{v}),$$

with $i = 1,2,3$, where the convex functionals j_i and J_0 are defined in (3.10b)–(3.10d) and (3.11) and $l(\boldsymbol{v}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, \mathrm{d}\Omega$ is a bounded linear functional. Suppose $\boldsymbol{u}_0$ and $\boldsymbol{u}_1$ are distinct minimizers, and define $\boldsymbol{w} = \boldsymbol{u}_1 - \boldsymbol{u}_0$. By the convexity of the J_i, it follows that $\boldsymbol{u}_\lambda = \boldsymbol{u}_0 + \lambda \boldsymbol{w}$ must also be a minimizer for $\lambda \in [0,1]$.

As shown in theorem 3.1, the mapping $\boldsymbol{A} \mapsto G(|\boldsymbol{A}|;\boldsymbol{x})$ is strictly convex on $\mathbb{R}^{3\times3}_{\mathrm{symm}}$ for all $\boldsymbol{x} \in \Omega$ except possibly on a set of measure zero. Hence $J_0(\boldsymbol{u}_0 + \lambda \boldsymbol{w}) = \int_\Omega G(|\boldsymbol{D}(\boldsymbol{u}_0 + \lambda \boldsymbol{w})|;\boldsymbol{x}) \, \mathrm{d}\Omega$ is strictly convex in λ and we cannot have $J_i(\boldsymbol{u}_0) = J_i(\boldsymbol{u}_\lambda) = J_i(\boldsymbol{u}_1)$ for $\lambda \in (0,1)$ unless $\boldsymbol{D}(\boldsymbol{w}) \equiv \boldsymbol{0}$. Hence we have as a necessary condition for non-uniqueness that $\boldsymbol{w} \in \mathcal{R}$.

Conversely, we have $J_0(\boldsymbol{u}_\lambda) = J_0(\boldsymbol{u}_0) = J_0(\boldsymbol{u}_1)$ if $\boldsymbol{w} \in \mathcal{R}$. Thus, if $\boldsymbol{u}_0$ is a minimizer of J_i, a necessary and sufficient condition for $\boldsymbol{u}_1 = \boldsymbol{u}_0 + \boldsymbol{w}$ to be a minimizer distinct from $\boldsymbol{u}_0$ is that $\boldsymbol{w} \in \mathcal{R} \setminus \{\boldsymbol{0}\}$ and that $l(\boldsymbol{u}_0) + j_i(\boldsymbol{u}_0) = l(\boldsymbol{u}_0 + \lambda \boldsymbol{w}) + j_i(\boldsymbol{u}_0 + \lambda \boldsymbol{w})$ for $\lambda \in [0,1]$, or equivalently,

$$\lambda l(\boldsymbol{w}) = j_i(\boldsymbol{u}_0 + \lambda \boldsymbol{w}) - j_i(\boldsymbol{u}_0). \tag{4.7}$$

Hence the necessary and sufficient condition for uniqueness is that the mapping $\lambda \mapsto j_i(\boldsymbol{u}_0 + \lambda \boldsymbol{w})$ is affine for $\lambda \in [0,1]$, and that

$$l(\boldsymbol{w}) = j_i(\boldsymbol{u}_0 + \boldsymbol{w}) - j_i(\boldsymbol{u}_0). \tag{4.8}$$

This is possible for j_3 only if

$$\int_\Gamma \tau_c (1+\epsilon)^{-1} |\boldsymbol{u}_0 + \lambda \boldsymbol{w}|^{1+\epsilon} \, \mathrm{d}\Gamma = 0 \tag{4.9}$$

for all $\lambda \in [0,1]$. This requires that $|\boldsymbol{u}_0 + \lambda \boldsymbol{w}| = 0$ on Γ_τ for all $\lambda \in [0,1]$, which in turn requires that $\boldsymbol{u}_0 = \boldsymbol{w} = \boldsymbol{0}$ a.e. on Γ_τ. However, if $\boldsymbol{w} \in \mathcal{R} \setminus \{\boldsymbol{0}\}$, then $\boldsymbol{w}$

is non-zero almost everywhere on Γ_τ, and (4.9) cannot hold unless $\mathrm{mes}\,(\Gamma_\tau) = 0$. Hence minimizers of J_3 must be unique if $\mathrm{mes}\,(\Gamma_\tau) > 0$.

Consider now j_1 and j_2, which we may generically write as $j_i(\boldsymbol{v}) = \int_\Gamma \tau_c \psi(|\boldsymbol{v}|;\boldsymbol{x})\,\mathrm{d}\Gamma$ if we put $\psi(v;\boldsymbol{x}) = v$ for $i = 1$. The mapping $\lambda \mapsto j_i(\boldsymbol{u}_0 + \lambda\boldsymbol{w})$ is then affine if and only if, for all $\boldsymbol{x} \in \Gamma_\tau$ except possibly on a subset of measure zero, $\lambda \mapsto |\boldsymbol{u}_0 + \lambda\boldsymbol{w}|$ is affine for $\lambda \in [0,1]$ and $\psi(u;\boldsymbol{x})$ is affine for u in the range

$$\min(|\boldsymbol{u}_0(\boldsymbol{x})|, |\boldsymbol{u}_0(\boldsymbol{x}) + \boldsymbol{w}(\boldsymbol{x})|) \leq u \leq \max(|\boldsymbol{u}_0(\boldsymbol{x})|, |\boldsymbol{u}_0(\boldsymbol{x}) + \boldsymbol{w}(\boldsymbol{x})|). \tag{4.10}$$

We will show that these conditions correspond to conditions (i)–(iii) of the theorem.

As $\boldsymbol{w} \in \mathcal{R} \setminus \{\boldsymbol{0}\}$ vanishes at most on a subset of Γ_τ of measure zero, we can write

$$|\boldsymbol{u}_0 + \lambda\boldsymbol{w}| = \sqrt{|\boldsymbol{w}|^2\left(\lambda + \frac{\boldsymbol{u}_0\cdot\boldsymbol{w}}{|\boldsymbol{w}|^2}\right)^2 + \left|\boldsymbol{u}_0 - \frac{\boldsymbol{u}_0\cdot\boldsymbol{w}}{|\boldsymbol{w}|^2}\boldsymbol{w}\right|^2},$$

which is affine in λ in the range $0 \leq \lambda \leq 1$ if and only if $\boldsymbol{u}_0 - (\boldsymbol{u}_0\cdot\boldsymbol{w})\boldsymbol{w}/|\boldsymbol{w}|^2 = 0$ and $\lambda + \boldsymbol{u}_0\cdot\boldsymbol{w}/|\boldsymbol{w}|^2$ does not change sign. Putting $\zeta = (\boldsymbol{u}_0\cdot\boldsymbol{w})/|\boldsymbol{w}|^2$, we have equivalently that $\boldsymbol{u}_0 = \zeta\boldsymbol{w}$ and that $\lambda + \zeta$ does not change sign for $\lambda \in [0,1]$. Conditions (i)–(ii) follow, by redefining ζ on a subset of Γ of measure zero if necessary. Moreover, the range in (4.10) is that spanned by $|\boldsymbol{w}(\boldsymbol{x})||\zeta(\boldsymbol{x}) + \lambda|$ for $0 \leq \lambda \leq 1$, so condition (iii) follows. Clearly, the function $\psi(v;\boldsymbol{x}) = v$ is affine, so this condition is satisfied trivially for J_1.

Lastly, condition (iv) of the theorem is simply the same as equation (4.8). □

The physical interpretation of this theorem in the Coulomb friction case runs as follows: If a solution $\boldsymbol{u}$ is such that one can add a non-zero rigid body motion $\boldsymbol{r}$ to it such that friction at the base of the flow remains the same, then the stress field in the domain Ω is unchanged and shear stress at the boundary still matches the amount friction experienced by the flow. Hence the $\boldsymbol{u}+\boldsymbol{r}$ is then also a solution. The crucial point here is that $\boldsymbol{u}$ and $\boldsymbol{u}+\boldsymbol{r}$ must generate the same amount of friction at the base of the flow. With a Coulomb friction law, this is the case if and only if $\boldsymbol{u}$ and $\boldsymbol{u}+\boldsymbol{r}$ have the same *direction* on the part of the base of the glacier on which the yield stress τ_c is non-zero, that is, on Γ_τ. By extension, $\boldsymbol{u}+\lambda\boldsymbol{r}$ will then also have the same direction as $\boldsymbol{u}$ on Γ_τ for any $\lambda \in [0,1]$. Moreover, the direction of $\boldsymbol{u}$ on Γ_τ must be the same as, or opposite to, the direction of the rigid body motion $\boldsymbol{r}$. This accounts for conditions (i) and (ii), where the latter ensures that there is no change in direction of $\boldsymbol{u}+\lambda\boldsymbol{r}$ as λ passes from 0 to 1. Incidentally, this requires that $\boldsymbol{u}+\lambda\boldsymbol{r}$ does not vanish anywhere on Γ_τ when $0 < \lambda < 1$, so that there is sliding on every part of the bed that generates friction. Condition iv simply states that friction generated by sliding at the base of the glacier balances component of gravitational force or the torque associated with the translation or rotation represented by $\boldsymbol{r}$.

One can take this argument further, though this is not central to the main thrust of this paper, which we continue in the next section (§5). Dwelling on the issue of uniqueness, we note an interesting consequence of condition (iv) of theorem

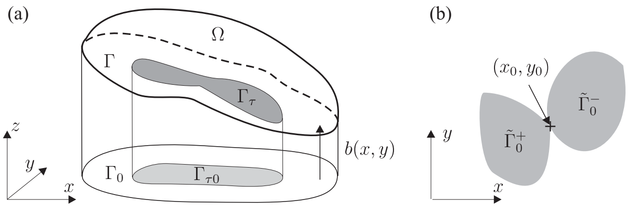

Fig. 3. Geometry for corollary 4.1. Panel (b) shows that $\tilde{\Gamma}_0^+ \cup \tilde{\Gamma}_0^- \cup \{(x_0, y_0)\}$ cannot be open.

(4.3): Γ^+ and Γ^- must both have positive measure for a non-unique solution to exist if the strong solvability condition (4.4) holds. In other words, in order for force and torque balance to be satisfied, there must be areas at the base of the glacier where the velocity field is aligned with $\boldsymbol{r}$ and areas where it points in the opposite direction to $\boldsymbol{r}$. For the Coulomb friction law this is immediately obvious: suppose to the contrary that $\operatorname{mes}(\Gamma^-) = 0$ so that, up to a set of measure zero, $\Gamma_\tau = \Gamma^+$. Then (4.5) reads $\int_\Gamma \tau_c |\boldsymbol{r}| \, \mathrm{d}\Gamma - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{r} \, \mathrm{d}\Omega = 0$, which contradicts (4.4). A similar argument shows that we also cannot have $\operatorname{mes}(\Gamma^+) = 0$. For the regularized Coulomb friction law, note that ψ is increasing and Lipschitz continuous with unit Lipschitz constant in its first argument. Suppose then that $\operatorname{mes}(\Gamma^-) = 0$, $\Gamma_\tau = \Gamma^+$ in (4.6). It follows that $\int_\Gamma \tau_c |\boldsymbol{r}| \, \mathrm{d}\Gamma - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{r} \, \mathrm{d}\Omega \leq 0$, which again contradicts (4.4). Hence we cannot have $\operatorname{mes}(\Gamma^-) = 0$, and a similar argument once more shows that Γ^+ also has non-zero measure.

