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Marine ice sheet stability

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We examine the stability of two-dimensional marine ice sheets in steady state. The dynamics of marine ice sheets is described by a viscous thin-film model with two Stefan-type boundary conditions at the moving boundary or 'grounding line' that marks the transition from grounded to floating ice. One of these boundary conditions constrains ice thickness to be at a local critical value for flotation, which depends on depth to bedrock at the grounding line. The other condition sets ice flux as a function of ice thickness at the grounding line. Depending on the shape of the bedrock, multiple equilibria may be possible. Using a linear stability analysis, we confirm a long-standing heuristic argument that asserts that the stability of these equilibria is determined by a simple mass balance consideration. If an advance in the grounding line away from its steady-state position leads to a net mass gain, the steady state is unstable, and stable otherwise. This also confirms that grounding lines can only be stable in positions where bedrock slopes downwards sufficiently steeply.

Key words: contact lines, ice sheets, lubrication theory

1. Introduction

Marine ice sheets are continent-sized ice masses that rest on bedrock below sea level. The most important present-day example is the West Antarctic Ice Sheet, which holds sufficient ice to raise sea levels by several metres were it to disintegrate. Portions of the much larger East Antarctic Ice Sheet as well as the Greenland Ice Sheet are also grounded below sea level. Marine ice sheets differ from land-based ice sheets in that they can lose mass at their edges through ice flow into floating ice shelves, from where icebergs subsequently break off. It is this behaviour that has led to intense interest in the dynamics of marine ice sheets. In particular, an extensively developed theory indicates that the rate of mass loss at the edges is controlled by bedrock geometry, with deeper bedrock corresponding to faster mass loss (Weertman 1974; Chugunov & Wilchinsky 1996; Schoof 2007*a*,*b*, 2011; Durand *et al.* 2009; Katz & Worster 2010; Robison, Huppert & Worster 2010). This result appears to be robust for marine ice sheets in one horizontal dimension, as the flow of the coupled ice shelves can play an active role in determining mass loss through changing stresses at the edges of the ice sheet only in two horizontal dimensions (MacAyeal & Barcilon 1988; Dupont & Alley 2005; Goldberg, Holland & Schoof 2009; Winkelmann *et al.* 2011).

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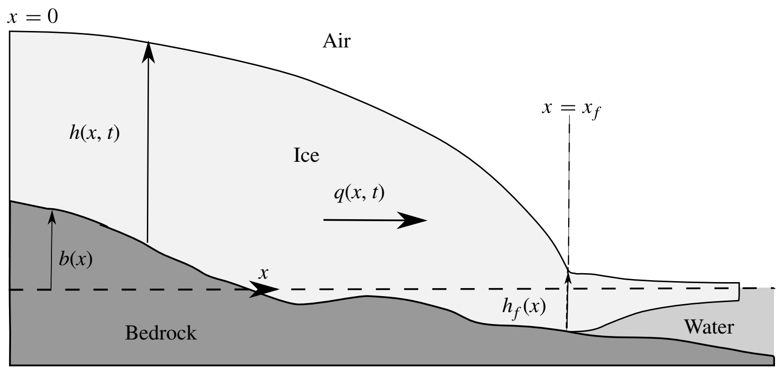

FIGURE 1. Geometry of a marine ice sheet, with the notation used in the text.

One of the inferences that has been widely drawn from this theory is that the stability of steady-state configurations for a two-dimensional marine ice sheet (i.e. with one horizontal dimension, see figure 1) should depend crucially on bed slope at the edge of the ice sheet (also known as the *grounding line*). The argument goes as follows: a slight increase in the size of the ice sheet will lead to an increase in snow accumulation due to a wider area over which snowfall is captured. If the bed at the grounding line does not slope downwards steeply enough in the flow direction, then this increase in snowfall will exceed the increase in mass loss through the grounding line, as the latter is related directly to depth to bedrock at the grounding line. The ice sheet will therefore gain mass, causing it to spread further and evolve away from the steady state.

