Dynamics of Ice Stream Temporal Variability: Supplementary Material

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1 Scaling the reduced model

We write the variables in their scaled forms

$$h = [h]h^* \tag{1}$$

$$t = [t]t^* \tag{2}$$

$$\tau_d = [\tau_d]\tau_d^* \tag{3}$$

$$\tau_b = [\tau_b]\tau_b^* \tag{4}$$

$$u_b = [u_b]u_b^*, \tag{5}$$

where dimensional scales are bracketed and dimensionless variables are starred.

The first choice of scales is

$$[\tau_d] = \frac{\rho_i g [h]^2}{L}, \tag{6}$$

such that

$$\tau_d^* = (h^*)^2 . \tag{7}$$

The form of (equation for sliding velocity) indicates that τ_d and τ_b will have the same scale. Thus, we set $[\tau_b] = [\tau_d]$

$$\tau_b^* = \nu \exp(-c(e - e_c)), \tag{8}$$

With $\nu = \frac{a'}{[\tau_d]}$. Leading to an expression for $[u_b]$

$$[u_b] = \frac{A_g W^{n+1}}{4^n (n+1)} \left(\frac{[\tau_d]}{[h]} \right)^n , \tag{9}$$

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and the dimensionless velocity is

$$u^* = \max\left[\left(h^* - \frac{\tau_b^*}{h^*}\right), 0\right]^n. \tag{10}$$

We can also go about setting the ice thickness scale using the steady state of the ice thickness equation, which gives

$$a_c L = [h][u_b], \tag{11}$$

and then solving for $[h]$

$$[h] = L\left[\frac{A_g W^{n+1}(\rho_i g)^n}{4^n(n+1)a_c}\right]^{-\frac{1}{n+1}}. \tag{12}$$

A timescale can also be chosen by balancing accumulation with ice thickness change

$$[t] = \frac{[h]}{a_c}. \tag{13}$$

These non-dimensionalizations give rise to the scaled form of the undrained system

$$\frac{dh^*}{dt^*} = 1 - h^* u_b^* \tag{14}$$

$$m = \begin{cases} \tau_b^* u_b^* + \beta - \frac{\gamma}{h^*} & \text{if } w^* > 0 \\ \beta - \frac{\gamma}{h^*} & \text{if } w^* = 0 \end{cases}, \tag{15}$$

$$\alpha\frac{dw^*}{dt^*} = \begin{cases} m & \text{if } w^* > 0 \text{ or } m > 0 \\ 0 & \text{otherwise} \end{cases}, \tag{16}$$

$$\tau_b^* = \begin{cases} \nu\exp(-c(e^* - e_c^*)) & \text{if } w^* > 0 \\ \infty & \text{otherwise} \end{cases}, \tag{17}$$

$$u^* = \max\left[\left(h^* - \frac{\tau_b^*}{h^*}\right), 0\right]^n, \tag{18}$$

$$e^* = \max\left[e_c^*, w^*\right], \tag{19}$$

with the dimensionless parameters

$$\alpha = \frac{\rho_i L_f}{[t][\tau_d][u_b]} = \frac{L_f}{g[h]^2} = \frac{L_f}{L^2 g}\left[\frac{A_g W^{n+1}(\rho_i g)^n}{4^n(n+1)a_c}\right]^{\frac{2}{n+1}} \tag{20}$$

$$\beta = \frac{G}{[\tau_d][u_b]} = \frac{G}{a_c\rho_i g[h]} = \frac{G}{a_c\rho_i g L}\left[\frac{A_g W^{n+1}(\rho_i g)^n}{4^n(n+1)a_c}\right]^{\frac{1}{n+1}} \tag{21}$$

$$\gamma = \frac{k_i\Delta T}{[h][\tau_d][u_b]} = \frac{k_i\Delta T}{a_c\rho_i g[h]^2} = \frac{k_i\Delta T}{a_c\rho_i g L^2}\left[\frac{A_g W^{n+1}(\rho_i g)^n}{4^n(n+1)a_c}\right]^{\frac{2}{n+1}}. \tag{22}$$

(The common bracketed term on the right-hand side above represents the inverse of the frictional heating scale.)

α is the ratio of bed relaxation rate to frictional heating rate. β is the ratio of geothermal heating to frictional heating. γ is the ratio of vertical heat conduction to frictional heating. Hereafter asterisks are dropped in subsequent calculations with the non-dimensional system.

2 Location of Hopf bifurcation (stability boundary)

The transition in between the two modes highlighted in the text is a Hopf bifurcation. Assuming that the system is not degenerate (which numerical simulations indicate is not the case), then we can determine the stability from the trace of the jacobian (which is also proportional to the real parts of the eigenvalues)

$$S_t \equiv \mathrm{Tr}(\mathrm{Jac}) = \left.\frac{dF_1}{dh}\right|_{h_0,w_0} + \left.\frac{dF_2}{dw}\right|_{h_0,w_0} \tag{23}$$

where $F_1 = \frac{dh}{dt}$, $F_2 = \frac{dw}{dt}$ and h_0 and w_0 are fixed points. When $S_t = 0$ the system undergoes a Hopf bifurcation from a stable fixed point ($S_t < 0$) to a stable limit cycle ($S_t > 0$). We can find where this transition occurs by solving for $S_t = 0$.

We start by solving for the fixed points. This is simply done by combining equations (68) and (69), resulting in

$$(h_0)^2 + \beta h_0 - (h_0)^{\frac{n-1}{n}} - \gamma = 0 \tag{24}$$

This is not trivially solved for h_0 (though it is nearly quadratic). We will return to this later on in this section.

Next we turn to the stability parameter itself

$$S_t = \frac{d}{dh}(1 - hu) + \frac{d}{dw}\left(\frac{\tau_b u_b + \beta - \frac{\gamma}{h}}{\alpha}\right) \tag{25}$$

Before proceeding with calculation of the stability parameter, we will find the scale of each term to determine if one can be dropped. Having scaled the variables, we can easily see that

$$\frac{d}{dh}(1 - hu) \sim O(1) \tag{26}$$

The second term is different:

$$\frac{d}{dw}\left(\frac{\tau_b u_b + \beta - \frac{\gamma}{h}}{\alpha}\right) \sim O(\alpha^{-1}) \tag{27}$$

for typical parameter values (as in Table 1), $\alpha \sim O(10^{-1})$. So, we see that this second term dominates the first term, which can be dropped from the stability calculation.