The decomposition of Γ_τ into two non-void disjoint sets Γ^+ and Γ^- in condition ii of theorem 4.3 then suggests that, under certain regularity assumptions, solutions must be unique when the set Γ_τ is open and connected in an appropriate topology (see figure 3; an example of this would be the case in which Γ_τ corresponds to the entire base of the glacier):

Corollary 4.1. *Let Γ_0 and $\Gamma_{\tau 0}$ be the projections of Γ and Γ_τ onto the xy-plane. Assume that there is a single-valued, continuous function b defined on Γ_0 such that $\Gamma = \{(x, y, z) \in \mathbb{R}^3 : z = b(x, y)\}$, and assume that $\Gamma_{\tau 0}$ is an open connected subset of $\mathbb{R}^2$. Let the problem (3.2a) or (3.2b) have a solution $\boldsymbol{u}$ that is continuous on Γ_τ. Then the solution is unique.*

Proof. Suppose to the contrary that $\boldsymbol{u}$ is non-unique, so that $\boldsymbol{u} = \zeta \boldsymbol{r}$ on Γ_τ for a suitable scalar function ζ and some non-zero rigid body motion $\boldsymbol{r}$. Note that $\boldsymbol{r}$ is continuous on $\bar{\Omega}$. If $\boldsymbol{r}$ does not vanish on Γ_τ, then $\zeta = \boldsymbol{r} \cdot \boldsymbol{u} / |\boldsymbol{r}|^2$ is also continuous on Γ_τ. As b is continuous and the intervals $[0, \infty)$ and $(-\infty, -1]$ are closed, it follows that the projections Γ_0^+ and Γ_0^- of Γ^+ and Γ^- onto the xy-plane are non-void, disjoint and closed in the relative topology of $\Gamma_{\tau 0}$ (viewed as a subspace of $\mathbb{R}^2$). As $\Gamma_{\tau 0} = \Gamma_0^+ \cup \Gamma_0^-$, this contradicts the assumption that $\Gamma_{\tau 0}$ is connected. Suppose then that $\boldsymbol{r}$ vanishes somewhere on Γ_τ; from the definition of rigid body motions and the assumptions about the geometry of Γ, $\boldsymbol{r}$ therefore vanishes at a single point

$\boldsymbol{x}_0 = (x_0, y_0, b(x_0, y_0))$ on Γ_τ. Let $\tilde{\Gamma}_0^+ = \Gamma_0^+ \setminus \{(x_0, y_0)\}$ and $\tilde{\Gamma}_0^- = \Gamma_0^- \setminus \{(x_0, y_0)\}$. ζ is still continuous on $\Gamma_\tau \setminus \{\boldsymbol{x}_0\}$, and hence $\tilde{\Gamma}_0^+$ and $\tilde{\Gamma}_0^-$ are non-void, disjoint and closed in the relative topology of $\Gamma_{\tau 0} \setminus \{(x_0, y_0)\}$. As their union is $\Gamma_{\tau 0} \setminus \{(x_0, y_0)\}$, they are also open in that topology, and because $\Gamma_{\tau 0} \setminus \{(x_0, y_0)\}$ is an open subset of $\mathbb{R}^2$, $\tilde{\Gamma}_0^+$ and $\tilde{\Gamma}_0^-$ are disjoint and open in $\mathbb{R}^2$. As $\Gamma_{\tau 0} = \tilde{\Gamma}_0^+ \cup \tilde{\Gamma}_0^- \cup \{(x_0, y_0)\}$ is connected, (x_0, y_0) must lie on the boundary of both, $\tilde{\Gamma}_0^+$ and $\tilde{\Gamma}_0^-$. However, $\Gamma_{\tau 0}$ is also open, and hence it must be possible to find an open annulus $A = \{(x, y) \in \mathbb{R}^2 : r_1 < |(x - x_0, y - y_0)| < r_2\}$ with non-zero inner radius r_1 and centered at (x_0, y_0) such that $A \subset \Gamma_{\tau 0}$ and A intersects both, $\tilde{\Gamma}_0^+$ and $\tilde{\Gamma}_0^-$. As the annulus does not contain (x_0, y_0), we have $A = (A \cap \tilde{\Gamma}_0^+) \cup (A \cap \tilde{\Gamma}_0^-)$. The intersections $A \cap \tilde{\Gamma}_0^+$ and $A \cap \tilde{\Gamma}_0^-$ are non-void, disjoint open sets, hence showing that the annulus A cannot be connected. This is however impossible, thus proving that $\boldsymbol{u}$ must be unique. □

5. Regularizations of Coulomb friction

Our next aim is to show that solutions to the regularized Coulomb friction problem (3.2b) and the power law problem (3.2c) converge to the solution of the Coulomb friction problem (3.2a) in appropriate parametric limits, for instance when $\|\Lambda\|_{L^p(\Gamma)} \to 0$ in (2.9) or $\epsilon \to 0$ in (2.8c). Regularized friction laws of the form (2.8b) have previously been studied in Refs. 4, 21 in the context of elasticity theory, under the assumption that $\boldsymbol{u}$ vanishes on some prescribed part of the boundary $\partial\Omega$. One of the main challenges below is that we dispense with this simplifying assumption, and that we also study the power law friction case (2.8c).

In order to obtain convergence results, we require an additional monotonicity constraint on the viscosity function μ. Recall that $g(t; \boldsymbol{x}) = 4\mu(t; \boldsymbol{x})t$ if $t > 0$, $g(0; \boldsymbol{x}) = 0$. We assume that, in addition to the conditions in definition 3.1, there is a $C > 0$ such that g satisfies

$$C(1 + t + s)^{p-2}(t - s) \leq g(t; \boldsymbol{x}) - g(s; \boldsymbol{x}) \tag{5.1}$$

for all $t \geq s \geq 0$, a.e. on Ω. Again, many choices of μ satisfy (5.1), such as Glen's law (2.3) as well as (3.9) (this can be shown from the mean value theorem) and the viscosity functions in Refs. 7, 6, 16. From (5.1) there is then a $C_6 > 0$ such that (see e.g. Ref. 22, lemma 2.1)

$$\begin{aligned} &4\left(\mu(|\boldsymbol{D}(\boldsymbol{u})|; \boldsymbol{x})\boldsymbol{D}(\boldsymbol{u}) - \mu(|\boldsymbol{D}(\boldsymbol{v})|; \boldsymbol{x})\boldsymbol{D}(\boldsymbol{v}), \boldsymbol{D}(\boldsymbol{u}) - \boldsymbol{D}(\boldsymbol{v})\right) \\ &\quad \geq C_6^{-1}\left[1 + |\boldsymbol{D}(\boldsymbol{u})| + |\boldsymbol{D}(\boldsymbol{v})|\right]^{p-2} |\boldsymbol{D}(\boldsymbol{u}) - \boldsymbol{D}(\boldsymbol{v})|^2 \end{aligned} \tag{5.2}$$

almost everywhere on Ω. We then have the following general convergence result:

Theorem 5.1. *Let $\psi_\varepsilon : (0, \infty) \times \Gamma \mapsto \mathbb{R}$ be a family of functions, depending on a real parameter ε, that satisfy the following conditions: for each ε, ψ_ε is convex in its first argument a.e. on Γ, and $\psi_\varepsilon(|u|; \boldsymbol{x})$ is measureable for all $u \in L^p(\Gamma)$ such that $\tau_c\psi_\varepsilon(|u|; \boldsymbol{x}) \in L^1(\Gamma)$. For all ε sufficiently small, let there be a minimizer $\boldsymbol{u}_\varepsilon \in X$*

of

$$J_\varepsilon(\boldsymbol{v}) = \int_\Omega G(|\boldsymbol{D}(\boldsymbol{v})|;\boldsymbol{x}) - \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\Omega + \int_\Gamma \tau_c\psi_\varepsilon(|\boldsymbol{v}|;\boldsymbol{x})\,\mathrm{d}\Gamma \tag{5.3}$$

such that $\|\boldsymbol{u}_\varepsilon\| \leq R$ *for some* $R > 0$ *that is independent of* ε. *Assume also that problem (3.2a) satisfies the conditions for the existence of a unique weak solution* $\boldsymbol{u}$ *in theorems 4.2 and 4.3, and that* $\int_\Gamma \tau_c[|\boldsymbol{v}| - \psi_\varepsilon(|\boldsymbol{v}|;\boldsymbol{x})]\,\mathrm{d}\Gamma \to 0$ *as* $\varepsilon \to 0$ *uniformly in* $\boldsymbol{v}$ *for bounded subsets of* X. *Then* $\boldsymbol{u}_\varepsilon \to \boldsymbol{u}$ *strongly in* X *as* $\varepsilon \to 0$.

To prove theorem 5.1, we need a technical lemma, whose proof is provided in A.2:

Lemma 5.1. *Let* X *be a normed space and* $\phi : X \mapsto \mathbb{R}$ *be uniformly continuous on bounded subsets of* X. *Then* $\Phi(r) = \inf_{\|x\|=r}\phi(x)$ *is continuous.*

Using this, we have

Proof. (Theorem 5.1) To prove convergence, note that $\boldsymbol{u}$ satisfies (3.2a), while $\boldsymbol{u}_\varepsilon$ minimizes J_ε and therefore satisfies

$$\int_\Omega 4\mu(|\boldsymbol{D}(\boldsymbol{u}_\varepsilon)|;\boldsymbol{x})\,(\boldsymbol{D}(\boldsymbol{u}),\boldsymbol{D}(\boldsymbol{v}-\boldsymbol{u}_\varepsilon)) - \boldsymbol{f}\cdot(\boldsymbol{v}-\boldsymbol{u}_\varepsilon)\,\mathrm{d}\Omega + \int_\Gamma \tau_c\,(\psi_\varepsilon(|\boldsymbol{v}|;\boldsymbol{x}) - \psi_\varepsilon(|\boldsymbol{u}_\varepsilon|;\boldsymbol{x}))\,\mathrm{d}\Gamma \geq 0 \tag{5.4}$$

for all $\boldsymbol{v} \in X$. Putting $\boldsymbol{v} = \boldsymbol{u}_\varepsilon$ in (3.2a) and $\boldsymbol{v} = \boldsymbol{u}$ in (5.4) and adding, we have on rearranging that

$$\int_\Omega 4\,(\mu(|\boldsymbol{D}(\boldsymbol{u})|;\boldsymbol{x})\boldsymbol{D}(\boldsymbol{u}) - \mu(|\boldsymbol{D}(\boldsymbol{u}_\varepsilon)|;\boldsymbol{x})\boldsymbol{D}(\boldsymbol{u}_\varepsilon), \boldsymbol{D}(\boldsymbol{u}) - \boldsymbol{D}(\boldsymbol{u}_\varepsilon))\,\mathrm{d}\Omega \leq \int_\Gamma \tau_c\,[|\boldsymbol{u}_\varepsilon| - \psi_\varepsilon(|\boldsymbol{u}_\varepsilon|;\boldsymbol{x}) + \psi_\varepsilon(|\boldsymbol{u}|;\boldsymbol{x}) - |\boldsymbol{u}|]\,\mathrm{d}\Gamma. \tag{5.5}$$

Without loss of generality, assume that $\|\boldsymbol{u}\| < R$. For any $\nu > 0$, we can find $\varepsilon_0 > 0$ such that

$$\left|\int_\Gamma \tau_c\,[|\boldsymbol{v}| - \psi_\varepsilon(|\boldsymbol{v}|;\boldsymbol{x})]\,\mathrm{d}\Gamma\right| < \nu \tag{5.6}$$

when $\|\boldsymbol{v}\| < R$, $|\varepsilon| < \varepsilon_0$. By (5.2), Hölder's inequality and (5.6), combined with the growth estimate (3.11) and the bound on $\|\boldsymbol{u}\|$ and $\|\boldsymbol{u}_\varepsilon\|$, it therefore follows that

$$\left[\int_\Omega |\boldsymbol{D}(\boldsymbol{u}_\varepsilon - \boldsymbol{u})|^p\,\mathrm{d}\Omega\right]^{2/p} \leq 3^{2-p}C_6\left[(1+2C_1)^{1/p}\mathrm{mes}\,(\Omega)^{1/p} + (3C_1)^{1/p}R\right]^{2-p}\nu \tag{5.7}$$

when $|\varepsilon| < \varepsilon_0$. Writing $\boldsymbol{w}_\varepsilon = \boldsymbol{u}_\varepsilon - \boldsymbol{u}$ and defining $\boldsymbol{w}_{\varepsilon\mathcal{R}} = \pi_\mathcal{R}(\boldsymbol{w}_\varepsilon)$, $\tilde{\boldsymbol{w}}_\varepsilon = \boldsymbol{w}_\varepsilon - \boldsymbol{w}_{\varepsilon\mathcal{R}}$, it follows from theorem 4.1 that $\|\tilde{\boldsymbol{w}}_\varepsilon\| \to 0$ as $\varepsilon \to 0$. In fact, convergence occurs at least at the rate $\sqrt{\nu}$. It remains to show that $\boldsymbol{w}_{\varepsilon\mathcal{R}}$ also tends to zero in norm.