However, this argument has mostly been put heuristically (Weertman 1974; Schoof 2007*a*,*b*). Implicit in the argument is that the gain in mass associated with a perturbation of the grounding line position necessarily causes the ice sheet to spread further laterally through viscous flow. In other words, we require a stability analysis that resolves the flow of the ice sheet rather than relying on a simple mass balance argument. An analysis of the stability of steady states in a marine ice sheet model appears in Wilchinsky (2009). However, this is limited to identifying the unique conditions under which a zero eigenvalue can be obtained, and solving the problem numerically when these conditions are not satisfied (see also Wilchinsky 2001). It is not immediately clear if a continuation argument allows general conditions for stability and instability to be determined from the zero-eigenvalue result in Wilchinsky (2009), especially as his condition for a zero eigenvalue involves a data function (the derivative of depth to bedrock under the ice sheet) rather than a scalar parameter. In this paper, we apply a linear stability analysis to a wider class of marine ice sheet models than was done previously, and derive very general stability and instability criteria that establish the validity of the heuristic argument given in the previous paragraph.

2. A model for marine ice sheet dynamics

As a template for the model we will analyse, we use the description of marine ice sheet dynamics in one horizontal dimension given by Schoof (2007*a*). The grounded portion of the ice sheet (which is thick enough not to float) occupies a moving domain $(0, x_f(t))$, on which the ice thickness $h(x, t)$ satisfies a thin-film flow model of the form

(see also figure 1)

$$h_t + q_x = a, \tag{2.1a}$$
$$q = -(\rho g/C)^{1/m} h^{1+1/m} |(h+b)_x|^{1/m-1} (h+b)_x \tag{2.1b}$$

where subscripts x and t denote partial differentiation with respect to horizontal coordinate and time; ρ is the density of ice, g acceleration due to gravity, while $C > 0$ and $m > 0$ are constants describing the sliding motion of the ice, a is ice accumulation rate, and $b(x)$ is the elevation of the ice sheet bed, which we assume to be fixed (see also Gomez *et al.* 2010, for complications arising when this assumption is dispensed with). We assume the ice sheet to be symmetrical about the origin, so that ice flux vanishes there. Consequently

$$q = 0 \quad \text{at } x = 0. \tag{2.1c}$$

The moving boundary $x_f(t)$ marks the location where ice begins to float, also known as the grounding line. We have two boundary conditions, one representing flotation and the other describing how coupling with a floating ice shelf controls flux through the grounding line. These take the form

$$h = -\frac{\rho_w}{\rho} b, \tag{2.1d}$$
$$q = \left(\frac{A(\rho g)^{n+1} (1 - \rho/\rho_w)^n}{4^n C} \right)^{1/(m+1)} h^{(m+n+3)/(m+1)} \quad \text{at } x = x_f(t), \tag{2.1e}$$

where ρ_w is the density of water, and $A > 0$ and $n > 0$ are constants describing the power-law rheology of ice.

Generically, we have a diffusion model with a free boundary that is similar in nature to a Stefan problem. The quantity subject to diffusion is ice thickness h, which attains a known value $-\rho_w b(x_g)/\rho$ at the grounding line, and the flux q associated with h therefore also attains a known value from (2.1*e*); an important feature of the model is that these known values of h and q in fact depend on the location of the moving boundary. It is this feature which allows multiple steady states with different stability properties to arise.

The model above can be justified by means of matched asymptotic expansions for the coupled flow of grounded and floating ice in the presence of rapid basal slip (Schoof 2007*b*, 2011). There are other models for different amounts of slip at the base of the ice (Chugunov & Wilchinsky 1996; Wilchinsky 2009; Schoof 2011) that differ in detail but can be posed in the same generic form. In particular, the precise details of ice flux within the domain, expressed as a function of local ice thickness and surface slope, and of discharge at the grounding line, expressed as a function of ice thickness (or equivalently, of depth to bedrock), may differ from the formulae given above. We therefore generalize (2.1) as follows: on the moving domain, we have

$$h_t + q_x = a, \tag{2.2a}$$
$$q = Q(h, -h_x, x), \tag{2.2b}$$

where the function Q increases in its first and second arguments, and its dependence on the third argument is dictated by the shape of the ice sheet bed. At the origin, we still have zero flux,

$$q = 0 \quad \text{at } x = 0, \tag{2.2c}$$

while ice thickness h and flux q attain values at the moving boundary that depend purely on the position of the moving boundary,

$$h(x_f(t), t) = h_f(x_f(t)), \tag{2.2d}$$

$$q(x_f(t), t) = Q_f(x_f(t)), \tag{2.2e}$$

where h_f and Q_f are known functions whose form depends only on fixed physical parameters and on the shape of the bed underlying the ice sheet. In particular, $h_f(x)$ and $Q_f(x)$ increase with depth to bedrock $-b(x)$ at any given position x. It is straightforward to see that the model (2.1) can be cast in this form, as can the models in Wilchinsky (2001, 2009) and Schoof (2011). For completeness, we add the requirement that ice thickness exceed its local flotation value everywhere in the moving domain, so

$$h(x, t) > h_f(x) \tag{2.2f}$$

for $x \in (0, x_f(t))$.