Utilizing equations (17) and (19), we can proceed with the calculation of the stability parameter

$$S_t = \frac{d}{dw}\left(\frac{\tau_b u_b + \beta - \frac{\gamma}{h}}{\alpha}\right) \tag{28}$$

$$= \frac{1}{\alpha}\left[-c\tau_b u_b + \frac{nc\,(\tau_b)^2}{h}\left(h - \frac{\tau_b}{h}\right)^{n-1}\right] \tag{29}$$

$$= \frac{c\tau_b u_b}{\alpha}\left[\frac{n\tau_b}{(h)^2 - \tau_b} - 1\right] \tag{30}$$

To find the stability boundary, we set this to zero and rearrange

$$0 = \frac{c\tau_b u_b}{\alpha}\left[\frac{n\tau_b}{(h)^2 - \tau_b} - 1\right] \tag{31}$$

$$(h)^2 - (n+1)\tau_b = 0 \tag{32}$$

Substituting in equation (69)

$$(h)^2 - (n+1)(\gamma - \beta h) = 0 \tag{33}$$

$$(h)^2 - \gamma(n+1) + \beta(n+1)h = 0 \tag{34}$$

$$\beta = \frac{\gamma}{h} - \frac{h}{n+1}. \tag{35}$$

Returning now to equation (24), we can substitute in the above expression for β and solve for h

$$(h_0)^2 + \left(\frac{\gamma}{h_0} - \frac{h_0}{n+1}\right) h_0 - (h_0)^{\frac{n-1}{n}} - \gamma = 0 \tag{36}$$

$$(h_0)^2 - \frac{(h_0)^2}{n+1} - (h_0)^{\frac{n-1}{n}} = 0 \tag{37}$$

$$h = \left(\frac{n+1}{n}\right)^{\frac{n}{n+1}}. \tag{38}$$

On the parameter plane, the stability boundary is thus found at

$$\beta = \left(\frac{n+1}{n}\right)^{-\frac{n}{n+1}} \gamma - \frac{\left(\frac{n+1}{n}\right)^{\frac{n}{n+1}}}{n+1}. \tag{39}$$

3 The form of the Hopf bifurcation

When (h, w) are near (h_0, w_0), the dynamics are (generically) governed by the linearization of equations (15) and (16) about these points. Thus we let $h = h_0 + h'$ and $w = w_0 + w'$ and consider the linear system

$$\frac{d}{dt}\begin{pmatrix} h' \\ w' \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} h' \\ w' \end{pmatrix}, \tag{40}$$

where the matrix M depends on α, β, and γ. Explicitly,

$$M = \begin{pmatrix} -\left(nh_0^{1/n}(1+\tau_0/h_0^2) + 1/h_0\right) & nh_0^{-1+1/n} \\ -\frac{cw_0\tau_0}{\alpha h_0^2}\left(\gamma + n\tau_0 h_0^{1+1/n}(1+\tau_0/h_0^2)\right) & -\frac{cw_0\tau_0}{\alpha h_0}(1 - n\tau_0 h_0^{-1+1/n}) \end{pmatrix}, \tag{41}$$

where $\tau_0 = ae^{-be_0}$ and $\tau' = ae^{-be'}$.

The next step is to make a linear transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} h' \\ \tau' \end{pmatrix} \tag{42}$$

such that, along the stability boundary, the system of equations (15) and (16) takes the form

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}, \tag{43}$$

with f and g strictly nonlinear functions of x and y. Straightforward computations lead to the simple choice

$$\begin{pmatrix} ab \\ cd \end{pmatrix} = \begin{pmatrix} M_{21} - M_{11} \\ -M_{11} - M_{12} \end{pmatrix} \tag{44}$$

$$\omega = \det M \tag{45}$$

The coefficient which indicates whether the transition is supercritical or subcritical is (**?**)

$$a = \partial_x^3 f + \partial_x \partial_y^2 f + \partial_x^2 \partial_y g + \partial_y^3 g + \frac{1}{\omega}\left[(\partial_x\partial_y f)\Delta f - (\partial_x\partial_y g)\Delta g - (\partial_x^2 f)(\partial_x^2 g) + (\partial_y^2 f)(\partial_y^2 g)\right], \tag{46}$$

where everything is evaluated at $h' = w' = 0$. If $a > 0$ ($a < 0$), then the transition is subcritical (supercritical). Here we compute a to leading order in α.

The basic observation is that along $Tr(M) = 0$ we have

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = \begin{pmatrix} O(1) & O(1) \\ O(\alpha^{-1}) & O(1) \end{pmatrix} \tag{47}$$

$$\begin{pmatrix} ab \\ cd \end{pmatrix} = \begin{pmatrix} O(\alpha^{-1})O(1) \\ O(1)O(1) \end{pmatrix} \tag{48}$$

This implies

$$\frac{dx}{dt} = O(\alpha^{-1}) \tag{49}$$

$$\frac{dy}{dt} = O(\alpha^{-1}) \tag{50}$$

$$\frac{\partial}{\partial x} = O(\alpha) \tag{51}$$

$$\frac{\partial}{\partial y} = O(1) \tag{52}$$

Noting that $\omega = O(\alpha^{-1})$, we count powers of α and see that only $\partial_y^3 g$ and $(\partial_y^2 f)(\partial_y^2 g)$ can contribute to a. To leading order,

$$a = \partial_y^3 g + \frac{1}{\omega}(\partial_y^2 f)(\partial_y^2 g). \tag{53}$$

To leading order, $\partial/\partial y = (M_{21}/\omega)\,(\partial/\partial\tau')$, $f = M_{21}\,(dh'/dt) - M_{11}\,(d\tau'/dt)$, and $g = -M_{12}\,(d\tau'/dt)$, where we ignore linear terms in f and g which will vanish after differentiation in a. Hence we need to compute

$$a = -\left(\frac{M_{21}}{\omega}\right)^3 M_{12}\frac{\partial^3\phi}{\partial(\tau')^3} + \left(\frac{M_{21}}{\omega}\right)^4 \frac{M_{12}^2}{\omega}\left[\frac{\partial^2\phi}{\partial(\tau')^2}\right]\left[M_{21}\frac{\partial^2\psi}{\partial(\tau')^2} - M_{11}\frac{\partial^2\phi}{\partial(\tau')^2}\right] \tag{54}$$

where $\psi = dh'/dt$ and $\phi = d\tau'/dt$. Because we take two derivatives of τ', not all terms in ψ and ϕ are needed. It is sufficient to take

$$\psi(h', \tau') = -h_0 u'(h', \tau') \tag{55}$$

$$\phi(h', \tau') = -\frac{cw_0}{\alpha}(\tau_0 + \tau')^2(u_0 + u'(h', \tau')), \tag{56}$$

with

$$u_0 + u'(h', \tau') = \frac{1}{h_0^n}(h_0^2 - \tau_0 - \tau')^n \tag{57}$$

$$u_0 + u'(h', \tau') = \frac{1}{h_0^n}(h_0^{(n-1)/n} - \tau')^n \tag{58}$$

$$u_0 + u'(h', \tau') = \frac{1}{h_0}\left(1 - \frac{\tau'}{h_0^{1-1/n}}\right)^n. \tag{59}$$

Note that

$$\frac{\partial^2 \phi}{\partial(\tau')^2} = -\frac{cw_0}{\alpha}\left[2h_0^{-1} + 4\tau_0\frac{\partial u'}{\partial \tau'} + \tau_0^2\frac{\partial^2 u'}{\partial(\tau')^2}\right] \tag{60}$$

$$\frac{\partial^3 \phi}{\partial(\tau')^3} = -\frac{cw_0}{\alpha}\left[6\frac{\partial u'}{\partial \tau'} + 6\tau_0^2\frac{\partial^2 u'}{\partial(\tau')^2} + 3\tau_0^3\frac{\partial^3 u'}{\partial(\tau')^3}\right] \tag{61}$$

$$\frac{\partial^2 \psi}{\partial(\tau')^2} = -h_0\frac{\partial^2 u'}{\partial(\tau')^2}, \tag{62}$$

up to terms which vanish at $h' = \tau' = 0$.