We do so in two steps. First, putting $\boldsymbol{v} = \boldsymbol{u}$ in (5.4) and rearranging yields

$$\int_\Gamma \tau_c \left[|\boldsymbol{u} + \boldsymbol{w}_{\varepsilon\mathcal{R}}| - |\boldsymbol{u}|\right] \mathrm{d}\Gamma + \int_\Omega \boldsymbol{f} \cdot \boldsymbol{w}_{\varepsilon\mathcal{R}} \,\mathrm{d}\Omega$$
$$\leq -\int_\Omega 4\mu(|\boldsymbol{D}(\boldsymbol{u}_\varepsilon)|;\boldsymbol{x})\,(\boldsymbol{D}(\boldsymbol{u}_\varepsilon), \boldsymbol{D}(\tilde{\boldsymbol{w}}_\varepsilon)) - \boldsymbol{f} \cdot \tilde{\boldsymbol{w}}_\varepsilon \,\mathrm{d}\Omega$$
$$+ \int_\Gamma \tau_c \left[\psi_\varepsilon(|\boldsymbol{u}|;\boldsymbol{x}) - |\boldsymbol{u}| + |\boldsymbol{u}_\varepsilon| - \psi_\varepsilon(|\boldsymbol{u}_\varepsilon|;\boldsymbol{x}) + |\boldsymbol{u}_\varepsilon - \tilde{\boldsymbol{w}}_\varepsilon| - |\boldsymbol{u}_\varepsilon|\right] \mathrm{d}\Gamma \quad (5.8)$$

The right-hand side tends to zero as ε does: for the integral over Γ, this is clear from (5.6), the bound on $\boldsymbol{u}_\varepsilon$ and $\boldsymbol{u}$ and the fact that $\|\tilde{\boldsymbol{w}}_\varepsilon\| \to 0$. For the integral over Ω, note simply that $(\mu(|\boldsymbol{D}(\boldsymbol{u}_\varepsilon)|;\boldsymbol{x})\boldsymbol{D}(\boldsymbol{u}_\varepsilon), \boldsymbol{D}(\boldsymbol{w}_{\varepsilon\mathcal{R}})) \leq |\mu(|\boldsymbol{D}(\boldsymbol{u}_\varepsilon)|;\boldsymbol{x})\boldsymbol{D}(\boldsymbol{u}_\varepsilon)||\boldsymbol{D}(\boldsymbol{w}_{\varepsilon\mathcal{R}})|$, and apply Hölder's inequality; by the growth bound (3.4) and the fact that $g(t) = 4\mu(t)t$ is an increasing function, it follows that $\mu(t)t \leq C_1' + pC_1 t^{p-1}/4$ for some $C_1' > 0$, and hence from the definition of $\boldsymbol{D}$ and the norm on X,

$$\int_\Omega |\mu(|\boldsymbol{D}(\boldsymbol{u}_\varepsilon)|;\boldsymbol{x})\boldsymbol{D}(\boldsymbol{u}_\varepsilon)|^{p/(p-1)} \,\mathrm{d}\Omega \leq 2^{1/(p-1)} C_1'^{p/(p-1)} \mathrm{mes}\,(\Omega) + \frac{3[pC_1]^{p/(p-1)}}{2^{p/(p-1)}} R^p.$$

Next, define $\phi : \mathcal{R} \mapsto \mathbb{R}$ by

$$\phi(\boldsymbol{r}) := J_1(\boldsymbol{u} + \boldsymbol{r}) - J_1(\boldsymbol{u}) = \int_\Gamma \tau_c \left[|\boldsymbol{u} + \boldsymbol{r}| - |\boldsymbol{u}|\right] \mathrm{d}\Gamma - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{r} \,\mathrm{d}\Omega. \quad (5.9)$$

From (5.8), we have that $\phi(\boldsymbol{w}_{\varepsilon\mathcal{R}}) \to 0$ as $\varepsilon \to 0$. We will show that this implies that $\|\boldsymbol{w}_{\varepsilon\mathcal{R}}\| \to \boldsymbol{0}$. As $\boldsymbol{u}$ uniquely minimizes J_1, it follows that $\phi(\boldsymbol{r}) > 0$ if $\boldsymbol{r} \neq 0$. Define $\Phi(r) = \inf_{\|\boldsymbol{r}\|=r} \phi(\boldsymbol{r})$, so that $0 \leq \Phi(\|\boldsymbol{w}_{\varepsilon\mathcal{R}}\|) \leq \phi(\boldsymbol{w}_{\varepsilon\mathcal{R}})$ and hence $\Phi(\|\boldsymbol{w}_{\varepsilon\mathcal{R}}\|) \to 0$ as $\varepsilon \to 0$. As $\mathcal{R}$ is finite-dimensional and ϕ is continuous, $\phi(\boldsymbol{r}) = \Phi(r)$ for some $\boldsymbol{r}$ with $\|\boldsymbol{r}\| = r$, and therefore $\Phi(r) > 0$ if $r > 0$ while $\Phi(0) = 0$. Moreover, from lemma 5.1, Φ is continuous. As ϕ is also convex, $\Phi(r)$ is strictly increasing in r, and hence Φ has a continuous inverse defined on some interval $[0, a)$, with $\Phi^{-1}(0) = 0$. From $\Phi(\|\boldsymbol{w}_{\varepsilon\mathcal{R}}\|) \to 0$, it therefore follows that $\|\boldsymbol{w}_{\varepsilon\mathcal{R}}\| \to 0$, as required. □

For regularized Coulomb friction laws of the form (2.8b), the following conditions suffice in order for theorem 5.1 to apply:

Lemma 5.2. *Let f_ε satisfy the conditions on f in definition 3.1, and let $\psi_\varepsilon(v;\boldsymbol{x}) = \int_0^v f_\varepsilon(u;\boldsymbol{x})\,\mathrm{d}u$. Assume that $\int_\Gamma \tau_c[|\boldsymbol{v}| - \psi_\varepsilon(|\boldsymbol{v}|;\boldsymbol{x})]\,\mathrm{d}\Gamma \to 0$ as $\varepsilon \to 0$ uniformly for $\boldsymbol{v}$ in some neighbourhoood of $\boldsymbol{0}$ in X. Then, for ε sufficiently small, $\int_\Gamma \tau_c[|\boldsymbol{v}| - \psi_\varepsilon(|\boldsymbol{v};\boldsymbol{x})]\,\mathrm{d}\Gamma \to 0$ as $\varepsilon \to 0$ uniformly for $\boldsymbol{v}$ in bounded subsets of X, and there is an $R > 0$ independent of ε such that minimizers $\boldsymbol{u}_\varepsilon$ of J_ε defined in (5.3) satisfy $\|\boldsymbol{u}_\varepsilon\| \leq R$.*

Proof. There is an $r_0 > 0$ such that, given any $\nu > 0$, we can find an ϵ_0 such that $0 \leq \int_\Gamma \tau_c\,[|\boldsymbol{v}| - \psi_\varepsilon(|\boldsymbol{v}|;\boldsymbol{x})]\,\mathrm{d}\Gamma \leq \nu/r_0$ when $\|\boldsymbol{v}\| < r_0$ and $|\epsilon| < \epsilon_0$. Recall further (see the proof of theorem 4.2) that for $\lambda > 0$, $\int_\Gamma \tau_c \left[|\boldsymbol{v}| - \psi_\varepsilon(\lambda|\boldsymbol{v}|;\boldsymbol{x})/\lambda\right] \mathrm{d}\Gamma$ is non-increasing in λ for any $\boldsymbol{v} \in X$. Thus, for all $\boldsymbol{v} \in X$,

$$\left| \int_\Gamma \tau_c \left[|\boldsymbol{v}| - \psi_\varepsilon(|\boldsymbol{v}|;\boldsymbol{x})\right] \mathrm{d}\Gamma \right| \leq \nu \|\boldsymbol{v}\| + \nu/r_0, \tag{5.10}$$

and the first half of the lemma follows.

Defining $J_\varepsilon(\boldsymbol{v}) = \int_\Omega G(|\boldsymbol{D}(\boldsymbol{v})|;\boldsymbol{x}) - \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\Omega + \int_\Gamma \tau_c\psi_\varepsilon(|\boldsymbol{v}|;\boldsymbol{x})\,\mathrm{d}\Gamma$, we again show that there is an $R > 0$ such that $J_\varepsilon(\boldsymbol{v}) > J_\varepsilon(\boldsymbol{0}) = 0$ when $\|\boldsymbol{v}\| > R$, and the second half of the lemma follows. Defining $\tilde{\boldsymbol{v}}$ and $\boldsymbol{v}_{\mathcal{R}}$ as before, we have (cf. inequality (A.6)):

$$J_\varepsilon(\boldsymbol{v}) \geq C_2C_3\|\tilde{\boldsymbol{v}}\|^p - (\|\boldsymbol{f}\| + C_5\|\tau_c\|)\|\tilde{\boldsymbol{v}}\| + \delta\|\boldsymbol{v}_{\mathcal{R}}\| - \int_\Gamma \tau_c(|\boldsymbol{v}_{\mathcal{R}}| - \psi_\varepsilon(|\boldsymbol{v}_{\mathcal{R}}|;\boldsymbol{x}))\,\mathrm{d}\Gamma - C_4. \tag{5.11}$$

Taking $\nu = \delta/2$ in (5.10),

$$J_\varepsilon(\boldsymbol{v}) \geq C_2C_3\|\tilde{\boldsymbol{v}}\|^p - (\|\boldsymbol{f}\| + C_5\|\tau_c\|)\|\tilde{\boldsymbol{v}}\| + \frac{\delta}{2}\|\boldsymbol{v}_{\mathcal{R}}\| - \left(C_4 + \frac{\delta}{2r_0}\right) \tag{5.12}$$

for $|\varepsilon|$ sufficiently small. This is of the same form as (A.3), and the existence of the bound R follows from (A.5). □