3. Steady states

In what follows, we assume that accumulation rate is known as a function of position. Let $h_0(x)$, $q_0(x)$ and x_{f0} denote steady-state ice thickness, flux and grounding line position. Then

$$q_{0x} = a(x) \quad \text{on } (0, x_{f0}) \tag{3.1a}$$

$$q_0 = Q(h_0, -h_{0x}, x) \quad \text{on } (0, x_{f0}) \tag{3.1b}$$

$$q_0 = 0 \quad \text{at } x = 0 \tag{3.1c}$$

$$h_0 = h_f(x_{f0}) \quad \text{at } x = x_{f0} \tag{3.1d}$$

$$q_0 = Q_f(x_{f0}) \quad \text{at } x = x_{f0} \tag{3.1e}$$

combined with

$$h_0(x) > h_f(x) \tag{3.1f}$$

for $x \in (0, x_{f0})$.

Steady states are straightforward to determine in principle. Integrating (3.1*a*) from $x = 0$ with (3.1*c*) gives

$$q_0(x) = \int_0^x a(x')\,\mathrm{d}x', \tag{3.2}$$

and (3.1*e*) then determines possible steady-state grounding line positions x_{f0} implicitly:

$$\int_0^{x_{f0}} a(x')\,\mathrm{d}x' = Q_f(x_{f0}). \tag{3.3}$$

Once x_{f0} has been determined, ice thickness $h_0(x)$ can be determined by recognizing (3.1*b*) as a first-order ordinary differential equation for h_0,

$$Q(h_0, -h_{0x}, x) = \int_0^x a(x')\,\mathrm{d}x', \tag{3.4}$$

and integrating this from $x = x_{f0}$, using $h(x_{f0}) = h_f(x_{f0})$ as an initial condition. Compliance with (3.1*f*) must then be checked *a posteriori*, and any solutions that do not satisfy the constraint rejected. To be definite, we also insist that $q_0(x) > 0$ for

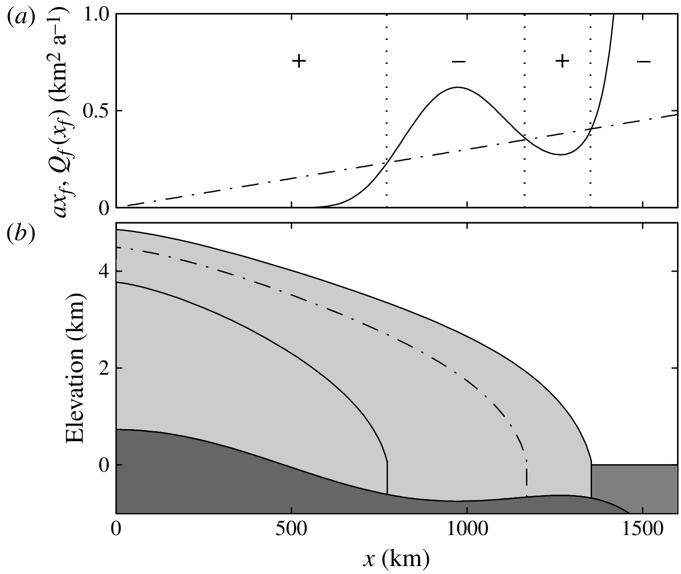

FIGURE 2. Steady states for the model given by (2.1), parameter choices as in Schoof (2007*a*), see figure 5 therein. Accumulation rate a is constant. (*a*) $Q_f(x)$ as given by (2.1*e*) (solid curve) and $\int_0^x a(x')\,\mathrm{d}x' = ax$ (dot-dashed line) against x. (*b*) The bedrock shape and the corresponding steady state ice sheet shapes. Q_f is an increasing function of depth to bedrock, and the trough in the bed in (*b*) ensures that $Q_f(x)$ is an S-shaped curve. Points of intersection between the two curves in (*a*) (marked by vertical dotted lines) are steady-state grounding line positions. The $+$ and $-$ signs in (*a*) indicate grounding line positions for the which the ice sheet will gain ($+$) and lose ($-$) mass. By the simple mass balance argument leading up to (3.5), the intermediate steady state (dot-dashed line in *b*) is unstable, while the other two are stable.

any position intermediate between the 'ice divide' at the origin and the grounding line at $x = x_{f0}$.

