Final: EOSC 250

19 April, 2016

This exam consists of four questions worth ten marks each. Questions 2–4 have optional parts worth an additional bonus point each. Available marks for each part of a question are indicated in brackets; these are a guide to the level of detail expected. Attempt **THREE** questions. **READ THE QUESTIONS CAREFULLY.** You have 2 hours and forty-five minutes.

1. (Vector calculus) Let S be the triangle with corners $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$, with $\hat{\mathbf{n}}$ being the upward-pointing unit normal to S. Let C be the boundary curve to S, traversed in an anticlockwise direction when viewed from above. That is, C consists of the three line segments connecting the vertices of the triangle. Let

$$\mathbf{v} = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}.$$

   (a) (1 point) Sketch S and C, indicating the orientation of $\hat{\mathbf{n}}$ and (with an arrow) the direction in which C is traversed.

   (b) (2 points) Compute $\nabla \times \mathbf{v}$. Use the equation sheet if necessary.

   (c) (4 points) Compute $\int_S (\nabla \times \mathbf{v}) \cdot \hat{\mathbf{n}}\, \mathrm{d}S$

   (d) (3 points) Compute $\int_C \mathbf{v} \cdot \mathrm{d}\mathbf{r}$, and verify that Stokes' theorem holds. (Integrate along each line segment separately to do this integral.)

2. (Vector calculus and Poisson's equation) Consider an infinitely long cylinder centred on the z-axis, containing electical charge with uniform charge density ρ_0. The electrical potential ϕ due to this cylinder satisfies

$$-\epsilon_0 \nabla^2 \phi = \rho_0.$$

Let

$$\mathbf{r}(x, y, z) = x\mathbf{i} + y\mathbf{j}, \qquad r(x, y, z) = \sqrt{x^2 + y^2} = |\mathbf{r}(x, y, z)|.$$

so r is distance of the point (x, y, z) from the symmetry axis of the cylinder. Recall the product and chain rules for divergences and gradients,

$$\nabla f(g(x, y, z)) = \frac{\mathrm{d}f}{\mathrm{d}g} \nabla g, \qquad \nabla(fg) = g\nabla f + f\nabla g, \qquad \nabla \cdot (f\mathbf{v}) = f\nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla f.$$

Assume that the symmetry of the cylinder means that ϕ can be written as $\phi = \phi(r)$.

(a) (2 points) Show that

$$\nabla r = \frac{\mathbf{r}}{r}$$

(b) (1 point) Show that

$$\nabla \cdot \mathbf{r} = 2$$

(c) (1 point) Show that

$$\nabla \phi = \frac{\mathrm{d}\phi}{\mathrm{d}r} \frac{\mathbf{r}}{r}$$

(d) (3 points) Show that

$$\nabla^2 \phi = \nabla \cdot (\nabla \phi) = \frac{\mathrm{d}^2 \phi}{\mathrm{d}r^2} + \frac{1}{r} \frac{\mathrm{d}\phi}{\mathrm{d}r} = \frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}r} \left(r \frac{\mathrm{d}\phi}{\mathrm{d}r} \right).$$

(e) (3 points) Find a general solution to Poisson's equation

$$-\epsilon_0 \frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}r} \left(r \frac{\mathrm{d}\phi}{\mathrm{d}r} \right) = \rho_0$$

containing two constants of integration. Do not attempt to impose any boundary conditions.

(f) (1 point) BONUS: If a point charge gives rise to an 'inverse square law' electrical field (meaning $|\mathbf{E}|$ is proportional to one over the square of distance from the point source), how does the electrical field strength vary with distance from a line of electrical charge?

3. (Conservation laws and the heat equation) Consider a sphere of radius R. Let the heat production rate density in the sphere be a, where a is constant. At the surface of the sphere, heat is received from space through incoming radiation at a fixed rate of q_0 per unit surface area. Treat q_0 as a constant, with dimensions of energy over time and area. Heat is also lost into space through radiation from the surface. Per unit surface area, the rate of heat loss is σT_s^4, where T_s is the surface temperature, and σ is a constant known as the Stefan-Boltzmann constant. Treat T_s as having the same value everywhere on the surface of the sphere.

   (a) (2 points) Assume the sphere is in a steady state. Without solving any differential equations, use a simple energy balance argument and algebra / geometry to compute the surface temperature of the sphere in terms of a, q_0, σ and R. Show that

$$T_s = \left(\frac{q_0 + aR/3}{\sigma} \right)^{1/4}.$$

   (b) (8 points) Assume that the the temperature field in the sphere $T(r)$ depends only on the distance r of a point from the centre of the sphere, and that heat transport occurs purely be conduction. $T(r)$ satisfies

$$-\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 k \frac{\mathrm{d}T}{\mathrm{d}r} \right) = a$$

   with boundary condition

$$-k\frac{\mathrm{d}T}{\mathrm{d}r} = \sigma T^4 - q_0 \qquad \text{at } r = R.$$

   Solve for $T(r)$, being careful in how you derive the constants of integration. Clearly state any further assumptions you make. Show that your solution at $T(R)$ equals T_s computed in part a.