Many regularizations of Coulomb friction satisfy the conditions of lemma 5.2 (see also Refs. 4, 21). Here we focus on (2.9), (3.7) and (3.8). For (2.9) with $p \in (1,2)$, note that

$$\begin{aligned} |\boldsymbol{v}| - \psi(|\boldsymbol{v}|;\boldsymbol{x}) &= \int_0^{|\boldsymbol{v}|} 1 - \left(\frac{u}{u+\Lambda}\right)^{p-1} \mathrm{d}u \leq \int_0^{|\boldsymbol{v}|} \left(\frac{\Lambda}{u+\Lambda}\right)^{p-1} \mathrm{d}u \\ &= \frac{1}{p}\left[(|\boldsymbol{v}|+\Lambda)^{2-p}\Lambda^{p-1} - \Lambda\right] \leq \frac{1}{p}|\boldsymbol{v}|^{2-p}\Lambda^{p-1}, \end{aligned} \tag{5.13}$$

and hence $\int_\Gamma \tau_c\left[|\boldsymbol{u}| - \psi(|\boldsymbol{u}|;\boldsymbol{x})\right] \mathrm{d}\Gamma \leq p^{-1}C_5\|\tau_c\|\|\boldsymbol{v}\|^{2-p}\|\Lambda\|^{p-1} \to 0$ uniformly on any bounded subset of X as $\|\Lambda\| \to 0$, where C_5 is again the norm of the trace operator. For (3.7), we have $\psi(u;\boldsymbol{x}) = \sqrt{u^2+\Lambda^2} - |\Lambda|$ and so

$$\begin{aligned} \int_\Gamma \tau_c\left[|\boldsymbol{u}| - \psi(|\boldsymbol{u}|;\boldsymbol{x})\right] \mathrm{d}\Gamma &= \int_\Gamma \tau_c\left[|\boldsymbol{u}| + |\Lambda| - \sqrt{|\boldsymbol{u}|^2+\Lambda^2}\right] \mathrm{d}\Gamma \\ &\leq \int_\Gamma \tau_c|\Lambda|\,\mathrm{d}\Gamma \leq \|\tau_c\|\|\Lambda\|, \end{aligned} \tag{5.14}$$

which converges uniformly on X as $\|\Lambda\| \to 0$ Lastly, (3.8) is a generalization of equation (2.6) in Ref. 4. We have

$$\psi(u;\boldsymbol{x}) = \begin{cases} u - \frac{\tau_c}{2C} & \text{if } u \geq \frac{\tau_c}{C}, \\ \frac{Cu^2}{\tau_c} & \text{if } u < \frac{\tau_c}{C}. \end{cases}$$

Let $\Gamma_C = \{\boldsymbol{x} \in \Gamma : |\boldsymbol{u}(\boldsymbol{x})| < \tau_c(\boldsymbol{x})/C(\boldsymbol{x})\}$. Then

$$\begin{aligned}\int_\Gamma \tau_c\left[|\boldsymbol{u}| - \psi(|\boldsymbol{u}|;\boldsymbol{x})\right] \mathrm{d}\Gamma &= \int_{\Gamma_C} \tau_c\left[|\boldsymbol{u}| - \psi(|\boldsymbol{u}|;\boldsymbol{x})\right] \mathrm{d}\Gamma + \int_{\Gamma\setminus\Gamma_C} \tau_c\left[|\boldsymbol{u}| - \psi(|\boldsymbol{u}|;\boldsymbol{x})\right] \mathrm{d}\Gamma \\ &= \int_{\Gamma_C}\left[\tau_c|\boldsymbol{u}| - \frac{C|\boldsymbol{u}|^2}{2}\right]\mathrm{d}\Gamma + \int_{\Gamma\setminus\Gamma_C}\frac{\tau_c^2}{2C}\,\mathrm{d}\Gamma \\ &\leq \int_\Gamma \frac{\tau_c^2}{2C}\,\mathrm{d}\Gamma \leq \frac{1}{2}\left\|\tau_c\right\|^2\left\|C^{-1}\right\|, \end{aligned} \tag{5.15}$$

which again converges uniformly on X as $\left\|C^{-1}\right\| \to 0$. More generally, we have the following general result, which covers all three examples above, although in somewhat weaker form as it relies on the dominated convergence theorem and does not give an explicit rate of convergence:

Theorem 5.2. *Suppose that f_ε satisfies the conditions on f in definition 3.1, and let $\psi_\varepsilon(v;\boldsymbol{x}) = \int_0^v f_\varepsilon(u;\boldsymbol{x})\,\mathrm{d}u$. Assume that $\psi_\varepsilon(u;\boldsymbol{x}) \to u$ as $\varepsilon \to 0$ pointwise in u and a.e. on Γ. Then $\int_\Gamma \tau_c[|\boldsymbol{v}| - \psi_\varepsilon(|\boldsymbol{v}|;\boldsymbol{x})]\,\mathrm{d}\Gamma \to 0$ as $\varepsilon \to 0$ uniformly in $\boldsymbol{v}$ on bounded subsets of X.*

Proof. Given any $\delta > 0$ and $\varepsilon > 0$, let $\Gamma_\varepsilon = \{\boldsymbol{x} \in \Gamma : 1 - \psi_\varepsilon(u;\boldsymbol{x})/u < \delta$ when $u \geq \delta\}$. As $1 - \psi_\varepsilon(u/\boldsymbol{x})/u$ is a decreasing function of u, we have $\Gamma_\varepsilon = \{\boldsymbol{x} \in \Gamma : 1 - \psi_\varepsilon(u;\boldsymbol{x})/u < \delta$ when $u = \delta\}$. Let $\chi_{\Gamma_\varepsilon}$ be the characteristic function of Γ_ε, $\chi_{\Gamma_\varepsilon}(\boldsymbol{x}) = 1$ if $\boldsymbol{x} \in \Gamma_\varepsilon$ and 0 otherwise. From the assumptions of the theorem, $\chi_{\Gamma_\varepsilon} \to 0$ as $\varepsilon \to 0$ at fixed δ a.e. on Γ. Given $\boldsymbol{v} \in X$, let $\Gamma_1 = \{\boldsymbol{x} \in \Gamma \setminus \Gamma_\varepsilon : |\boldsymbol{v}(\boldsymbol{x})| \geq \delta$ in the trace sense$\}$, $\Gamma_2 = \Gamma \setminus (\Gamma_\varepsilon \cup \Gamma_1)$. Then

$$\begin{aligned}\int_\Gamma \tau_c[|\boldsymbol{v}| - \psi_\varepsilon(|\boldsymbol{v}|;\boldsymbol{x})]\,\mathrm{d}\Gamma &= \left(\int_{\Gamma_1} + \int_{\Gamma_2} + \int_{\Gamma_\varepsilon}\right)\tau_c[|\boldsymbol{v}| - \psi_\varepsilon(|\boldsymbol{v};\boldsymbol{x})]\,\mathrm{d}\Gamma \\ &\leq \int_{\Gamma_1}\tau_c|\boldsymbol{v}|\delta\,\mathrm{d}\Gamma + \int_{\Gamma_2}\tau_c\delta\,\mathrm{d}\Gamma + \int_{\Gamma_\varepsilon}\tau_c|\boldsymbol{v}|\,\mathrm{d}\Gamma \\ &\leq \delta\left[C_5\left\|\tau_c\right\|\left\|\boldsymbol{v}\right\| + \left\|\tau_c\right\|\mathrm{mes}\,(\Gamma)^{1/p}\right] \\ &\quad + C_5\left[\int_\Gamma \tau_c^{p/(p-1)}\chi_{\Gamma_\varepsilon}\,\mathrm{d}\Gamma\right]^{(p-1)/p}\left\|\boldsymbol{v}\right\|, \end{aligned} \tag{5.16}$$

By the dominated convergence theorem, the last term on the last line of (5.16) tends to zero as $\varepsilon \to 0$ for any fixed δ. The desired result follows. □

Next, we turn to the power-law friction case (3.2c), and consider the limit $\epsilon \to 0$. Again, we can show convergence to the solution of (3.2a) under very general conditions:

Theorem 5.3. *Let $\tau_c \in L^{p_{\epsilon_0}/(p_{\epsilon_0}-1)}$ for some $p_{\epsilon_0} = p/(1+\epsilon_0)$, where $p-1 \geq \epsilon_0 > 0$, and $\tau_c \geq 0$ a.e. on Γ. With τ_c fixed, let problem (3.2a) satisfy the conditions for the existence of a unique solution $\boldsymbol{u}$ in theorems 4.2 and 4.3. Let $\boldsymbol{u}_\epsilon$ solve problem (3.2c) with $0 < \epsilon < \epsilon_0$. Then $\boldsymbol{u}_\epsilon \to \boldsymbol{u}$ strongly in X as $\epsilon \to 0$.*

Proof. Let $\psi_\epsilon(v) = v^{1+\epsilon}/(1+\epsilon)$. We will show that ψ_ϵ satisfies the conditions of theorem 5.1. $\boldsymbol{u}_\epsilon$ minimizes

$$J_\epsilon(\boldsymbol{v}) = \int_\Omega G(|\boldsymbol{D}(\boldsymbol{v})|;\boldsymbol{x}) - \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\Omega + \int_\Gamma \tau_c\psi_\epsilon(|\boldsymbol{v}|)\,\mathrm{d}\Gamma. \tag{5.17}$$

First, we prove that there is an $R>0$ such that $\|\boldsymbol{u}_\epsilon\| \leq R$ for all ϵ sufficiently small. Define

$$\tilde{\psi}_\epsilon(u) = \begin{cases} u^{1+\epsilon}/(1+\epsilon) & \text{when } 0 \leq u < 1, \\ u - \epsilon/(1+\epsilon) & \text{when } u \geq 1, \end{cases}$$

so that $\psi_\epsilon(v) \geq \tilde{\psi}_\epsilon(v)$ for $v \geq 0$. $\tilde{\psi}_\epsilon$ satisfies the conditions of theorem 5.2, and from the proof of lemma 5.2, we see that for ϵ sufficiently small there is an R inependent of ϵ such that

$$J_\epsilon(\boldsymbol{v}) \geq \int_\Omega G(|\boldsymbol{D}(\boldsymbol{v})|;\boldsymbol{x}) - \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\Omega + \int_\Gamma \tau_c\tilde{\psi}_\epsilon(|\boldsymbol{v}|)\,\mathrm{d}\Gamma > 0 \tag{5.18}$$

when $|\boldsymbol{v}| > R$. Hence $\|\boldsymbol{u}_\epsilon\| \leq R$

Next, we show that $\int_\Gamma \tau_c\,[|\boldsymbol{v}| - \psi_\epsilon(|\boldsymbol{v}|)]\,\mathrm{d}\Gamma \to 0$ as $\epsilon \to 0$ uniformly in $\boldsymbol{v}$ on bounded subsets of X. Given $\boldsymbol{v} \in X$, define $\Gamma_1 = \{\boldsymbol{x} \in \Gamma : |\boldsymbol{v}(\boldsymbol{x})| > 1/(\epsilon|\log\epsilon|)\}$, $\Gamma_2 = \{\boldsymbol{x} \in \Gamma : \epsilon|\log\epsilon| \leq |\boldsymbol{v}(\boldsymbol{x})| \leq 1/(\epsilon|\log\epsilon|)\}$ and $\Gamma_3 = \{\boldsymbol{x} \in \Gamma : |\boldsymbol{v}(\boldsymbol{x})| < \epsilon|\log\epsilon|\}$, the inequalities in each case being understood in the trace sense. We will show that each term of the form $\left|\int_{\Gamma_i} \tau_c\,[|\boldsymbol{v}| - \psi_\epsilon(|\boldsymbol{v}|)]\,\mathrm{d}\Gamma\right|$, $i = 1,2,3$, converges uniformly as $\epsilon \to 0$ on bounded subsets of X.