A number of numerical examples are shown for instance in Schoof (2007*a*). In figure 2 we show a case in which there are multiple steady-state solutions, which naturally raises the question of whether these are all stable or not.

The usual heuristic argument for identifying stable steady states goes as follows (Weertman 1974; Schoof 2007*a*): suppose the steady state is perturbed, with x_f increased by a small amount to $x_{f0} + \varepsilon x_{f1}$. Now we can ask whether the ice sheet in its perturbed state gains or loses mass. The increase in accumulation due to a wider ice sheet is given by $\varepsilon a(x_{f0})x_{f1}$, while the increase in mass loss through the grounding line is given by $\varepsilon Q_f'(x_{f0})x_{f1}$, where the prime indicates differentiation. The ice sheet therefore gains mass provided

$$a(x_{f0}) - Q_f'(x_{f0}) > 0. \tag{3.5}$$

The argument now goes that the ice sheet will therefore continue to grow, causing the grounding line x_f to advance further, leading to yet more mass gain in a positive feedback. This would suggest that the steady state is unstable if (3.5) is satisfied, and stable if the inequality sign is reversed. If true, then the intermediate steady state in figure 2, with its grounding line located on an upward bed slope, should be unstable, while the smaller and larger steady states, with grounding lines on downward slopes, should be stable. Direct numerical solutions of the full model (2.1) indeed seem confirm this, see Schoof (2007*a*).

However, the difficulty with the general argument above is that it does not immediately follow that grounding line x_f has to advance if the ice sheet gains mass, or that it has to retreat if the ice sheet loses mass, and yet this behaviour is essential to the positive feedback mechanism. Below, we perform a standard linear stability analysis to show that (3.5) indeed furnishes the correct instability criterion, and that stability is ensured if the inequality is reversed.

4. Linear stability analysis

Consider a small perturbation to the steady-state solutions computed above,

$$h(x,t) = h_0(x) + \varepsilon h_1(x)\mathrm{e}^{\sigma t}, \quad q(x,t) = q_0(x) + \varepsilon q_1(x)\mathrm{e}^{\sigma t}, \quad x_f(t) = x_{f0} + \varepsilon x_{f1}\mathrm{e}^{\sigma t}. \tag{4.1}$$

Linearizing the model (2.2) about the steady state (i.e. retaining only terms of $O(\varepsilon)$) we get

$$\sigma h_1 + q_{1x} = 0 \quad \text{on } (0, x_{f0}), \tag{4.2a}$$

$$q_1(x) = Q_1(x)h_1(x) - Q_2(x)h_{1x}(x) \quad \text{on } (0, x_{f0}), \tag{4.2b}$$

$$q_1 = 0 \quad \text{at } x = 0, \tag{4.2c}$$

$$h_{0x}x_{f1} + h_1 = h_f'(x_{f0})x_{f1} \quad \text{at } x = x_{f0}, \tag{4.2d}$$

$$q_{0x}x_{f1} + q_1 = Q_f'(x_{f0})x_{f1} \quad \text{at } x = x_{f0}, \tag{4.2e}$$

where

$$Q_1(x) = \left.\frac{\partial Q(h,p,x)}{\partial h}\right|_{h=h_0(x),p=-h_{0x}(x)}, \quad Q_2 = \left.\frac{\partial Q(h,p,x)}{\partial p}\right|_{h=h_0(x),p=-h_{0x}(x)}. \tag{4.3}$$

As above, we assume that flux increases with ice thickness and with surface slope, so $Q_1 \geqslant 0$ and $Q_2 \geqslant 0$.

We have assumed that the perturbations in question can be represented in separable form. We will show next that (4.2) is in fact of Sturm–Liouville form, and therefore has a complete set of eigenfunctions that allows an arbitrary initial condition to be represented as a superposition of these eigenfunctions, justifying the separation of variables in (4.1).