   (c) (1 point) BONUS: For a planet, q_0 is usually not constant over the surface of the planet. A constant q_0 would imply for instance incoming sunlight everywhere falling *vertically* on the planetary surface. This is not the case: for the Earth, less light is received per unit area of surface near the poles because sunlight does not reach the surface vertically. Assume instead that parallel rays of light reach the planetary surface from one side, and that these rays of light carry energy at a rate q_s per unit area perpendicular to the rays. Again using basic geometry and algebra, show why

$$\sigma T_s^4 = \frac{aR}{3} + \frac{q_s}{4}.$$

4. (Differential equations) This question is about modelling population size. Let $n(t)$ be the number of living individuals in a population at time t (which could be bacteria in a petri dish or humans in a society). Assume that in a given time interval δt, a constant *fraction* $\lambda \delta t$ of individuals that are alive will reproduce successfully. Assume also that the fraction of individuals that will die in a given time interval δt increases with population size: individuals are more likely to die in a crowded population, where competition for resources becomes more intense. Assume that the *fraction* of initially living individuals that die in the interval is $\mu n(t) \delta t$.

   (a) (2 points) Explain carefully why

   $$\delta n = (\lambda n - \mu n^2)\delta t.$$

   Be sure to explain why it is *not true* that

   $$\delta n = (\lambda - \mu n)\delta t.$$

   Why does δt have to be small? Be succinct and precise.

   (b) (1 point) The rate of change in the population size is therefore

   $$\frac{\mathrm{d}n}{\mathrm{d}t} = \lambda n - \mu n^2. \tag{1}$$

   What are the possible steady states for the population? Assume λ and μ are constants.

   (c) (1 point) As a preliminary step in solving equation (1), demonstrate the identity

   $$\frac{1}{n(a-n)} = \frac{1}{a}\left(\frac{1}{n} + \frac{1}{a-n}\right).$$

   You can either show that the expression on the right equals the expression on the left, or vice versa.

   (d) (6 points) Solve (1) for $n(t)$, assuming that $n(0) = n_0$ is given.

   (e) (1 point) BONUS: One of the steady states you have identified in part b is 'unstable'. Which one, and (physically) why?

EOSC 250 - Geophysical Fields and Fluxes
Equation Summary

$$A = \pi r^2$$

$$A = 2\pi r h$$

$$A = 4\pi r^2$$

$$A = \frac{1}{2} \times \text{base} \times \text{height}$$

$$V = \pi r^2 h$$

$$V = \frac{4}{3}\pi r^3$$

$$V = \frac{1}{3} \times \text{base} \times \text{height}$$

$$M = \int_V \rho \, \mathrm{d}V$$

$$E = \int_V e \, \mathrm{d}V$$

$$\hat{\mathbf{n}} = \pm \frac{\mathbf{k} - \frac{\partial h}{\partial x}\mathbf{i} - \frac{\partial h}{\partial y}\mathbf{j}}{\sqrt{1 + \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2}}$$

$$\mathrm{d}S = \sqrt{1 + \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2} \, \mathrm{d}x \, \mathrm{d}y$$

$$\int \mathbf{q} \cdot \hat{\mathbf{n}} \, \mathrm{d}S = \int \int q_z - q_x \frac{\partial h}{\partial x} - q_y \frac{\partial h}{\partial y} \, \mathrm{d}y \, \mathrm{d}x$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V \rho \, \mathrm{d}V = - \int_S \rho \mathbf{v} \cdot \hat{\mathbf{n}} \, \mathrm{d}S$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V e \, \mathrm{d}V = - \int_S e\mathbf{v} \cdot \hat{\mathbf{n}} \, \mathrm{d}S - \int_S \mathbf{q}_c \cdot \hat{\mathbf{n}} \, \mathrm{d}S + \int_V a \, \mathrm{d}V$$

$$\nabla \cdot \mathbf{q} = \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}, \qquad \mathbf{q} = q_x \mathbf{i} + q_y \mathbf{j} + q_z \mathbf{k}$$

$$\int_S \mathbf{q} \cdot \hat{\mathbf{n}} \, \mathrm{d}S = \int_V \nabla \cdot \mathbf{q} \, \mathrm{d}V$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial e}{\partial t} + \nabla \cdot (e\mathbf{v}) + \nabla \cdot \mathbf{q}_c = a$$

$$\mathbf{q}_c = -k\nabla T, \qquad e = \rho c T$$

$$\nabla T = \mathbf{i}\frac{\partial T}{\partial x} + \mathbf{j}\frac{\partial T}{\partial y} + \mathbf{k}\frac{\partial T}{\partial z}$$