First, we bound $\mathrm{mes}\,(\Gamma_1)$. We have $\mathrm{mes}\,(\Gamma_1)/(\epsilon|\log\epsilon|)^p < \int_{\Gamma_1} |\boldsymbol{v}|^p\,\mathrm{d}\Gamma \leq C_5^p\,\|\boldsymbol{v}\|^p$, where C_5 is the norm of the trace operator as before. Hence $\mathrm{mes}\,(\Gamma_1) < C_5^p\,\|\boldsymbol{v}\|^p\,(\epsilon|\log\epsilon|)^p$, and

$$\begin{aligned}\left|\int_{\Gamma_1} \tau_c\,[|\boldsymbol{v}| - \psi_\epsilon(|\boldsymbol{v}|)]\,\mathrm{d}\Gamma\right| &\leq \left|\int_{\Gamma_1} \tau_c\frac{|\boldsymbol{v}|^{1+\epsilon}}{1+\epsilon}\,\mathrm{d}\Gamma\right| \\ &\leq \|\tau_c\|_{L^{p\epsilon_0/(p\epsilon_0-1)}(\Gamma)}\,\mathrm{mes}\,(\Gamma_1)^{(\epsilon_0-\epsilon)/p}C_5^{1+\epsilon}\,\|\boldsymbol{v}\|^{1+\epsilon} \\ &\leq C_5^{1+\epsilon_0}\,\|\tau_c\|_{L^{p\epsilon_0/(p\epsilon_0-1)}(\Gamma)}\,\|\boldsymbol{v}\|^{1+\epsilon_0}\,(\epsilon|\log\epsilon|)^{\epsilon_0-\epsilon}, \end{aligned} \tag{5.19}$$

which clearly converges uniformly for bounded $\boldsymbol{v}$. Next, we have

$$\begin{aligned}\left|\int_{\Gamma_2} \tau_c\,[|\boldsymbol{v}| - \psi_\epsilon(|\boldsymbol{v}|)]\,\mathrm{d}\Gamma\right| &\leq \max\left(1-(\epsilon|\log\epsilon|)^\epsilon, \left(\frac{\epsilon|\log\epsilon|}{\epsilon_0}\right)^{-\epsilon} - 1\right)\int_{\Gamma_2}\tau_c|\boldsymbol{v}|\,\mathrm{d}\Gamma \\ &\leq \max\left(1-(\epsilon|\log\epsilon|)^\epsilon, (\epsilon|\log\epsilon|)^{-\epsilon} - 1\right) C_5\,\|\tau_c\|\,\|\boldsymbol{v}\|. \end{aligned} \tag{5.20}$$

For ϵ small, $\max\left(1-(\epsilon|\log\epsilon|)^\epsilon, (\epsilon|\log\epsilon|)^{-\epsilon} - 1\right) \sim \epsilon|\log(\epsilon|\log\epsilon|)|$ and the last term in (5.20) converges uniformly for bounded $\boldsymbol{v}$. Lastly, we have

$$\begin{aligned}\left|\int_{\Gamma_3} \tau_c\,[|\boldsymbol{v}| - \psi_\epsilon(|\boldsymbol{v}|)]\,\mathrm{d}\Gamma\right| &\leq \left|\int_{\Gamma_3}\tau_c|\boldsymbol{v}|\,\mathrm{d}\Gamma\right| \\ &\leq \|\tau_c\|_{L^{p/(p-1)}(\Gamma)}\,\mathrm{mes}\,(\Gamma)^{1/p}\epsilon|\log\epsilon|, \end{aligned} \tag{5.21}$$

which converges uniformly on X. The desired result follows. □

Thus we obtain convergence to the Coulomb friction law solution for both, regularized Coulomb friction laws and power laws. In closing, it is worth pointing out that there is practical uncertainty not only over the form of the subglacial friction law, but also over the precise rheology of ice, with various viscosity functions μ having been applied in the context of Blatter's model.[7,16,28] All of these are essentially regularized forms of the power law (2.3). A question which can be addressed using the same approach as above is: does the solution of Blatter's model with a regularized Glen's law viscosity converge to the solution for Glen's law (e.g., (3.9) in the limit $D_0 \to 0$)? Using the same method as above (though we do not present a proof here due to space restrictions), one can show that the answer is yes if the regularized viscosity μ_ε is such that $\int_\Omega |\mu(|\boldsymbol{D}(\boldsymbol{v})|;\boldsymbol{x})\boldsymbol{D}(\boldsymbol{v}) - \mu_\varepsilon(|\boldsymbol{D}(\boldsymbol{v})|;\boldsymbol{x})\boldsymbol{D}(\boldsymbol{v})|^{p/(p-1)}\,\mathrm{d}\Gamma \to 0$ uniformly in X as $\varepsilon \to 0$.

6. Remarks on finite element approximations

Extensive studies of finite element approximations of Blatter's model with simpler boundary conditions than those employed here can be found in Refs. 7, 6, 16, 28, while results on Coulomb friction problems posed in $[H^1(\Omega)]^3$ can be found in Campos *et al.*[4] and Kikuchi and Odean[21], chapter 10, again under the assumption that the solution $\boldsymbol{u}$ vanishes on a fixed part of the boundary of positive measure. As the use Blatter's model is mostly as a numerical simulation tool, it seems appropriate to adapt these results to the sliding scenarios considered here. The main difficulty lies in the complications caused by dispensing with the assumption[4,21] of $\boldsymbol{u}$ vanishing on a fixed part of $\partial\Omega$.

Let $X_h \subset X$ be a family of finite dimensional subspaces of X with the property that $\lim_{h\to 0}\inf_{\boldsymbol{v}_h\in X_h}\|\boldsymbol{v}-\boldsymbol{v}_h\|_X = 0$ for all $\boldsymbol{v} \in X$. As an approximation of the minimizer $\boldsymbol{u}$ of any one of the J_i's ($i = 1,2,3$) over X, we consider the $\boldsymbol{u}_h \in X_h$ that minimizes the same J_i over X_h. It is straightforward to show that such a minimizer exists for each subspace X_h, and that the bound on $\boldsymbol{u}$ in theorem 4.2 also serves as a bound on $\boldsymbol{u}_h$, independently of h. Moreover, it can be shown that $\boldsymbol{u}_h \to \boldsymbol{u}$ strongly as $h \to 0$. We demonstrate this for minimizers of J_2 and J_1, along the same lines as the proof of theorem 5.1 above. Note that $\boldsymbol{u}_h$ satisfies

$$\int_\Omega 4\mu(|\boldsymbol{D}(\boldsymbol{u}_h)|;\boldsymbol{x})(\boldsymbol{D}(\boldsymbol{u}_h),\boldsymbol{D}(\boldsymbol{v}_h-\boldsymbol{u}_h)) - \boldsymbol{f}\cdot(\boldsymbol{v}_h-\boldsymbol{u}_h)\,\mathrm{d}\Gamma + \int_\Gamma \tau_c\,[\psi(|\boldsymbol{v}_h|;\boldsymbol{x}) - \psi(|\boldsymbol{u}_h|;\boldsymbol{x})]\,\mathrm{d}\Gamma \geq 0 \qquad \forall \boldsymbol{v}_h \in X_h \tag{6.1}$$

while $\boldsymbol{u}$ satisfies the equivalent statement (3.2), which can also be cast in the from (6.1) with $\boldsymbol{u}$ replacing $\boldsymbol{u}_h$ and $\boldsymbol{v}_h$ ranging over all of X. In (6.1), ψ either satisfies the conditions in definition 3.1, or $\psi(|\boldsymbol{v}|;\boldsymbol{x}) := |\boldsymbol{v}|$ as appropriate for the Coulomb friction law. In either case, we note that ψ is Lipschitz continuous with unit Lipschitz constant in its first argument. Let $\boldsymbol{v}_h \in X$ be the closest approximation to $\boldsymbol{u}$ in X_h, and put $\boldsymbol{v} = \boldsymbol{u}_h$ in the equivalent statement to (6.1) for $\boldsymbol{u}$. Adding the left-hand sides of these inequalities and re-arranging,

$$\begin{aligned}\int_\Omega 4\left(\mu(|\boldsymbol{D}(\boldsymbol{u}_h)|;\boldsymbol{x})\boldsymbol{D}(\boldsymbol{u}_h) - \mu(|\boldsymbol{D}(\boldsymbol{u})|;\boldsymbol{x})\boldsymbol{D}(\boldsymbol{u}), \boldsymbol{D}(\boldsymbol{u}_h - \boldsymbol{u})\right) \mathrm{d}\Omega \\ \leq \int_\Omega 4\mu(|\boldsymbol{D}(\boldsymbol{u})|;\boldsymbol{x})(\boldsymbol{D}(\boldsymbol{u}), \boldsymbol{D}(\boldsymbol{v}_h - \boldsymbol{u})) - \boldsymbol{f}\cdot(\boldsymbol{v}_h - \boldsymbol{u})\,\mathrm{d}\Omega \\ + \int_\Gamma \tau_c\left[\psi(|\boldsymbol{v}_h|;\boldsymbol{x}) - \psi(|\boldsymbol{u}|;\boldsymbol{x})\right] \mathrm{d}\Gamma \\ + \int_\Omega 4\left(\mu(|\boldsymbol{D}(\boldsymbol{u}_h)|;\boldsymbol{x})\boldsymbol{D}(\boldsymbol{u}_h) - \mu(|\boldsymbol{D}(\boldsymbol{u})|;\boldsymbol{x})\boldsymbol{D}(\boldsymbol{u}), \boldsymbol{D}(\boldsymbol{v}_h - \boldsymbol{u})\right) \mathrm{d}\Omega\end{aligned} \tag{6.2}$$

Let $\boldsymbol{w}_h = \boldsymbol{u}_h - \boldsymbol{u}$, and let $\boldsymbol{w}_{h\mathcal{R}} = \pi_\mathcal{R}(\boldsymbol{w}_h)$, $\tilde{\boldsymbol{w}}_h = \boldsymbol{w}_h - \boldsymbol{w}_{h\mathcal{R}}$ as before. Using $|\psi(|\boldsymbol{v}_h|;\boldsymbol{x}) - \psi(|\boldsymbol{u}|;\boldsymbol{x})| \leq |\boldsymbol{v}_h - \boldsymbol{u}|$ and applying Hölder's inequality on the right-hand side of (6.2) as well as using (5.2) combined with Hölder's inequality and the bound on $\boldsymbol{u}$ and $\boldsymbol{u}_h$ on the left-hand side as in the derivation of (5.7), it follows from $\lim_{h\to 0}\|\boldsymbol{u} - \boldsymbol{v}_h\| = 0$ that $\tilde{\boldsymbol{w}}_h \to 0$ in norm. To demonstrate that we also obtain $\|\boldsymbol{w}_{h\mathcal{R}}\| \to 0$, we simply note that, from (6.1),

$$\begin{aligned}\int_\Gamma \tau_c\left[\psi(|\boldsymbol{u} + \boldsymbol{w}_{h\mathcal{R}}|;\boldsymbol{x}) - \psi(|\boldsymbol{u}|;\boldsymbol{x})\right] \mathrm{d}\Gamma - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{w}_{h\mathcal{R}}\,\mathrm{d}\Omega \\ \leq \int_\Omega 4\mu(|\boldsymbol{D}(\boldsymbol{u}_h)|;\boldsymbol{x})\left(\boldsymbol{D}(\boldsymbol{u}_h), \boldsymbol{D}(\boldsymbol{v}_h - \boldsymbol{u} - \tilde{\boldsymbol{w}}_h)\right) - \boldsymbol{f}\cdot(\boldsymbol{v}_h - \boldsymbol{u} - \tilde{\boldsymbol{w}}_h)\,\mathrm{d}\Omega \\ + \int_\Gamma \tau_c\left[\psi(|\boldsymbol{v}_h|;\boldsymbol{x}) - \psi_\varepsilon(|\boldsymbol{u}|;\boldsymbol{x}) + \psi(|\boldsymbol{u}_h - \tilde{\boldsymbol{w}}_h|;\boldsymbol{x}) - \psi(|\boldsymbol{u}_h|;\boldsymbol{x})\right] \mathrm{d}\Gamma,\end{aligned} \tag{6.3}$$

where the right-hand side tends to zero as $h \to 0$ as in (5.8) if we note again the Lipschitz continuity of ψ. Using this, we can follow the last part of the proof of theorem 5.1, putting

$$\phi(\boldsymbol{r}) := \int_\Gamma \tau_c\left[\psi(|\boldsymbol{u} + \boldsymbol{r}|;\boldsymbol{x}) - \psi(|\boldsymbol{u}|;\boldsymbol{x})\right] \mathrm{d}\Gamma - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{r}\,\mathrm{d}\Omega.$$

Explicit *a priori* error estimates in terms of maximum element size h are harder to come by. As the behaviour near the origin of the function $\Phi(r) := \inf_{\boldsymbol{r}\in\mathcal{R}:\|\boldsymbol{r}\|=r}\phi(\boldsymbol{r})$ cannot be constrained further without additional specifications, such an estimate can only be found for $\tilde{\boldsymbol{w}}_h$. From (6.2), we can directly find such an estimate in the form

$$\|\tilde{\boldsymbol{w}}_h\|^2 \leq C\,\|\boldsymbol{v}_h - \boldsymbol{u}\| \tag{6.4}$$

for some $C > 0$. This estimate, for which we have only required the constraint (5.1), is however suboptimal as it implies only a rate of convergence as $\sqrt{\|\boldsymbol{v}_h - \boldsymbol{u}\|}$. Hence we cannot guarantee that $\tilde{\boldsymbol{w}}_h$ tends to zero at the same rate as the difference between $\boldsymbol{u}$ and its closest approximation $\boldsymbol{v}_h \in X_h$.