Substituting (4.2*b*) into (4.2*a*) gives

$$\sigma h_1 + (Q_1 h_1)_x - (Q_2 h_{1x})_x = \sigma h_1 + Q_{1x}h_1 + Q_1 h_{1x} - Q_{2x}h_{1x} - Q_2 h_{1xx} = 0. \tag{4.4}$$

Through the standard use of an integrating factor, this can be turned into

$$(\alpha h_1')' + \beta h_1 - \sigma\gamma h_1 = 0, \tag{4.5}$$

where the weight functions α, β and γ are defined through

$$\alpha = \exp\left(\int \frac{Q_{2x} - Q_1}{Q_2}\,\mathrm{d}x\right), \quad \beta = -Q_{1x}\alpha/Q_2, \quad \gamma = \alpha/Q_2. \tag{4.6}$$

The boundary condition (4.2*c*) with (4.2*b*) is clearly homogeneous in h_1. We can further reduce (4.2*d*) and (4.2*e*) to homogeneous form by eliminating x_{f1}. Note from (3.1*a*) that $q_{0x} = a$, so we get

$$q_1 = Q_1(x_{f0})h_1 - Q_2(x_{f0})h_{1x} = \frac{Q_f'(x_{f0}) - a(x_{f0})}{h_f'(x_{f0}) - h_{0x}(x_{f0})}h_1 \tag{4.7}$$

at $x = x_{f0}$, which is again homogeneous.

There are some technical points to attend to. For simplicity, we assume that we have a regular Sturm–Liouville problem. Note that we already have that $Q_2 \geqslant 0$; a regular Sturm–Liouville problem then requires that Q_{1x} and Q_2 be continuous on $[0, x_{f0}]$ with a sufficiently smooth derivative Q_{2x}, and that Q_2 does not vanish on $[0, x_{f0}]$.

Furthermore, consider the requirement (3.1*f*) that ice thickness exceed its flotation value. Linearizing about $x = x_{f0}$, we have $h_0(x) \sim h_0(x_{f0}) + (x - x_{f0})h_{0x}(x_{f0})$ and $h_f(x) \sim h_f(x_{f0}) + (x - x_{f0})h_f'(x_{f0})$. But from (3.1*d*), $h_0(x_{f0}) = h_f(x_{f0})$. Moreover, for a point x in the domain, $(x - x_{f0}) < 0$ and hence $h_0(x) > h_f(x)$ requires

$$h_f'(x_{f0}) \geqslant h_{0x}(x_{f0}). \tag{4.8}$$

We will assume the strong form of this inequality holds,

$$h_f'(x_{f0}) > h_{0x}(x_{f0}), \tag{4.9}$$

which ensures that the denominator on the right-hand side of (4.7) is positive.

Our subsequent analysis will be based on the following observation: integrate (4.2*a*) from 0 to x_{f0} and use (4.2*c*) as well as (4.7). This yields

$$\sigma \int_0^{x_{f0}} h_1(x)\,\mathrm{d}x = -q_1\,|_0^{x_{f0}} = -\frac{Q_f'(x_{f0}) - a(x_{f0})}{h_f'(x_{f0}) - h_{0x}(x_{f0})} h_1(x_{f0}), \tag{4.10}$$

or, assuming $\int_0^{x_{f0}} h_1(x)\,\mathrm{d}x \neq 0$,

$$\sigma = \frac{a(x_{f0}) - Q_f'(x_{f0})}{h_f'(x_{f0}) - h_{0x}(x_{f0})} \frac{h_1(x_{f0})}{\displaystyle\int_0^{x_{f0}} h_1\,\mathrm{d}x}. \tag{4.11}$$

We are trying to demonstrate that (3.5) determines whether a steady state is unstable. In other words, when $a(x_{f0}) - Q_f'(x_{f0}) > 0$, we would like to show that there is at least one positive eigenvalue, whereas all eigenvalues must be negative when $a(x_{f0}) - Q_f'(x_{f0}) < 0$. By relating eigenvalues σ to the quantity $a(x_{f0}) - Q_f'(x_{f0})$, (4.11) provides the means to do this.