$$\rho c \frac{\partial T}{\partial t} + \rho c \mathbf{v} \cdot \nabla T - \nabla \cdot (k \nabla T) = a$$

$$-\nabla \cdot (k\nabla T) = a$$

$$\nabla \cdot \nabla T = \nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

$$-k\nabla^2 T = a$$

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(k\frac{\mathrm{d}T}{\mathrm{d}x}\right) = a(x),$$

$$-\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(rk\frac{\mathrm{d}T}{\mathrm{d}r}\right) = a(r)$$

$$-\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 k\frac{\mathrm{d}T}{\mathrm{d}r}\right) = a(r)$$

$$\frac{\mathrm{d}q}{\mathrm{d}x} = a(x) \qquad q(x) = -k\frac{\mathrm{d}T}{\mathrm{d}x}$$

$$\frac{1}{r}\frac{\mathrm{d}(rq)}{\mathrm{d}r} = a(r), \qquad q(r) = -k\frac{\mathrm{d}T}{\mathrm{d}r}$$

$$\frac{1}{r^2}\frac{\mathrm{d}(r^2 q)}{\mathrm{d}r} = a(r) \qquad q(r) = -k\frac{\mathrm{d}T}{\mathrm{d}r}$$

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}, \qquad r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}, \qquad \hat{\mathbf{r}} = \frac{\mathbf{r}}{r}.$$

$$\nabla T(r) = \frac{\mathrm{d}T}{\mathrm{d}r}\hat{\mathbf{r}}$$

$$\nabla \cdot [q(r)\hat{\mathbf{r}}] = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left[r^2 q(r)\right].$$

$$\mathbf{q}(\mathbf{r}) = \frac{Q_0}{4\pi r^2}\hat{\mathbf{r}}.$$

$$T(\mathbf{r}) = \frac{Q_0}{4\pi k r}$$

$$\mathbf{q}(\mathbf{r}) = \frac{Q_0}{4\pi|\mathbf{r} - \mathbf{r}_0|^2}\frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|}$$

$$T(\mathbf{r}) = T_\infty + \frac{Q_0}{4\pi k|\mathbf{r} - \mathbf{r}_0|}$$

$$\mathbf{q}(\mathbf{r}) = \sum_i \frac{Q_i}{4\pi|\mathbf{r}-\mathbf{r}_i|^2}\frac{(\mathbf{r}-\mathbf{r}_i)}{|\mathbf{r}-\mathbf{r}_i|}$$

$$T(\mathbf{r}) = T_\infty + \sum_i \frac{Q_i}{4\pi k|\mathbf{r}-\mathbf{r}_i|}$$

$$\int_C \mathbf{f}\cdot \mathrm{d}\mathbf{r} = \int_{t_1}^{t_2} \left[f_x(x(t),y(t),z(t))\frac{\mathrm{d}x}{\mathrm{d}t} + f_y(x(t),y(t),z(t))\frac{\mathrm{d}y}{\mathrm{d}t} + f_z(x(t),y(t),z(t))\frac{\mathrm{d}z}{\mathrm{d}t}\right] \mathrm{d}t$$

$$\nabla\times\mathbf{f} = \left(\frac{\partial f_z}{\partial y} + \frac{\partial f_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right)\mathbf{k}$$

$$\nabla\times\mathbf{f} = \begin{vmatrix} \mathbf{j} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix}$$

$$\int_S (\nabla\times\mathbf{f})\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \int_C \mathbf{f}\cdot\,\mathrm{d}\mathbf{r}$$

$$\phi = -\int_C \mathbf{f}\cdot\,\mathrm{d}\mathbf{r}$$

$$\phi(\mathbf{r}_B) - \phi(\mathbf{r}_A) = \int_C \nabla\phi\cdot\,\mathrm{d}\mathbf{r}$$

$$\mathbf{f} = -\nabla\phi$$

$$\mathbf{f} = -\frac{Gm}{r^2}\hat{\mathbf{r}}$$

$$\phi = -\frac{Gm}{r}$$

$$\nabla^2\phi = 4\pi G\rho, \qquad \mathbf{g} = -\nabla\phi$$

$$-\epsilon\nabla^2\phi = \rho_c, \qquad \mathbf{E} = -\nabla\phi$$

$$\nabla(fg) = (\nabla f)g + f(\nabla g)$$

$$\nabla f(g) = \frac{\mathrm{d}f}{\mathrm{d}g}\nabla g$$

$$\nabla\cdot(\phi\mathbf{f}) = \phi\nabla\cdot\mathbf{f} + \mathbf{f}\cdot\nabla\phi$$

$$\nabla\times(\phi\mathbf{f}) = (\nabla\phi)\times\mathbf{f} + \phi\nabla\times\mathbf{f}$$

$$\nabla\times\nabla\phi = \mathbf{0}$$

$$\nabla\cdot(\nabla\times\mathbf{f}) = 0$$