We notice that the Newtonian case ($n = 1$) is singular. Indeed, in this case

$$a(n=1) = -\frac{cw_0}{\alpha}\left(\frac{M_{21}}{\omega}\right)^3 M_{12}h_0^{-1} - \frac{c^2w_0^2}{\alpha^2}\left(\frac{M_{21}}{\omega}\right)^4 \frac{M_{11}M_{12}^2}{\omega}\left[2h_0^{-1} - 4\tau_0 h_0^{-1}\right]^2 \tag{63}$$

Since $M_{11} < 0, M_{21} < 0$, and $M_{12} > 0$, both terms are positive. Hence $a(n = 1) > 0$ and the transition is always subcritical.

In the limit $n \to \infty$ (implicitly we are assuming $n < 1/\alpha$), we can count powers of n to see the dominant terms. With respect to n, $h_0 = O(1), \tau_0 = O(1)$, and all elements of M are $O(n)$. Each derivative of u' brings down a power of n, hence we see that both terms in a are $O(n)$. We find

$$a(n\to\infty) = -\frac{cw_0}{\alpha}\left(\frac{M_{21}}{\omega}\right)^3 M_{12}\frac{3\tau_0^3 n^3}{h_0^{4-3/n}} + \frac{c^2w_0^2}{\alpha^2}\left(\frac{M_{21}}{\omega}\right)^4 \frac{M_{12}^2}{\omega}\left(\frac{\tau_0^2 n^2}{h_0^{3-2/n}}\right)^2 \times\left[-h_0M_{21} + M_{11}\frac{cw_0}{\alpha}\tau_0^2\right] + O(1). \tag{64}$$

One can compute that $-h_0M_{21} + M_{11}\frac{cw_0}{\alpha}\tau_0^2 = O(1)$, rather than $O(n)$ as expected, and hence only the first term in a contributes as $n \to \infty$. This is positive, hence the transition is again subcritical.

Calculation of this parameter for $n = 3$ is non-trivial and likely does not have an analytic form. Numerical experiments appear to indicate that for all $n > 1$ there is a subcritical Hopf bifurcation.

4 Boundary between steady-streaming with and without drainage

To find the condition dividing the steady-streaming with drainage from steady-streaming without drainage regimes, we start by looking for fixed points in the undrained limiting case which satisfy

$$0 = 1 - hu \tag{65}$$

$$0 = \tau_b u_b + \beta - \frac{\gamma}{h} \tag{66}$$

$$u = \left(h - \frac{\tau_b}{h}\right)^n. \tag{67}$$

We combine equations (65) and (67) to

$$\tau_b = (h)^2 - (h)^{\frac{n-1}{n}} \tag{68}$$

and combine equations (65) and (66) to

$$\tau_b = \gamma - \beta h. \tag{69}$$

We know that in the steady-streaming regime without drainage, $\tau_b > 0$. From equations 68 and 69 we can say

$$h > 1 \tag{70}$$

$$h < \frac{\gamma}{\beta}. \tag{71}$$

Ultimately, we find a condition for the steady-streaming regime without drainage

$$\gamma > \beta. \tag{72}$$

When this condition is violated, we are in the steady-streaming with drainage regime that results in $\tau_b \to 0$, a zero-strength bed.

5 Approximating steady-streaming velocity without drainage

We need to solve for the fixed point of u in the case that $\tau_b > 0$ in order to determine the steady streaming velocity in the case without drainage. We start by combining equations 68 and 69 to get

$$h^2 + \beta h - \gamma - h^{1-\frac{1}{n}} = 0. \tag{73}$$

This is a quadratic equation in h with an extra term. Equation (65) gives us $u = h^{-1}$, so all we need in order to solve for u is an expression for h.

Our approach here is to solve equation (73) using a perturbation method. (For an overview of this approach see **?**). This follows from the observation that if we replace the last term in equation (73) with $h^{1+\epsilon}$ where $\epsilon = -\frac{1}{n}$, then we can find an exact solution in the limit that $\epsilon \to 0$. That is, the solution of

$$h^2 + \beta h - \gamma - h = 0, \tag{74}$$

is

$$h_0 = \frac{1}{2}(1-\beta) + \frac{1}{2}\sqrt{(\beta-1)^2 + 4\gamma}. \tag{75}$$

We can now find corrections to this zero-order approximation by assuming that equation (73) has the following solution

$$h(\epsilon) = h_0 + a_1\epsilon + a_2\epsilon^2 + \ldots \tag{76}$$

Retaining just the $O(\epsilon^2)$ terms

$$h(\epsilon) \approx h_0 + a_1\epsilon + a_2\epsilon^2, \tag{77}$$

we insert this expression into equation (73), collecting terms in orders of ϵ

$$h_0^2 + \beta h_0 - \gamma + (2h_0 a_1 + \beta a_1)\,\epsilon + \left(2h_0 a_2 + a_1^2 + \beta a_2\right)\epsilon^2 - \left(h_0 + a_1\epsilon + a_2\epsilon^2\right)^{1+\epsilon} = 0. \tag{78}$$

The last term on the LHS of this equation requires more care in expansion. We first expand the generic expression $h^{1+\epsilon}$ with a Taylor series retaining only $O(\epsilon^2)$ terms

$$h^{1+\epsilon} \approx h + \epsilon h ln h + \frac{1}{2}\epsilon^2 h ln^2 h. \tag{79}$$

Inserting equation (77) above, collecting terms in ϵ and neglecting higher order terms, we have

$$h^{1+\epsilon} \approx h_0 + (a_1 + h_0 ln h_0)\,\epsilon + \left(a_2 + a_1 ln h_0 + a_1 + \frac{1}{2} h_0 ln^2 h_0\right)\epsilon^2. \tag{80}$$

Combining this back with equation (78)

$$\begin{aligned} &\left[h_0^2 + (\beta - 1)\,h_0 - \gamma\right] + (2h_0 a_1 + \beta a_1 - a_1 - h_0 ln h_0)\,\epsilon \\ &+ \left(2h_0 a_2 + a_1^2 + \beta a_2\right)\epsilon^2 + h_0 + (a_1 + h_0 ln h_0)\,\epsilon + \left(a_2 + a_1 ln h_0 + a_1 + \frac{1}{2} h_0 ln^2 h_0\right)\epsilon^2 = 0. \end{aligned} \tag{81}$$

The first bracketed term on the LHS here is exactly zero, from equation (74). Thus, we can now set each of the ϵ coefficients to zero and solve for a_1 and a_2

$$a_1 = \frac{h_0 ln h_0}{2h_0 + \beta - 1} \tag{82}$$

$$a_2 = \frac{a_1\,(ln h_0 + 1 - a_1) + \frac{1}{2} h_0 ln^2 h_0}{2h_0 + \beta - 1} \tag{83}$$

Together with equations (75) and (77), keeping in mind that $\epsilon = -\frac{1}{n}$ and $u = h^{-1}$ this gives us zero, first and second order approximations on the non-dimensional equilibrium sliding velocity for steady-streaming without drainage (accurate, respectively to 5%, 1% and 0.1% of numerically determined values). The zero-order approximation is reproduced in the text.