We already know that $h_f'(x_{f0}) - h_{0x}(x_{f0}) > 0$; all we have to do is to determine the sign of $h_1(x_{f0})/\int_0^{x_{f0}} h_1\,\mathrm{d}x$. We can also immediately show that this latter quantity must be non-zero: from the mixed boundary condition (4.7) with $Q_2 \neq 0$, it is immediately clear that we cannot have $h_1 = 0$ at x_{f0}, as this would then simultaneously imply that $h_{1x} = 0$ there, and a non-trivial solution to (4.5) cannot admit two independent homogeneous boundary conditions at the same boundary. Moreover, the integral of $\int_0^{x_f} h_1(x)\,\mathrm{d}x$ must be finite or zero by standard Sturm–Liouville theory, so $h_1(x_{f0})/\int_0^{x_{f0}} h_1\,\mathrm{d}x$ is non-zero; by extension, the integral $\int_0^{x_{f0}} h_1\,\mathrm{d}x$ cannot vanish except possibly when $a(x_f) - Q'(x_f) = 0$, as the eigenvalue σ must also be finite.

4.1. *Instability:* $a(x_{f0}) - Q_f'(x_{f0}) > 0$

We can now straightforwardly demonstrate that we get at least one positive eigenvalue when $a(x_{f0}) - Q_f'(x_{f0}) > 0$, implying that the corresponding steady states are unstable as predicted by the heuristic argument leading up to (3.5). By Sturm's oscillation theorem, there exists a solution of (4.5) that does not have any zeros in $(0, x_{f0})$, implying that h_1 has the same sign throughout the interval as at x_{f0}. It follows that, for this eigenfunction,

$$\frac{h_1(x_{f0})}{\int_0^{x_{f0}} h_1\, \mathrm{d}x} > 0 \tag{4.12}$$

and hence (4.11) shows that the corresponding eigenvalue $\sigma > 0$ as required.

4.2. *Stability:* $a(x_{f0}) - Q_f'(x_{f0}) < 0$

We want to prove that every eigenvalue is negative when $a(x_{f0}) - Q_f'(x_{f0}) < 0$. Assume the contrary, that some eigenfunction h_1 corresponds to a positive eigenvalue σ. We can then show that such an eigenfunction also cannot change sign in $(0, x_{f0})$. Without loss of generality, assume that $h_1 > 0$ near $x = 0$, and suppose that $x = x_0 \in (0, x_{f0})$ were a point at which h_1 changes sign. Integrating (4.2*a*) from 0 to x_0 and using (4.2*c*) as well as (4.2*b*) gives

$$0 = \sigma \int_0^{x_0} h_1\, \mathrm{d}x + q_1 \,|_0^{x_0} = \sigma \int_0^{x_0} h_1\, \mathrm{d}x - Q_2(x_0) h_{1x}(x_0), \tag{4.13}$$

so

$$h_{1x}(x_0) = Q_2\,(x_0)^{-1}\, \sigma \int_0^{x_0} h_1\, \mathrm{d}x. \tag{4.14}$$

But by assumption, $\sigma > 0$ and $\int_0^{x_0} h_1\, \mathrm{d}x > 0$. We also know that $Q_2 > 0$, and hence $h_{1x}(x_0) > 0$, contrary to the assumption that h_1 changes sign from positive to negative at x_0. It follows that h_1 cannot change sign. But then we once more have that

$$\frac{h_1(x_{f0})}{\int_0^{x_{f0}} h_1\, \mathrm{d}x} > 0. \tag{4.15}$$

However, in that case (4.11) shows that σ cannot be positive, contrary to assumption. We can also exclude zero eigenvalues. As already noted by Wilchinsky (2009), (4.10) shows that a zero eigenvalue is only possible if $a(x_{f0}) - Q_f'(x_{f0}) = 0$. Hence, as predicted by the heuristic argument leading up to (3.5), steady states with $a(x_{f0}) - Q_f'(x_{f0}) < 0$ are stable.

5. Discussion and conclusions

The work above has confirmed that a simple argument about the mass balance of the ice sheet suffices to determine the stability of steady states in one horizontal dimension. If an expansion of the ice sheet away from the steady state leads to a positive mass balance, then the steady state is unstable, while the converse implies a stable steady state (Weertman 1974; Schoof 2007*a*). In particular, if $Q_f(x_f)$ is outflow through the grounding line at position x_f, then a steady state with its grounding line at x_f is linearly unstable if and only if

$$a(x_f) - Q_f'(x_f) > 0 \tag{5.1}$$

where a is accumulation rate as a function of position, and the prime denotes differentiation. Typically, outflow $Q_f(x_f)$ increases with depth to the bed at the grounding line. Instability is therefore associated with beds that slope upward in the flow direction near the grounding line, or beds that have at least an insufficient downward slope. In that case, $Q_f'(x_f)$ is negative for an upward-sloping bed, or positive but of insufficient magnitude to render $a(x_f) - Q_f'(x_f)$ negative.