As we show in theorem 6.1 below, it is possible to improve on this estimate for certain viscosity functions and if the solution $\boldsymbol{u}$ satisfies certain regularity requirements. In order to apply the theory of Ref. 2, we make the additional continuity assumption on μ that there is a $C > 0$ such that

$$|g(t;\boldsymbol{x}) - g(s;\boldsymbol{x})| \leq C|t-s|[1+t+s]^{p-2}, \tag{6.5}$$

for all s, $t \geq 0$ a.e. on Γ. This is satisfied by (3.9) with D_0 = constant and by the viscosity functions in Refs. 7, 6, 16 but not by Glen's power law (2.3). Then (see lemma 2.1 of Ref. 22) there is a $C_7 > 0$ such that

$$4|\mu(|\boldsymbol{D}(\boldsymbol{u})|;\boldsymbol{x})\boldsymbol{D}(\boldsymbol{u}) - \mu(|\boldsymbol{D}(\boldsymbol{v})|;\boldsymbol{x})\boldsymbol{D}(\boldsymbol{v})| \leq C_7|\boldsymbol{D}(\boldsymbol{u}-\boldsymbol{v})|[1+|\boldsymbol{D}(\boldsymbol{u})|+|\boldsymbol{D}(\boldsymbol{v})|]^{p-2} \tag{6.6}$$

a.e. on Γ for all $\boldsymbol{u}$, $\boldsymbol{v} \in X$. Following the proofs of theorem 1 in Ref. 16 and of theorem 3.1 in [28] we can use (5.2), (6.6) and lemma 2.2 of Ref. 2 to show that (6.2) implies the existence of an $M > 0$, depending on the bound on $\|\boldsymbol{u}\|$ and $\|\boldsymbol{u}_h\|$, such that

$$\begin{aligned}\|\tilde{\boldsymbol{w}}_h\|^2 &\leq \left(C_3^{-1}\int_\Omega |\boldsymbol{D}(\boldsymbol{u}_h-\boldsymbol{u})|^p \ \mathrm{d}\Omega\right)^{2/p} \\ &\leq M\Big(\|\boldsymbol{v}_h-\boldsymbol{u}\|^2 + \int_\Omega 4\mu(|\boldsymbol{D}(\boldsymbol{u})|;\boldsymbol{x})(\boldsymbol{D}(\boldsymbol{u}),\boldsymbol{D}(\boldsymbol{v}_h-\boldsymbol{u})) - \boldsymbol{f}\cdot(\boldsymbol{v}_h-\boldsymbol{u})\,\mathrm{d}\Omega \\ &\quad + \int_\Gamma \tau_c|\boldsymbol{v}_h-\boldsymbol{u}|\,\mathrm{d}\Gamma\Big)\end{aligned} \tag{6.7}$$

This still remains suboptimal because of the second and third terms on the right-hand side, which again indicate a rate of convergence as $\sqrt{\|\boldsymbol{v}_h-\boldsymbol{u}\|}$. Adapting theorem 10.5 of Ref. 21, we can however show that higher regularity at the boundary finally leads to a better convergence estimate.

Theorem 6.1. *Let the X_h consist of piecewise linear functions defined on regular tetrahedronizations of the domain Ω, which we assume to be polyhedral, and let h denote mesh size. Assume that $\boldsymbol{u} \in H^2(\Omega)$, in which case we can define a shear stress $\boldsymbol{\Sigma}^t(\boldsymbol{u}) \in [H^{-1/2}(\partial\Omega)]^3$ generated by $\boldsymbol{u}$ on $\partial\Omega$ such that (cf. [21], theorem 5.9)*

$$\int_\Omega 4\mu(|\boldsymbol{D}(\boldsymbol{u})|;\boldsymbol{x})(\boldsymbol{D}(\boldsymbol{u}),\boldsymbol{D}(\boldsymbol{v}_h-\boldsymbol{u}))-\boldsymbol{f}\cdot(\boldsymbol{v}_h-\boldsymbol{u})\,\mathrm{d}\Omega = \int_\Gamma \boldsymbol{\Sigma}^t(\boldsymbol{u})\cdot(\boldsymbol{u}-\boldsymbol{v}_h)\,\mathrm{d}\Gamma. \tag{6.8}$$

Assume that $\boldsymbol{\Sigma}^t(\boldsymbol{u}) \in [H^{1/2}(\partial\Omega)]^3$ and $\tau_c = H^{1/2}(\Gamma)$ Then $\|\tilde{\boldsymbol{w}}_h\| \leq Ch$, with C dependent on the bound on $\|\boldsymbol{u}\|$ and $\|\boldsymbol{u}_h\|$ but independent of h.

Proof. From (6.8)

$$\begin{aligned}&\left|\int_\Omega 4\mu(|\boldsymbol{D}(\boldsymbol{u})|;\boldsymbol{x})(\boldsymbol{D}(\boldsymbol{u}),\boldsymbol{D}(\boldsymbol{v}_h-\boldsymbol{u})) - \boldsymbol{f}\cdot(\boldsymbol{v}_h-\boldsymbol{u})\,\mathrm{d}\Omega + \int_\Gamma \tau_c|\boldsymbol{v}_h-\boldsymbol{u}|\,\mathrm{d}\Gamma\right| \\ &\qquad \leq \left(\left\|\boldsymbol{\Sigma}^t(\boldsymbol{u})\right\|_{[H^{1/2}(\partial\Omega)]^3} + \|\tau_c\|_{H^{1/2}(\Gamma)}\right)\|\boldsymbol{v}_h-\boldsymbol{u}\|_{[H^{-1/2}(\partial\Omega)]^3},\end{aligned} \tag{6.9}$$

and from standard interpolation estimates for $\boldsymbol{u} \in H^2(\Omega)$, $\|\boldsymbol{v}_h-\boldsymbol{u}\|_{[H^{-1/2}(\partial\Omega)]^3} \leq Ch^2\|\boldsymbol{u}\|_{[H^{3/2}(\partial\Omega)]^3}$, while $\|\boldsymbol{v}_h-\boldsymbol{u}\|_X \leq Ch^2\|\boldsymbol{u}\|_{H^2(\Omega)}$. The desired result follows.□

We reiterate, however, that this estimate does not apply to the rigid body motion part $\boldsymbol{w}_{h\mathcal{R}}$ of the difference between $\boldsymbol{u}_h$ and $\boldsymbol{u}$. Similarly, we note that one can adapt the *a posteriori* error estimator in Ref. 1 to the present problem to bound $\tilde{\boldsymbol{w}}_h$ (following theorem 3.2 of Ref. 28), but there is no obvious way of extending this to $\boldsymbol{w}_{h\mathcal{R}}$.

7. Numerical examples and conclusions

This paper has focused on two aspects of the friction problem described in the introduction. Firstly, we considered the existence and uniqueness of solutions to this friction problem for a variety of friction problems. Existence of solutions is ensured by a classical force and torque balance inequality constraint of the form (4.4), while non-unique solutions are possible under very specialized circumstances described in theorem 4.3, with its corollary 4.1 providing sufficient conditions for a unique solution to exist. Secondly, we presented a convergence analysis for a variety of different friction laws in §5. This analysis showed that solutions obtained for these different friction laws converge to the solution for the Coulomb friction law in appropriate parametric limits. From a practical perspective, this is important as it allows numerical solutions to be computed using smoother friction laws than a classical Coulomb friction law. To extend these results, we also applied the approach taken in the convergence analysis to the analysis of finite element approximations as described in §6. The main challenge in all parts of the paper was to deal with rigid body motions, i.e., to understand the relevance of translations and rotations that leave the stress field unchanged. As mentioned, rigid body motions can generate non-unique solutions under some very specialized circumstances as shown in theorem 4.3. More practically, they complicate the task of studying convergence under parameter changes and of deriving quantitative *a priori* error estimates for numerical solutions.

In closing, we present a few numerically computed ice flow solutions to illustrate some of our results. As our main focus in this paper has not been numerical, we have not attempted a numerical error estimation along the lines of other previous work.[6,16,28] Instead, our aim is to illustrate graphically the possibility of multiple solutions, the conditionality of existence of solutions on a force balance criterion of the form (4.3), and the convergence of results for parametric limits. Some of these are much simpler to illustrate in the restriction of the Blatter's model to two dimensions, as studied in Refs. 7, 6, 16, which corresponds to finding a (scalar) minimizer $v \in W^{1,p}(\Omega)$

$$J_i(v) = \int_\Omega G\left(\sqrt{\left(\frac{\partial v}{\partial x}\right)^2 + \frac{1}{4}\left(\frac{\partial v}{\partial z}\right)^2}\right) - fv \, \mathrm{d}\Omega + j_i(v), \tag{7.1}$$

where Ω is a two-dimensional domain and Γ part of its boundary curve $\partial\Omega$, while the j_i are in (3.10b)–(3.10d) are adapted appropriately. All results in the present paper carry over to this case, if we note that the set of rigid body motions simply becomes $\mathcal{R} = \{r \in W^{1,p}(\Omega) : r = \text{ constant}\}$, and Korn's inequality is replaced by Poincaré's inequality.

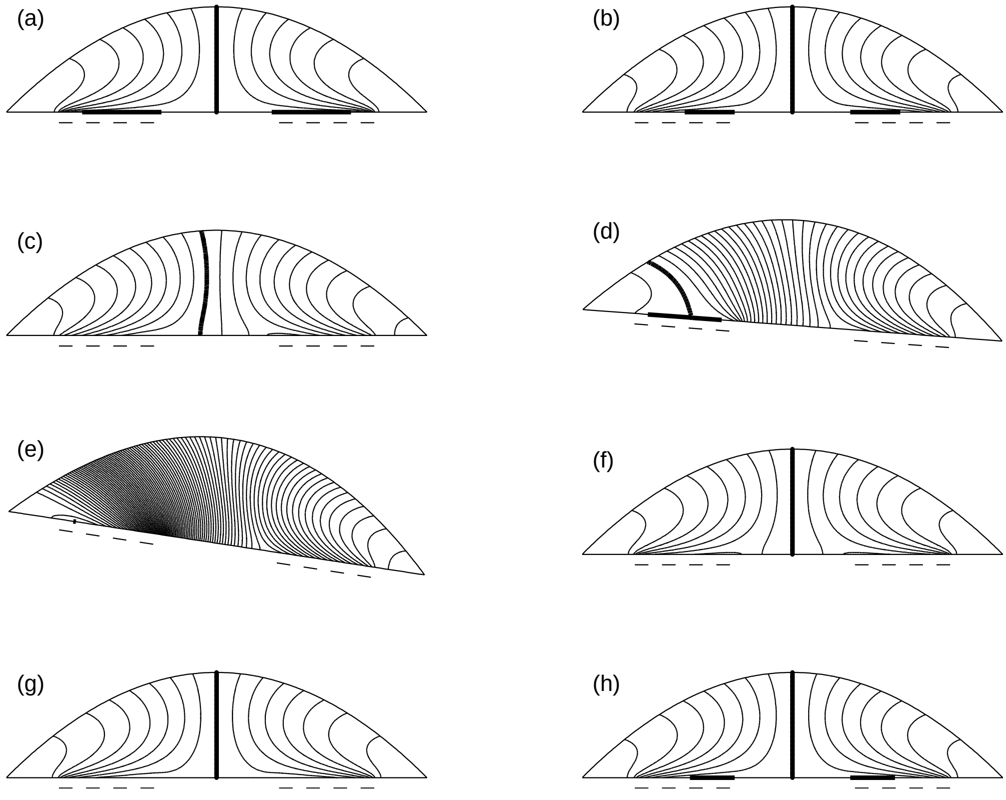

Fig. 4. Contour plots of velocity u as described in the text.