6 Asymptotics of small α relaxation oscillations

Below, we sketch the leading order structure of the solution to our ice stream model in more detail for the case of a rapidly-adjusting bed water content (small α). This allows us to show that oscillatory solutions consists of two distinct phases, one in which the ice stream is stagnant and thickening, and another in which the ice stream is active and thinning, with velocity computable simply as a function of ice thickness on both. This is analogous to the glacier surge model proposed by **?**. The details of the relationship between basal shear stress and water content of the bed are then no longer germane to the the structure of the ice thickness oscillation, except during brief transients when the ice stream switches on or off. In fact, even our choice of a Coulomb friction law rather than a power law to relate τ_b to velocity and bed water content is largely irrelevant. These simplifications also allows us to derive some simplified formulas for oscillation period and amplitude in some parametric limits.

We neglect drainage and basal cooling, treat the heat capacity of basal ice as negligible, take β, γ, τ_0 and c as $O(1)$ constants, and treat α as small. The reduced model is then given by equations (15)-(22).

6.1 Stagnant and active branches

With $\alpha \ll 1$, equation (16) will be reduced to its steady-state version except during brief transients. There are two such steady states, corresponding to a stagnant and an active ice stream, respectively. For a stagnant ice stream,

$$u = 0 \tag{84a}$$

$$\tau_b < h^2 \tag{84b}$$

$$h < \gamma/\beta \tag{84c}$$

in which case equation (15) becomes

$$\dot{h} = 1 \tag{84d}$$

so h increases linearly with time. We call this the zero-velocity or stagnant branch.

The other case is the active ice stream with $u > 0$ so $h^2 > \tau_b > 0$, In this case $h^2 - \tau_b = hu^{1/n}$ so $\tau_b = h^2 - hu^{1/n}$. Substituting for τ_b in equation (16) yields, at leading order in α

$$(h^2 - hu^{1/n})u = \gamma/h - \beta, . \tag{85}$$

Note that the friction law, equation (17), was not involved at all in this calculation, so the fact that τ_b is independent of u (which is one of the defining characteristics of a Coulomb friction law) is irrelevant to the behaviour of u for the active ice stream. In particular, equation (85) defines $u = U(h)$ implicitly as a function of h for some range of h. Note that $\tau_b = h^2 - hu^{1/n} \geq 0$ and so we require $h \leq \gamma/\beta$. We will show next that there is also a lower bound h_c on h for a solution to equation (85) to exist. This is the ice thickness attained at surge termination.

In equation (85), the left-hand side has a global maximum with respect to u at $u = (n/(n+1))^n h^n$, where the left-hand side attains a value of $(n^n/(n+1)^{n+1})h^{n+2}$. In order for equation (85) to have a solution, this must be greater than $\gamma/h - \beta$. In other words, we must have

$$\frac{n^n}{(n+1)^{n+1}} h^{n+3} + \beta h \geq \gamma, \tag{86}$$

which defines the critical value h_c by turning the weak inequality sign into an equality. There is a single, non-zero solution $u_c = U(h_c) = (n/(n+1))^n h_c^n$ to equation (85) at $h = h_c$.

The left-hand side of equation (85) is a concave function (with negative second derivative) for $u > 0$, and is positive for $0 < u < h^n$, attaining 0 at the end points of that interval. Therefore, in general, there will be either two values of u in $(0, h^n)$ that satisfy equation (85), or none at all. It is straightforward to show that, in order to ensure stability on the fast α^{-1} time scale for changes in w, we must choose the larger of the two solutions where solutions exist, and that this solution also satisfies $\mathrm{d}U/\mathrm{d}h > 0$.

Note that from equation (17), we also require $h^2 - hu^{1/n} = \tau_b < \tau_0$. It is therefore possible that the non-zero steady state velocity $U(h)$ will cease to exist at values $h > h_c$, namely if $\tau_0 u_c < \gamma/h_c - \beta$. Physically, this would correspond to an inability to generate sufficient friction to keep dissipation rates high enough to balance conductive heat loss. For simplicity, we ignore this possibility below, assuming that τ_0 is large enough for this possiblity not to become an issue.

Given $u = U(h)$, h then evolves according to

$$\dot{h} = 1 - hU(h) \tag{87}$$

on this solution branch, which we call the surge or active branch of the solution. Repeated oscillations can only occur if this evolution equation does not have a stable steady-state solution. This requires that we have $hU(h) > 1$ for all thicknesses that admit a non-zero velocity solution $U(h)$. since U increases with h, this is tantamount to $h_cU(h_c) > 0$.

The two evolution equations (84) and (87) combined with the requirement that $h > h_c$ on the active branch and $h < \gamma/\beta$ on the stagnant branch are sufficient to describe the overall dynamics predicted in our model. We exploit this in the next section to show how in certain limits of β and γ, we are able to give estimates of the oscillation amplitude and period, or able to predict that no oscillations will occur. We then still need to show that rapid transitions can indeed occur between the two branches as envisaged, which we defer to the end of our discussion as the relevant analysis is a great deal more complicated than the material that follows immediately below.

Before we proceed, we note that the structure of the solution we are constructing is that of a standard relaxation oscillation such as that produced by the van der Pol osillator (**?**). In both cases, the solution remains on the nullcline for the rapidly evolving variable during most of the limit cycle except for the rapid transition phases we are about to describe below. In the present case, this rapidly evolving variable is w. The primary difference between our oscillator and the canonical van der Pol oscillator is that our variable w satisfies an evolution equation that is non-smooth, as $\dot{w}$ changes discontinuously when w reaches zero from above (in which case $\tau_b u + \beta - \gamma/h > 0$).

6.2 The stagnant and active branches in parametric limits of β and γ

Next, we show that if β and γ are large or small (but not so much as to invalidate the asymptotic structure developed for small α above), we can draw several conclusions about the resulting surge cycle (or indeed, whether surges occur at all) from the behaviour of the active and stagnant branches in those parametric limits. We work through several of these limits in turn. For the case of β, $\gamma = O(1)$, no such results are possible analytically, as the active branch solution cannot be found analytically.