Note that this interpretation of (5.1) assumes that there is net accumulation even at the grounding line. Our theory however remains equally valid even when there is net mass loss (or *ablation*) on parts of the ice sheet surface, in which case $a(x)$ becomes negative locally. In that case, it is possible to have a stable grounding line in locations where $Q_f'(x_f)$ is negative, corresponding to an upward-sloping bed; all that is required is that $a(x_f)$ is more negative, i.e. that ablation is sufficiently intense.

Some open problems remain. The main limitation to the work above comes from the assumption that the Sturm–Liouville problem (4.5) is regular. Principally, we have to assume that Q_2 is not only continuous on the closed interval $[0, x_{f0}]$, but also that it does not vanish. Physically, Q_2 is the derivative of ice flux with respect to the downward surface slope in the flow direction. As such, we expect Q_2 to be non-negative, but there are situations in which Q_2 may vanish at the origin, where flux vanishes. For instance, if we assume a prescription for flux of the form (2.1*b*), then

$$Q_2(x) = -\frac{1}{m}\left(\rho g/C\right)^{1/m} h^{1+1/m} \left|(h+b)_x\right|^{1/m-1} = \frac{1}{m}\frac{\rho g}{C} h^{m+1} |q|^{1-m}, \tag{5.2}$$

which vanishes at $q = 0$ when $m < 1$ (Schoof 2007*a* for instance uses $m = 1/3$, and much of the theory of glacier sliding suggests that $m < 1$, see e.g. Fowler 1981). It is of course possible to avoid issues of this type by insisting on a regularization of the flux prescription, for instance by replacing (2.1*b*) with

$$q = -\frac{1}{m}\left(\rho g/C\right)^{1/m} h^{1+1/m} \left[(h_x + b_x)^2 + \delta^2\right]^{(1-2m)/(2m)} (h+b)_x, \tag{5.3}$$

where δ is a constant regularization parameter. Regularizations of this type are typically applied in numerical ice flow simulations. Alternatively, it may be possible to use stretched coordinates (Hindmarsh 1997) or to appeal to the theory of singular Sturm–Liouville problems to obtain the same results as above. However, the nature of the singularities encountered then generally depends on the specifics of the parameter choices, and different cases may need to be treated separately.

More complicated to tackle is the following problem: most of the derivations of grounding-line flux conditions such as (2.1*e*) or more generally (2.2*e*) are based on asymptotic expansions that actually require that depth to bedrock at the grounding line be small compared with ice thickness in the bulk of the ice sheet (Chugunov & Wilchinsky 1996; Schoof 2007*b*, 2011). At leading order, this is equivalent to having $h_f = 0$ in (2.2*d*). Typical thin-film flux relationships such as (2.1*b*) however give flux as a product of a power of ice thickness and a power of surface slope. A finite flux $Q_f(x_f)$ at the grounding line accompanied by vanishing ice thickness $h_f = 0$ therefore requires an infinite surface slope at the grounding line. This cannot be captured by the straightforward linearization in § 4, but may be amenable to an approach using stretched coordinates (Halfar 1981; Fowler 2001). We leave a consideration of this problem to future work.

If we accept the basic form of the problem (2.1) with a non-zero flotation thickness and with a regularization of the form (5.3), if necessary to make the analysis in this paper applicable, the general stability problem in two horizontal dimensions still remains open (Goldberg *et al.* 2009; Katz & Worster 2010). There are several possible complications that can arise. In the absence of an extensive ice shelf, one may still obtain a relation between ice flux and ice thickness at the grounding line. However, complex three-dimensional bed topogaphy may render a simple stability result of the form (3.5) impossible. Katz & Worster (2010) for instance present a steady state whose grounding line cuts across a trough in the bed topography such that (3.5) is satisfied along parts of the grounding line, while the inequality is reversed on others, and the steady state appears to be stable. Katz & Worster argue that curvature terms retained in their model for grounding-line flux are responsible for stabilizing the grounding line on an upward bed slope. However, it is also conceivable that transverse ice flow could help to stabilize the grounding line for the type of bed topography they consider in the absence of their curvature terms. The problem becomes more complex still when there is a shelf of significant size attached to the ice sheet. In two horizontal dimensions, a shelf can affect stresses at the grounding line, and this in turn can stabilize grounding lines on upward slopes (Goldberg *et al.* 2009). In that case, processes that affect the size of the shelf, such as melting or iceberg calving, also become of primary importance. These complications are beyond the scope of our work, and it is not clear that it will be possible to treat them analytically in the same way as the one-dimensional ice sheet problem considered here.