To illustrate our results, we choose the domain bounded by the curves $z = 0$ (which also defnes the lower boundary Γ) and $z = h(x) = 1 - x^2/4$, and use $f = f_0 - h'(x) = f_0 - x/2$. In terms of Blatter's original formulation,[3] f_0 can be interpreted as a downslope component of gravity due to a mean glacier inclination, and $h'(x)$ as the effect of surface slope deviating from this mean slope, causing an additional pressure gradient that drives the flow (note, however, that our domain probably does not qualify as a 'thin film', for which Blatter's model was devised; the chosen domain simply makes it easier to visualize the results). We also use μ defined by (3.9) with $p = 4/3$, $B = 0.1$, $D_0 = 10^{-4}$ and let $\tau_c(x) = \tau_0$ if $1/2 < |x| < 3/2$, $\tau_c(x) = 0$ otherwise, where $\tau_0 > 0$ is a constant. Note that the set $\Gamma_\tau = (-3/2, -1/2) \cup (1/2, 3/2)$ is deliberately chosen not to be connected, as this potentially allows multiple solutions.

Numerical results are displayed in figure 4. The figure shows contour plots of velocity. Heavy solid lines indicate the zero velocity contour and the parts of the boundary on which $|u| < 10^{-4}$ (which we interpret as parts of Γ on which there is no sliding in the Coulomb friction case). Contour intervals are 0.5, with positive velocities to the right of the zero contours and negative velocities to the left. Dashed lines below the bottom of the domain indicate Γ_τ.

Panels (a-e) show results for the Coulomb friction law (2.8a), computed through a sequence of regularised problems using (3.7), terminating with $\Lambda \equiv 10^{-6}$. Panels (a-c) have $f_0 = 0$, $\tau_0 = 0.3$ (a), 0.25 (b) and 0.2 (c). With $f_0 = 0$, we have $\int_\Gamma f r \, \mathrm{d}\Gamma = 0$ for all $r \equiv$ constant, and hence solvability is ensured by (4.4). However, as τ_0 is lowered, sliding occurs on larger parts of Γ (panels (a) and (b)), until sliding occurs everywhere at the base in panel (c), with u positive on $(3/2, 1/2)$ and negative on $(-3/2, -1/2)$. For the Coulomb friction law, this allows multiple solutions as the conditions of theorem 4.3 are satisfied: the solution in (c) is non-unique. In particular, we may note the asymmetry in panel (c). The solution shown was computed numerically using a mesh that is not perfectly symmetric, and symmetric solution as in (a) and (b) can also be obtained by subtracting a constant positive velocity $r \equiv 0.374$ from that displayed in panel (c). Panels (d) and (e) have $\tau_0 = 0.2$ and $f_0 = 0.075$ (e), 0.15 (f). To indicate the physical origin of the asymmetry, we have tilted the plots in panels (e) and (f) at angles θ such that $f_0 = \sin\theta$. The inequality (4.3) is satisfied only marginally in panel (e) (where $f_0 = 3\tau_0/4$), and there is sliding everywhere on Γ_τ in the positive x-direction except at a single spot at $x = -3/2$. A further increase in f_0 leads to force balance failure. Lastly, panels (f–h) use $f_0 = 0$, $\tau_0 = 0.25$, and the power law (2.8c) with $\epsilon = 1/3$ (f), 0.1 (g) and 0.01 (h). As expected, we see good agreement between panel (h) and panel (b), indicating the convergence of the power law solution to the Coulomb friction solution.

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Appendix A. Proofs of auxiliary lemmas and of theorem 4.2

A.1. *Theorem 4.2*

To prove theorem 4.2, we require the following lemma:

Lemma Appendix A.1. *Let X be a compact topological space and $C(X)$ be the set of continuous functionals defined on X. Let $f : (0,\infty)\times X \mapsto \mathbb{R}$ be such that for every $\lambda > 0$, $f(\lambda,\cdot) \in C(X)$ and let $\lim_{\lambda\to\infty} f(\lambda, x) = g(x)$ for every $x \in X$, where $g \in C(X)$. Assume also that f is monotonically decreasing in λ, $f(\mu, x) \geq f(\lambda, x)$ if $\mu < \lambda$ for every $x \in X$. Then $f \to g$ uniformly, i.e., for every $\delta > 0$ there is a Λ such that $|f(\lambda, x) - g(x)| < \delta$ for every $x \in X$ provided $\lambda > \Lambda$.*

Proof. Assume that the assertion is false, so that there exists a $\delta > 0$ such that for every $n \in \mathbb{N}$, there is a $\lambda_n > n$ and an $x_n \in X$ such that $|f(\lambda_n, x_n) - g(x_n)| > 2\delta$, or, due to monotonicity, $f(\lambda_n, x_n) - g(x_n) > 2\delta$. By compactness, we can extract a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ with limit $x_\infty \in X$. Assume without loss of generality that the λ_{n_i} are ordered, $\lambda_{n_1} < \lambda_{n_2} < \ldots$. It follows that $f(\lambda_{n_i}, x) \geq f(\lambda_{n_j}, x) \geq g(x)$ for all $x \in X$ if $i > j$. By the continuity of g, there is an i_0 such that $|g(x_{n_i}) - g(x_\infty)| < \delta$ whenever $i > i_0$. Hence $f(\lambda_{n_i}, x_{n_i}) - g(x_\infty) > \delta$ for every $i > i_0$. We also have $f(\lambda_{n_i}, x_{n_j}) \geq f(\lambda_{n_j}, x_{n_j})$ for $i < j$, and hence

$$f(\lambda_{n_i}, x_{n_j}) - g(x_\infty) > \delta$$

for $j > i > i_0$. Taking the limit $j \to \infty$, this leaves

$$|f(\lambda_{n_i}, x_\infty) - g(x_\infty)| = f(\lambda_{n_i}, x_\infty) - g(x_\infty) > \delta,$$

which contradicts the assumption of pointwise convergence, $\lim_{i\to\infty} f(\lambda_{n_i}, x_\infty) = g(x_\infty)$. □

Using this lemma, we have the following

Proof. (Theorem 4.2) We need to prove only that the functionals J_1, J_2 and J_3 defined in (3.10) are coercive, which we do by showing that generically there is an $R > 0$ such that $J(\boldsymbol{v}) > J(\boldsymbol{0}) = 0$ whenever $\|\boldsymbol{v}\| > R$. For any $\boldsymbol{v} \in X$, define $\boldsymbol{v}_{\mathcal{R}} = \pi_{\mathcal{R}}(\boldsymbol{v})$, $\tilde{\boldsymbol{v}} = \boldsymbol{v} - \boldsymbol{v}_{\mathcal{R}}$. By the growth constraint (3.5) and theorem 4.1, we have

$$\int_\Omega G(|\boldsymbol{D}(\boldsymbol{v})|; \boldsymbol{x})\,\mathrm{d}\Omega \geq C_2 \int_\Omega |\boldsymbol{D}(\boldsymbol{v})|^p\,\mathrm{d}\Omega - C_2 \int_\Omega |s_0|^p\,\mathrm{d}\Omega \geq C_2 C_3 \|\tilde{\boldsymbol{v}}\|^p - C_4, \quad \text{(A.1)}$$

where $C_4 = C_2 \|s_0\|^p_{L^p(\Omega)}$. Furthermore, as $\mathcal{R}$ is finite dimensional,

$$\delta := \inf_{\boldsymbol{r}\in\mathcal{R}:\|\boldsymbol{r}\|=1} \left(\int_\Gamma \tau_c |\boldsymbol{r}|\,\mathrm{d}\Gamma - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{r}\,\mathrm{d}\Omega \right) > 0,$$

and by homogeneity, for all $\boldsymbol{r} \in \mathcal{R}$,

$$\int_\Gamma \tau_c|\boldsymbol{r}|\,\mathrm{d}\Gamma - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{r}\,\mathrm{d}\Omega \geq \delta\,\|\boldsymbol{r}\|\,. \tag{A.2}$$

Hence, using $|\boldsymbol{v}| \geq |\boldsymbol{v}_\mathcal{R}| - |\tilde{\boldsymbol{v}}|$ and applying Hölder's inequality

$$\begin{aligned} J_1(\boldsymbol{v}) &\geq C_2C_3\,\|\tilde{\boldsymbol{v}}\|^p - C_4 - \int_\Omega \boldsymbol{f}\cdot\tilde{\boldsymbol{v}}\,\mathrm{d}\Omega - \int_\Gamma \tau_c|\tilde{\boldsymbol{v}}|\,\mathrm{d}\Gamma \\ &\quad + \left(\int_\Gamma \tau_c|\boldsymbol{v}_\mathcal{R}|\,\mathrm{d}\Gamma - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}_\mathcal{R}\,\mathrm{d}\Omega\right) \\ &\geq C_2C_3\,\|\tilde{\boldsymbol{v}}\|^p - (\|\boldsymbol{f}\| + C_5\,\|\tau_c\|)\,\|\tilde{\boldsymbol{v}}\| + \delta\,\|\boldsymbol{v}_\mathcal{R}\| - C_4 \\ &= C_2C_3\,\|\tilde{\boldsymbol{v}}\|^p - (\|\boldsymbol{f}\| + C_5\,\|\tau_c\| + \delta)\,\|\tilde{\boldsymbol{v}}\| + \delta R - C_4 \end{aligned} \tag{A.3}$$

where $R = \|\tilde{\boldsymbol{v}}\| + \|\boldsymbol{v}_\mathcal{R}\|$, C_5 is the norm of the trace operator from X into $L^p(\Gamma)$ and $\|\cdot\|$ denotes in each case the norm appropriate to the function spaces in definition 3.1. Clearly, $J_1(\boldsymbol{v})$ is positive when either $\|\tilde{\boldsymbol{v}}\|$ or $\|\boldsymbol{v}_\mathcal{R}\|$ is sufficiently large. Here we establish this explicitly as doing so also provides a bound on the minimizer $\boldsymbol{u}$ required in later work. The last line of (A.3) is minimized with respect to $\|\tilde{\boldsymbol{v}}\|$ at constant R when $\|\tilde{\boldsymbol{v}}\| = [(\|\boldsymbol{f}\| + C_5\,\|\tau_c\| + \delta)/(pC_2C_3)]^{1/p}$, and hence

$$J_1(\boldsymbol{v}) \geq -\frac{(p-1)[\|\boldsymbol{f}\| + C_5\,\|\tau_c\| + \delta]^{p/(p-1)}}{[C_2C_3]^{1/(p-1)}p^p} + \delta R - C_4 \tag{A.4}$$

Hence $J_1(\boldsymbol{v}) > J_1(\boldsymbol{0}) = 0$ when

$$R = \|\tilde{\boldsymbol{v}}\| + \|\boldsymbol{v}_\mathcal{R}\| \geq \|\boldsymbol{v}\| > R_0 := \frac{1}{\delta}\left\{C_4 + \frac{(p-1)[\|\boldsymbol{f}\| + C_5\,\|\tau_c\| + \delta]^{p/(p-1)}}{[C_2C_3]^{1/(p-1)}p^p}\right\}. \tag{A.5}$$

This establishes coercivity for J_1. Moreover, we have a bound on the minimizer $\boldsymbol{u}$ in $\|\boldsymbol{u}\| \leq R_0$.