Our arguments below are built around four observations. First, transitions from stagnant to active occur at at $h = \gamma/\beta$. Second, the reverse transition occurs at some value $h = h_c$ at which equation (86) holds with equality. Third, a solution to equation (85) that corresponds to $\tau_b > 0$ must have $h < \gamma/\beta$. Hence solutions on the stagnant branch have $h < \gamma/\beta$, while solutions on the active branch have $h_c < h < \gamma/\beta$. Fourth, in order for oscillations to occur, we must also have $hU(h) > 1$, or a stable steady state can form on the active branch.

6.2.1 $\gamma \sim \beta \ll 1$

When $\gamma \sim \beta \ll 1$, the transition from stagnant to active occurs at an $O(1)$ value γ/β. However, the active branch exists down to small values of h; from equation (86), we see that $h_c = O(\gamma^{1/(n+3)})$, and correspondingly, $u_c = O(\gamma^{n/(n+3)})$. But this implies that $u_c h_c = O(\gamma^{(n+1)/(n+3)}$, and hence $u_c h_c < 1$. A stable solution therefore exists on the active branch, corresponding to steady ice stream flow.

6.2.2 $\gamma = O(1)$, $\beta \ll 1$

The stagnant to active transition now occurs at a large $(O(\beta^{-1}))$ thickness γ/β, while from equation (86) we estimate that $h_c = O(1)$, and correspondingly $u_c = O(1)$. In this parameter regime, it is therefore possible that $u_c h_c < 1$ and that there is a stable on the active branch, but equally, there may not be. In the latter case, there will be oscillations, and we can find leading order expressions for their amplitude and period.

Both are dominated by the stagnant phase. First, the ampltidue of the oscillation is $\gamma/\beta - h_c \sim \gamma/\beta(1 - O(\beta))$ at leading order. Meanwhile equation (84) shows that, to build ice to a thickness γ/β from an initial thickness $h_c = O(1)$ takes a length of time $\sim \gamma/\beta(1 + O(\beta))$. By contrast, the surge phase has a much shorter duration. Initially, thickness h in the surge phase is $O(\beta^{-1})$, and correspondingly $u \sim h^n = O(\beta^{-n})$. The time scale for thinning early in the surge phase scales as $u^{-1} \sim \beta^n$. Subsequently, near the transition from active to stagnant, we have u and h of $O(1)$, corresponding to an $O(1)$ time scale. Both of these are much smaller than the length of time γ/β required at leading order for the stagnant phase, which therefore gives the leading order estimate for the period of oscillation as γ/β.

6.2.3 $\gamma \ll 1$, $\beta = O(1)$

Here, the transition from stagnant to fast occurs at a small thickness $h = \gamma/\beta = O(\gamma)$, corresponding to a small velocity $u = (\gamma^n)$. The velocity after transition to the active branch is therefore too *small*, and the ice stream will actually thicken rather than thin. This falls outside the remit of the asymptotic model we have formulated above. To allow hu to increase beyond γ/β is possible if, instead of insisting on equation (85) as the steady state version of equation (16), we allow $w \to \infty$. In that case, $\tau_b \sim 0$ from equation (17) and at leading order $u = h^n$ from equation (19). A steady state for h will then be attained when $uh = h^{n+1} = 1$, so $h = 1$,, with w going to infinity. Allowing drainage in the model would potentially alleviate w growing without bound.

6.2.4 $\gamma \gg 1$, $\beta = O(1)$

The transition from stagnant to active occurs at a large $h = \gamma/\beta = O(\gamma)$. From equation (86), we have $h_c = O(\gamma^{1/(3+n)})$ with $u_c = O(\gamma^{n/(3+n)})$. Hence $h_c u_c$ is large and oscillations are bound to occur. We can again find leading order expressions for amplitude and period, as we did for the case $\gamma = O(1)$, $\beta \ll 1$.

Specifically, the stagnant phase dominates again, h has to increase from $h_c = O(\gamma^{1/(n+3)})$ to γ/β, which at leading order takes an amount of time $\gamma/\beta(1 + O(\gamma^{-(n+2)/(n+3)})$, and the amplitude of the oscillation is also $\gamma/\beta - h_c = \gamma/\beta(1 + O(\gamma^{-(n+2)/(n+3)})$. Velocities at the beginning of the surge phase are of $O(\gamma^n)$, so the time scale for initial thinning is $O(\gamma^{-n})$. The transition from active to stagnant occurs when u is reduced to $O(\gamma^{1/(3+n)})$, with associated time scale $O(\gamma^{-1/(n+3})$, which is still much less than the error in the computation of the length of the stagnant phase above. The entire cycle therefore takes length $\gamma/\beta(1 + O(\gamma^{-(n+2)/(n+3)})$.

6.2.5 $\gamma = O(1)$, $\beta \gg 1$

As in the case $\gamma \ll 1$, $\beta = O(1)$, the transition from stagnant to active occurs at a small thickness $\gamma/\beta = O(\beta^{-1})$, and as in that previous case, we again expect a solution that settles into a steady state $h = 1$ for h, with w diverging to infinity.

6.2.6 $\gamma \sim \beta \gg 1$

The transition from stagnant to active now happens at an $O(1)$ value γ/β, corresponding to an $O(1)$ velocity. However, the range between h_c and γ/β is now very narrow: from equation (86), we can see that $h_c \sim \gamma/\beta + \beta^{-1}n^n/(n+1)^{n+1}(\gamma/\beta)^{n+3}$. There are three possiblities: either $hU(h) < 1$ at $h = \gamma/\beta$, and we have a situation analogous to the case $\gamma = O(1)$, $\beta \gg 1$ above, with h settling to a steady state at unity and w growing without bound. Alternatively, there could be an value of h between h_c and γ/β such that $hU(h) = 1$, with the ice stream settling into a steady state there. However, given that h_c is close to γ/β, this is an unlikely outcome. Lastly, it is possible that $h_cU(h_c) > 1$ and the ice stream will undergo oscillations. This will have very small amplitude $\gamma/\beta - h_c \sim \beta^{-1}$, with a period that also scales as β^{-1}.

6.3 Transitions between stagnant and active branches

Next, we look again at the case of oscillations, and complete the asymptotic analysis of limit cycle solutions by describing the leading order structure of the transitions between stagnant and active branches.

6.3.1 Transition from stagnant to active

At the time when h reaches γ/β, w will have attained 0 as $\beta < \gamma/h$ during the stagnant phase and, with $u = 0$, only the second case in equation (16) can be attained on setting the left-hand side to zero.