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REFERENCES

CHUGUNOV, V. A. & WILCHINSKY, A. V. 1996 Modelling of marine glacier and ice-sheet-ice-shelf transition zone based on asymptotic analysis. *Ann. Glaciol.* **23**, 59–67.

DUPONT, T. K. & ALLEY, R. B. 2005 Assessment of the importance of ice-shelf buttressing to ice sheet flows. *Geophys. Res. Lett.* **32**, L04503.

DURAND, G., GAGLIARDINI, O., DE FLEURIAN, B., ZWINGER, T. & LEMEUR, E. 2009 Marine ice sheet dynamics: Hysteresis and neutral equilibrium. *J. Geophys. Res.* **114**, F03009.

FOWLER, A. C. 1981 A theoretical treatment of the sliding of glaciers in the absence of cavitation. *Phil. Trans. R. Soc. Lond.* **298** (1445), 637–685.

FOWLER, A. C. 2001 Modelling the flow of glaciers and ice sheets. In *Continuum Mechanics and Applications in Geophysics and the Environment* (ed. B. Straughan, R. Greve, H. Ehrentraut & Y. Wang). pp. 276–304. Springer.

GOLDBERG, D., HOLLAND, D. M. & SCHOOF, C. 2009 Grounding line movement and ice shelf buttressing in marine ice sheets. *J. Geophys. Res.* **114**, F04026.

GOMEZ, N., MITROVICA, J. X., HUYBERS, P. & CLARK, P. U. 2010 Sea level as a stabilizing factor for marine-ice-sheet grounding lines. *Nature Geosci.* **3** (12), 850–853.

HALFAR, P. 1981 On the dynamics of ice sheets. *J. Geophys. Res.* **86** (NC11), 1065–1072.

HINDMARSH, R. C. A. 1997 Normal modes of an ice sheet. *J. Fluid Mech.* **502**, 17–40.

KATZ, R. F. & WORSTER, M. G. 2010 Stability of ice-sheet grounding lines. *Proc. R. Soc. Lond.* A **466** (2118), 1597–1620.

MACAYEAL, D. R. & BARCILON, V. 1988 Ice-shelf response to ice-stream discharge fluctuations: I. unconfined ice tongues. *J. Glaciol.* **34** (116), 121–127.

ROBISON, R. A. V., HUPPERT, H. E. & WORSTER, M. G. 2010 Dynamics of viscous grounding lines. *J. Fluid Mech.* **648**, 363–380.

SCHOOF, C. 2007*a* Ice sheet grounding line dynamics: steady states, stability and hysteresis. *J. Geophys. Res.* **112**, F03S28.

SCHOOF, C. 2007*b* Marine ice sheet dynamics. Part 1. The case of rapid sliding. *J. Fluid Mech.* **573**, 27–55.

SCHOOF, C. 2011 Marine ice sheet dynamics. Part 2. A Stokes flow contact problem. *J. Fluid Mech.* **679**, 122–155.

WEERTMAN, J. 1974 Stability of the junction of an ice sheet and an ice shelf. *J. Glaciol.* **13** (67), 3–13.

WILCHINSKY, A. V. 2001 Studying ice sheet stability using the method of separation of variables. *Geophys. Astrophys. Fluid Dyn.* **94**, 15–45.

WILCHINSKY, A. V. 2009 Linear stability analysis of an ice sheet interacting with the ocean. *J. Glaciol.* **55** (189), 13–20.

WINKELMANN, R., MARTIN, M. A., HASELOFF, M., ALBRECHT, T., BUELER, E., KHROULEV, C. & LEVERMANN, A. 2011 The potsdam parallel ice sheet model (pism-pik). Part 1. Model description. *The Cryosphere* **5** (3), 715–726.