In order to show coercivity for J_2, note that ψ defined in (3.6) is increasing and Lipschitz continuous in its first argument with unit Lipschitz constant by Rolle's theorem. Hence $\psi(|\boldsymbol{v}_\mathcal{R}|;\boldsymbol{x}) - \psi(|\boldsymbol{v}_\mathcal{R} + \tilde{\boldsymbol{v}}|;\boldsymbol{x}) \leq \psi(|\boldsymbol{v}_\mathcal{R}|;\boldsymbol{x}) - \psi(||\boldsymbol{v}_\mathcal{R}| - |\tilde{\boldsymbol{v}}||\,;\boldsymbol{x}) \leq |\tilde{\boldsymbol{v}}|$, so

$$\int_\Gamma \tau_c\psi(|\boldsymbol{v}|;\boldsymbol{x})\,\mathrm{d}\Gamma \geq \int_\Gamma \tau_c\psi(|\boldsymbol{v}_\mathcal{R}|;\boldsymbol{x})\,\mathrm{d}\Gamma - \int_\Gamma \tau_c|\tilde{\boldsymbol{v}}|.$$

By analogy with (A.3),

$$\begin{aligned} J_2(\boldsymbol{v}) &\geq C_2C_3\,\|\tilde{\boldsymbol{v}}\|^p - (\|\boldsymbol{f}\| + C_5\,\|\tau_c\|)\,\|\tilde{\boldsymbol{v}}\| \\ &\quad + \left(\int_\Gamma \tau_c\psi(|\boldsymbol{v}_\mathcal{R}|;\boldsymbol{x})\,\mathrm{d}\Gamma - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}_\mathcal{R}\,\mathrm{d}\Omega\right) - C_4 \\ &= C_2C_3\,\|\tilde{\boldsymbol{v}}\|^p - (\|\boldsymbol{f}\| + C_5\,\|\tau_c\|)\,\|\tilde{\boldsymbol{v}}\| \\ &\quad + \delta\,\|\boldsymbol{v}_\mathcal{R}\| - \int_\Gamma \tau_c\left(|\boldsymbol{v}_\mathcal{R}| - \psi(|\boldsymbol{v}_\mathcal{R}|;\boldsymbol{x})\right)\,\mathrm{d}\Gamma - C_4. \end{aligned} \tag{A.6}$$

It remains to deal with the penultimate term on the last line. For any $\boldsymbol{r} \in \mathcal{R}$ such that $\|\boldsymbol{r}\| = 1$ and for $\lambda > 0$, let

$$\eta(\lambda, \boldsymbol{r}) = \int_\Gamma \tau_c \frac{\lambda|\boldsymbol{r}| - \psi(\lambda|\boldsymbol{r}|;\boldsymbol{x})}{\lambda}\,\mathrm{d}\Gamma = \int_\Gamma \tau_c|\boldsymbol{r}|\left(1 - \frac{\psi(\lambda|\boldsymbol{r}|;\boldsymbol{x})}{\lambda|\boldsymbol{r}|}\right)\mathrm{d}\Gamma.$$

where the second equality holds because, owing to the definition of $\mathcal{R}$, an $\boldsymbol{r}$ with $\|\boldsymbol{r}\| = 1$ can vanish on at most a set of measure zero in Γ. We have $\psi(u;\boldsymbol{x})/u = \int_0^1 f(u\xi;\boldsymbol{x})\,\mathrm{d}\xi$. As f is monotonically increasing in its first argument with limit unity a.e. on Γ, it follows by the dominated convergence theorem that $\psi(u;\boldsymbol{x})/u$ is also monotonically increasing with $\lim_{u\to\infty}\psi(u;\boldsymbol{x})/u = 1$ a.e. on Γ. Moreover, since $0 \leq \psi(u;\boldsymbol{x}) \leq u$ for $u \geq 0$, the integrand $\tau_c(\lambda|\boldsymbol{r}| - \psi(\lambda|\boldsymbol{r}|;\boldsymbol{x}))/\lambda$ is dominated by $\tau_c|\boldsymbol{r}| \in L^1(\Gamma)$, so $0 < \eta(\lambda,\boldsymbol{r}) \leq \|\tau_c\|$. By the dominated convergence theorem, we obtain $\lim_{\lambda\to\infty}\eta(\lambda,\boldsymbol{r}) = 0$ for all $\boldsymbol{r}$. In addition, $\eta(\lambda,\boldsymbol{r})$ is monotonically decreasing in λ.

Define $\eta_\psi(\lambda) = \sup_{\|\boldsymbol{r}\|=1}\eta(\lambda,\boldsymbol{r})$, and it follows again that $0 < \eta_\psi(\lambda) \leq \|\tau_c\|$. We will show that $\lim_{\lambda\to\infty}\eta_\psi(\lambda) = 0$. Clearly, η is continuous in its second argument, and as $\mathcal{R}$ is finite dimensional, the unit sphere in $\mathcal{R}$ is compact. By lemma Appendix A.1, $\lim_{\lambda\to\infty}\eta(\lambda,\boldsymbol{r}) = 0$ uniformly in $\boldsymbol{r}$. Hence we can interchange limit and supremum, so $\lim_{\lambda\to\infty}\eta_\psi(\lambda) = 0$. From (A.6), and choosing K such that $\eta_\psi(\lambda) \leq \delta/2$ when $\lambda > K$,

$$\begin{aligned} J_2(\boldsymbol{v}) &\geq C_2C_3\|\tilde{\boldsymbol{v}}\|^p - (\|\boldsymbol{f}\| + C_5\|\tau_c\|)\|\tilde{\boldsymbol{v}}\| + \|\boldsymbol{v}_\mathcal{R}\|[\delta - \eta_\psi(\|\boldsymbol{v}_\mathcal{R}\|)] - C_4 \\ &\geq C_2C_3\|\tilde{\boldsymbol{v}}\|^p - (\|\boldsymbol{f}\| + C_5\|\tau_c\|)\|\tilde{\boldsymbol{v}}\| + \delta\|\boldsymbol{v}_\mathcal{R}\|/2 - (C_4 + \|\tau_c\|K) \end{aligned} \quad \text{(A.7)}$$

This bound is of the same form as the last line in (A.3), and J_2 is coercive.

In order to prove coercivity for J_3, consider first the case $\epsilon < p - 1$. By the convexity of the mapping $\boldsymbol{a} \mapsto |\boldsymbol{a}|^{1+\epsilon}$, have $|\boldsymbol{v}|^{1+\epsilon} \geq 2^{-\epsilon}|\boldsymbol{v}_\mathcal{R}|^{1+\epsilon} - |\tilde{\boldsymbol{v}}|^{1+\epsilon}$. Hence

$$\begin{aligned} J_3(\boldsymbol{v}) \geq{}& C_2C_3\|\tilde{\boldsymbol{v}}\|^p - \|\boldsymbol{f}\|\|\tilde{\boldsymbol{v}}\| - \int_\Gamma \frac{\tau_c}{1+\epsilon}|\tilde{\boldsymbol{v}}|^{1+\epsilon}\,\mathrm{d}\Gamma \\ & - \|\boldsymbol{f}\|\|\boldsymbol{v}_\mathcal{R}\| + \int_\Gamma \frac{\tau_c}{2^\epsilon(1+\epsilon)}|\boldsymbol{v}_\mathcal{R}|^{1+\epsilon}\,\mathrm{d}\Gamma - C_4 \\ \geq{}& C_2C_3\|\tilde{\boldsymbol{v}}\|^p - \|\tau_c\|_{L^{p/(p-1-\epsilon)}(\Gamma)}\|\tilde{\boldsymbol{v}}\|^{1+\epsilon} - \|\boldsymbol{f}\|\|\tilde{\boldsymbol{v}}\| \\ & + \frac{\|\boldsymbol{v}_\mathcal{R}\|^{1+\epsilon}}{2^\epsilon(1+\epsilon)} \inf_{\boldsymbol{r}\in\mathcal{R}:\|\boldsymbol{r}\|=1}\left(\int_\Gamma \tau_c|\boldsymbol{r}|^{1+\epsilon}\,\mathrm{d}\Gamma\right) - \|\boldsymbol{f}\|\|\boldsymbol{v}_\mathcal{R}\| - C_4. \end{aligned} \quad \text{(A.8)}$$

Let $j_3(\boldsymbol{r}) = \int_\Gamma \tau_c(1+\epsilon)^{-1}|\boldsymbol{r}|^{1+\epsilon}\,\mathrm{d}\Gamma > 0$ as in theorem 3.1. To show that J_3 is coercive, it suffices to prove that $\inf_{\boldsymbol{r}\in\mathcal{R}:\|\boldsymbol{r}\|=1} j_3(\boldsymbol{r}) > 0$. The functional j_3 is clearly continuous. As $\mathcal{R}$ is finite dimensional, we only need to show that $j_3(\boldsymbol{r}) > 0$ whenever $\|\boldsymbol{r}\| > 0$. From the definition of $\mathcal{R}$ in theorem 4.1, it follows that $|\boldsymbol{r}(\boldsymbol{x})| > 0$ almost everywhere on Γ whenever $\|\boldsymbol{r}\| > 0$. As $\tau_c \geq 0$ a.e. on Γ and $\int_\Gamma \tau_c\,\mathrm{d}\Gamma > 0$, the desired result follows.

For the case $\epsilon = p - 1$, it suffices to note that $\int_\Gamma \tau_c|\boldsymbol{v}|^p\,\mathrm{d}\Gamma \geq \int_\Gamma \tau_c|\boldsymbol{v}|^{1+\epsilon'}\,\mathrm{d}\Gamma - \int_\Gamma \tau_c\,\mathrm{d}\Gamma$ for any $\epsilon' < p - 1$, and the estimate (A.8) holds with ϵ replaced by ϵ' and with $C_4 = C_2\|s_0\|^p_{L^p(\Omega)} + \int_\Gamma \tau_c\,\mathrm{d}\Gamma$. □

A.2. *Lemma 5.1*

Proof. (Lemma 5.1) Given any $\delta > 0$, and $r > 0$, we can find $\varepsilon > 0$ such that $|\phi(x) - \phi(x')| < \delta/4$ when $\|x - x'\| < \varepsilon$ and $\|x\| < r$, $\|x'\| < r$ as ϕ is uniformly continuous on $\{x \in X : \|x\| \leq r\}$. Choose $r > r_2 > r_1 > 0$ such that $r_2 - r_1 < \varepsilon$ and let x_1 and x_2 be such that $\|x_1\| = r_1$, $\Phi(r_1) \leq \phi(x_1) \leq \Phi(r_1) + \delta/4$, $\|x_2\| = r_2$, $\Phi(r_2) \leq \phi(x_2) \leq \Phi(r_2) + \delta/4$. Let $y_1 = \|x_1\| / \|x_2\| \, x_2$, $y_2 = \|x_2\| / \|x_1\| \, x_1$. Then $-\delta/4 \leq \phi(y_1) - \phi(x_1) = \phi(y_1) - \phi(x_2) + \phi(x_2) - \phi(x_1)$. But $|y_1 - x_2| = r_2 - r_1$ and hence $|\phi(y_1) - \phi(x_2)| < \delta/4$, so $\phi(x_2) - \phi(x_1) > -\delta/2$. Similarly, $-\delta/4 \leq \phi(y_2) - \phi(x_2) = \phi(y_2) - \phi(x_1) + \phi(x_1) - \phi(x_2)$ and hence $\phi(x_2) - \phi(x_1) < \delta/2$. Hence $|\phi(x_2) - \phi(x_1)| < \delta/2$ and thus $|\Phi(r_2) - \Phi(r_1)| < \delta$ when $|r_2 - r_1| < \varepsilon$. The proof that Φ is continuous at 0 is easy. □