Suppose for simplicity that $\tau_0 > (\gamma/\beta)^2$. Physically, this means that the strength of the bed at the critical void ratio e_c is large enough to prevent sliding from recommencing at the critical thickness $h = \gamma/\beta$ at which melting begins. To reactivate sliding with h close to γ/β then requires a finite ($O(1)$) amount of melt to be generated first, so that τ_b drops to the critical value $(\gamma/\beta)^2$ at which sliding recommences. Label the finite amount of melt required to attain $\tau_b = \tau_0 \exp(-c(w - e_c)) = (\gamma/\beta)^2$ by $w_s = e_c + c^{-1}\log[(\gamma/\beta)^2/\tau_0]$. This finite amount of melt is generated by h rising slightly above γ/β, reducing conductive loss sufficiently to allow melt to take place.

Let $t = t_s$ be the time at which $h = \gamma/\beta$ is attained. Then define a fast time scale for this initial melt process as $T_1 = \alpha^{-1/2}(t - t_s)$, and let $h = \gamma/\beta + \alpha^{1/2}h_1$. With these rescalings, at leading order

$$\frac{\mathrm{d}h_1}{\mathrm{d}T_1} = 1, \qquad \frac{\mathrm{d}w}{\mathrm{d}T_1} = \frac{\beta^2}{\gamma}h_1.$$

Clearly h_1 and therefore w increase linearly with T_1. Eventually w_s is attained at a finite $T_1 = T_s = \gamma w_s/\beta^2$.

Sliding recommences at $T_1 = T_s$. There is an initial interval over which both frictional dissipation and conductive heating play similar roles, as velocity starts from zero so there is at first no dissipation. This is analogous to the corner layer of the van der Pol oscillator solution (**?**). In this initial sliding stage, we can rescale as $w = w_s + \alpha^{1/2n}W_2$, $T_2 = \alpha^{-1+(n-1)/2n}(t - t_s - \alpha^{1/2}T_s)$, $h = \gamma/\beta + \alpha^{1/2}T_s + O(\alpha^{-1-(n-1)/2n})$. equation (16) becomes at leading order

$$\frac{\mathrm{d}W_2}{\mathrm{d}T_2} = \left(\frac{\gamma}{\beta}\right)^{n+1} c^n W_2^n + \left(\frac{\beta^2}{\gamma}\right) h_1.$$

This solution to this equation initially grows linearly due to the second (conductive heating) term on the right hand side, but eventually the first (dissipation) term on the right-hand side becomes dominant. With $n > 1$, this eventually leads to finite-time blow-up at some $T_2 = T_b$, with W_2 behaving as

$$W_2 \sim \frac{\beta^{(n+1)/(n-1)}}{\gamma^{(n+1)/(n-1)} c^{n/(n-1)} [(n-1)(T_b - T_2)]^{1/(n-1)}} \tag{88}$$

This blow-up of course in reality corresponds to the main transition from the zero-velocity branch to the surge branch, during which w becomes much larger than w_s.

To capture this requires a further rescaling to the fast time scale α. for which we put $T_3 = \alpha^{-1}(t - t_s - \alpha^{1/2} T_s - \alpha^{-1+(n-1)/2n} T_b)$, $W_3 = w$, $U_3 = u$, $h = \gamma/\beta + \alpha^{1/2}$. With this rescaling, we get at leading order that $\tau_b = h^2 - hU^{1/n} = (\gamma/\beta)^2 - (\gamma/\beta)U^{1/n}$ and $\beta - \gamma/h = 0$ so that equation (16) becomes

$$\frac{\mathrm{d}W_3}{\mathrm{d}T} = [(\gamma/\beta)^2 - (\gamma/\beta)U^{1/n}]U, \tag{89}$$

with W_3 and U linked through equations (17) and (19) by $U = (\gamma/\beta)[1 - \exp(c(w_s - W))]^n$. U is therefore an increasing function of W, and the fixed point $U = 0$, $W = w_s$ is unstable. Matching with the corner layer solution through equation (88) leads to an initial condition

$$W_3 \sim \frac{\beta^{(n+1)/(n-1)}}{\gamma^{(n+1)/(n-1)} c^{n/(n-1)} [-(n-1)T_3]^{1/(n-1)}} \tag{90}$$

as $T_3 \to -\infty$ that describes the initial evolution away from the fixed point. The large T_3 solution is given by $U = (\gamma/\beta)^n$; this is the solution obtained from equation (19) by setting $\tau_b = 0$, and actually corresponds to $W_3 \to \infty$. $u = U = (\gamma/\beta)^n$ is also the larger of the two solutions to equation (85) when $h = \gamma/\beta$, and the large T_3 behavior therefore corresponds to $h \sim \gamma/\beta$, $U \sim U(h)$ as expected in order to match with the surge branch.

6.3.2 Transition from active to stagnant

The reverse transition from a surging to a non-surging ice stream occurs when h computed through equation (87) on the surge branch reaches the critical value h_c below which equation (85) has no solution. This transition is harder to describe as we cannot compute h_c analytically. However, generically we can write equation (85) in the form

$$F(u, h) = (h^2 - hu^{1/n})u + \gamma/h - \beta = 0,$$

and the critical thickness h_c corresponds to a saddle-node bifurcation point (h_c, u_c) at which

$$F(u_c, h_c) = 0, \qquad \frac{\partial F}{\partial u}(u_c, h_c) = 0. \tag{91}$$

With $u > 0$, the evolution equation equation (16) can also be written as

$$\alpha \frac{\mathrm{d}w}{\mathrm{d}t} = F(u, h), \tag{92}$$

with $w = e_c - c^{-1} \log[(h^2 - hu^{1/n})/\tau_0]$, so that $\mathrm{d}w/\mathrm{d}u > 0$.

Define

$$F_{uu} = \frac{\partial^2 F}{\partial u^2}(u_c, h_c), \qquad F_h = \frac{\partial F}{\partial h}(u_c, h_c), \qquad w_u = \frac{\mathrm{d}w}{\mathrm{d}u}(u_c, w_c).$$

We have $F_{uu} < 0$ while $F_h > 0$ and $u_w > 0$.

Let t_c be the time at which h computed from equation (87) reaches h_c. First, we need to describe the onset of shutdown, when the increase in conductive heating due to ongoing thinning contributes equally to reduction in bed water content as does the reduction in dissipative heating as u 'falls off' the surge branch. This is again analogous to the corner layer in the van der Pol oscillator. Rescale as

$$T_4 = \alpha^{-2/3}(t - t_c), \qquad (u - u_c) = \alpha^{1/3} U_4, \qquad h - h_c = \alpha^{2/3} H_4. \tag{93}$$

By Taylor expanding around (u_c, h_c), we can then show that the leading order form of equation (92) is now

$$w_u \frac{\mathrm{d}U_4}{\mathrm{d}T_4} = \frac{1}{2} F_{uu} U_4^2 + F_h H_4 \tag{94a}$$

$$\frac{\mathrm{d}H_4}{\mathrm{d}T} = 1 - u_c h_c. \tag{94b}$$

By assumption, we have $u_c h_c > 1$, so H_4 decreases linearly with time. As a matching condition with the surge branch computed from equations (85) and (87), we have $H_4 \sim (1 - u_c h_c) T_4$ as $T_4 \to -\infty$, and this therefore remains the solution for H_4 in time throughout the corner layer. We also have the matching condition

$$U_4 \sim \left[-\frac{2F_h}{F_{uu}} \right]^{1/2} H_4^{1/2}$$

as $T_4 \to -\infty$. When T_4 and hence H_4 is positive, U_4 will grow increasingly negative, initially due to the second term in equation (97) and later due to the first, quadratic term that once again leads to finite time blow-up at some time T_t, with U_4 behaving as

$$U_4 \sim \frac{F_{uu}}{2w_u(T_t - T_4)}. \tag{95}$$

Once more, finite time blow-up actually corresponds to a rapid transition, this time to the zero velocity branch. This is captured by the rescaling

$$T_5 = \alpha^{-1}(t - t_c - \alpha^{2/3} T_t), \qquad W_5 = w, \qquad h = h_c + O(\alpha^{2/3}), \tag{96}$$

which leads to the equivalent of equation (89), at leading order

$$\frac{\mathrm{d}W_5}{\mathrm{d}T_5} = [h_c^2 - h_c U^{1/n}] U + \gamma / h_c - \beta \tag{97}$$

now with $W_5 = e_c - c^{-1} \log[(h_c^2 - h_c U^{1/n})/\tau_0]$, so W_5 is again an increasing function of U. Matching with the solution of equation (94a) gives $U_5 \sim u_c - F_{uu}/(2w_u T_5)$ as $T_5 \to -\infty$. With $U < u_c$, the right-hand side of equation (97) will be negative (recall that it is zero when $U = u_c$). Hence W_5 will decrease, as will U. U reaches zero in finite time, completing the transition to the zero-velocity branch at $h = h_c$. This completes the limit cycle solution.

7 Approximating binge-purge period and critical thicknesses for realistic $\alpha = O(10^{-1})$

7.1 Critical stagnation thickness

Equation (86) gives us the exact location of the critical stagnation thickness on the active branch (for small α, though this applies equally well here). Here we rearrange and place a term ϵ on the linear term (perturbation expansions in other terms either fail or yield worse asymptotic approximation on h_s)

$$\eta h_s^{n+3} + \epsilon\beta h_s - \gamma = 0 \tag{98}$$

with $\eta = \frac{n^n}{(n+1)^{n+1}}$.

As above, in supplementary section 5, use a perturbation method. (For an overview of this approach see **?**.) We solve the $\epsilon = 0$, zero-order approximation first

$$\eta h_0^{n+3} - \gamma = 0 \tag{99}$$

$$\eta h_0 = \left(\frac{\gamma}{\eta}\right)^{\frac{1}{n+3}}. \tag{100}$$

We can now find corrections to this zero-order approximation by assuming that equation (86) has the following solution

$$h_s(\epsilon) = h_0 + a_1\epsilon + a_2\epsilon^2 + \dots \tag{101}$$

Retaining just the $O(\epsilon^2)$ terms

$$h_s(\epsilon) \approx h_0 + a_1\epsilon + a_2\epsilon^2, \tag{102}$$

We substitute this into equation equation (98)

$$\eta\left(h_0 + a_1\epsilon + a_2\epsilon^2\right)^{n+3} + \epsilon\beta\left(h_0 + a_1\epsilon + a_2\epsilon^2\right) - \gamma = 0, \tag{103}$$

expanding and only retaining terms in $O(\epsilon^2)$

$$\left[\eta h_0^{n+3} - \gamma\right] + (n+3)a_1\eta h_0^{n+2}\epsilon + (n+3)a_2\eta h_0^{n+2}\epsilon^2 + \binom{n+3}{2} a_1^2 h_0^{n+1}\epsilon^2 + \epsilon\beta h_0 + \beta a_1\epsilon^2 = 0. \tag{104}$$

The first bracketed term on the LHS is exactly the solution of the zero-order approximation. Otherwise, we collect terms in ϵ^1 and solve for a_1

$$\left[(n+3)a_1\eta h_0^{n+2} + \beta h_0\right]\epsilon = 0 \tag{105}$$

Substituting in h_0 and collecting terms in the relevant parameters yields

$$a_1 = -\frac{\beta}{n+3}\left(\eta^2\gamma^{n+1}\right)^{-\frac{1}{n+3}} \tag{106}$$

Similarly, we collect terms in ϵ^2 and solve for a_2

$$\left[(n+3)a_2\eta h_0^{n+2} + \binom{n+3}{2} a_1^2 h_0^{n+1}\epsilon^2 + \beta a_1\right]\epsilon^2 = 0 \tag{107}$$

Substituting in h_0 and a_1, we collecting terms in the relevant parameters and solve for a_2

$$a_2 = -\frac{n}{2(n+3)^2}\beta^2\left(\eta^3\gamma^{2n+3}\right)^{-\frac{1}{n+3}} \tag{108}$$

With these three terms in hand, we take $\epsilon \to 1$ and construct an approximation for (non-dimensional) h_s:

$$h_s = \left(\frac{\gamma}{\eta}\right)^{\frac{1}{n+3}} - \frac{\beta}{n+3}\left(\eta^2\gamma^{n+1}\right)^{-\frac{1}{n+3}} - \frac{n}{2\left(n+3\right)^2}\beta^2\left(\eta^3\gamma^{2n+3}\right)^{-\frac{1}{n+3}} \tag{109}$$

7.2 Stagnation period

When the ice stream falls off the active branch at h_s and onto the stagnant branch, the dynamics of the ice stream model become considerably simpler (as $u* = 0$). Notably, we have the following evolution of the till water content:

$$\alpha\frac{dw}{dt} = \beta - \frac{\gamma}{h}, \tag{110}$$

and a linear evolution in ice thickness:

$$h(t) = h_s + t, \tag{111}$$

(assuming that there is not any considerable change in h^* during the transition from the active to stagnant branch). We see that we can solve exactly for the evolution in till water content:

$$\alpha\frac{dw}{dt} = \beta - \frac{\gamma}{h_s + t} \tag{112}$$

$$w(t) = \frac{\beta}{\alpha}\left(h_s + t\right) - \frac{\gamma}{\alpha}\ln\left(h_s + t\right) + w_0 \tag{113}$$

Or rather, in reference to a time Δt since stagnation:

$$w(\Delta t) = \frac{\beta}{\alpha}\Delta t - \frac{\gamma}{\alpha}\ln\left(1 + \frac{\Delta t}{h_s}\right) \tag{114}$$

7.2.1 A quick estimate of period

To derive a quick estimate on the period (which we will use later to derive a more accurate estimate) we note that the change in driving stress, h^2, during the stagnant phase is relatively small in comparison to the change in basal shear strength. Thus, we attempt to find Δt by looking for the roots of equation (114). Also, we note that the trajectory of till water content is almost symmetric with a minimum value attained when $h = \frac{\gamma}{\beta}$. Thus, we can approximate equation (114) as the quadratic

$$w(\Delta t) = \Delta t^2 - 2\left(\frac{\gamma}{\beta} - h_s\right)\Delta t \tag{115}$$

This equation readily admits two roots, the first being $\Delta t = 0$ and the second being:

$$\Delta t = 2\left(\frac{\gamma}{\beta} - h_s\right) \tag{116}$$

7.2.2 Long way around

We now attempt to solve for the time since stagnation, Δt, at which the bed fails critically, which should occur when

$$h^2 = \tau_b, \tag{117}$$

or in terms of our time evolution for till water content and a linearly accumulating ice stream thickness:

$$(h_s + \Delta t)^2 = h_s^2 \exp\left(-b\left[\frac{\beta}{\alpha}\Delta t - \frac{\gamma}{\alpha}\ln\left(1 + \frac{\Delta t}{h_s}\right)\right]\right) \tag{118}$$

Rearranging and applying various log rules:

$$\Delta t - \left(\frac{\gamma}{\beta} - \frac{2\alpha}{b\beta}\right)\ln\left(1 + \frac{\Delta t}{h_s}\right) = 0 \tag{119}$$

Expanding in the first term of log about our earlier estimate of period, $\Delta t = 2\left(\frac{\gamma}{\beta} - h_s\right)$, we have

$$\Delta t - \left(\frac{\gamma}{\beta} - \frac{2\alpha}{b\beta}\right)\left[\ln\left(2\frac{\gamma}{\beta h_s} - 1\right) + \frac{\Delta t - 2\left(\frac{\gamma}{\beta} - h_s\right)}{2\frac{\gamma}{\beta} - h_s} + O(h^2)\right] = 0 \tag{120}$$

And then solving for Δt, we get

$$\Delta t = \frac{\gamma - \frac{2\alpha}{b}}{\beta - \frac{1}{2\frac{\gamma}{\beta} - h_s}\left(\gamma - \frac{2\alpha}{b}\right)}\left[\ln\left(2\frac{\gamma}{\beta h_s} - 1\right) - \frac{\frac{\gamma}{\beta} - h_s}{\frac{\gamma}{\beta} - \frac{h_s}{2}}\right] \tag{121}$$

For γ that is $O(1)$ it is the case that $\frac{2\alpha}{b} << \gamma$ unless the till layer is $O(10)$. This corresponds to a situation where the change in driving stress during stagnation is small compared to the change in bed strength, a valid assumption for all but the strongest binge-purge oscillations (for which this entire derivation does not apply anyway). The following simplification can be made:

$$\Delta t = \frac{\gamma}{\beta - \frac{\gamma}{2\frac{\gamma}{\beta} - h_s}}\left[\ln\left(2\frac{\gamma}{\beta h_s} - 1\right) - \frac{\frac{\gamma}{\beta} - h_s}{\frac{\gamma}{\beta} - \frac{h_s}{2}}\right] \tag{122}$$

This provides a good estimate on the period of binge-purge oscillations. In this approximation, we have neglected the purge phase, which only contributes significantly to the period near the stability boundary. Also, in the reduced model we are not taking into account any till-freezing or basal cooling, thus these approximations becomes increasingly poor for large REHF (in the strong binge-purge regime). However, this approximation is still within 10% of numerically derived period estimates in the weak binge-purge parameter regime away from the stability boundary.

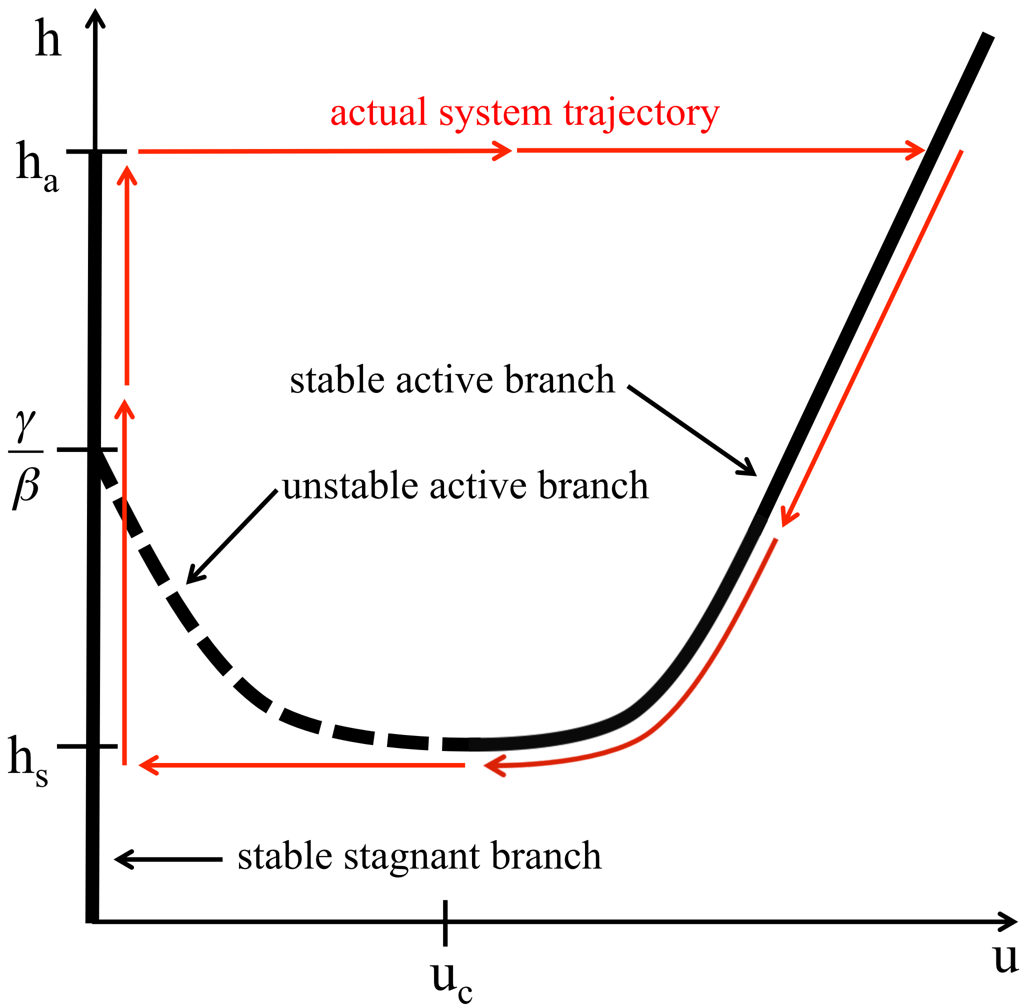

Figure 1: Phase portrait of the reduced model referenced in sections 3.1 and in the supplemental material.