Final: EOSC 250

19 April, 2016

This exam consists of four questions worth ten marks each. Questions 2–4 have optional parts worth an additional bonus point each. Available marks for each part of a question are indicated in brackets; these are a guide to the level of detail expected. Attempt **THREE** questions. **READ THE QUESTIONS CAREFULLY.** You have 2 hours and forty-five minutes.

1. (Vector calculus) Let S be the triangle with corners $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$, with $\hat{\mathbf{n}}$ being the upward-pointing unit normal to S. Let C be the boundary curve to S, traversed in an anticlockwise direction when viewed from above. That is, C consists of the three line segments connecting the vertices of the triangle. Let

$$\mathbf{v} = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}.$$

(a) (1 point) Sketch S and C, indicating the orientation of $\hat{\mathbf{n}}$ and (with an arrow) the direction in which C is traversed.

(b) (2 points) Compute $\nabla \times \mathbf{v}$. Use the equation sheet if necessary.

$$\nabla \times \mathbf{v} = -y\mathbf{i} - z\mathbf{j} - x\mathbf{k}$$

(c) (4 points) Compute $\int_S (\nabla \times \mathbf{v}) \cdot \hat{\mathbf{n}}\, \mathrm{d}S$

ANS: The surface is at $z = h(x, y) = 1 - x - y$, with bounds in the xy-plane of $0 < x < 1$, $0 < y < 1 - x$. We have, with an upward-pointing unit normal,

$$\hat{\mathbf{n}}\, \mathrm{d}S = \left(\mathbf{k} - \frac{\partial h}{\partial x}\mathbf{i} - \frac{\partial h}{\partial y}\mathbf{j}\right) \mathrm{d}y\, \mathrm{d}x = (\mathbf{i} + \mathbf{j} + \mathbf{k})\ \mathrm{d}y\, \mathrm{d}x$$

and so, with $z = 1 - x - y$,

$$\begin{aligned}
\int_S (\nabla \times \mathbf{v}) \cdot \hat{\mathbf{n}}\, \mathrm{d}S &= \int_0^1 \int_0^{1-x} (-y\mathbf{i} - (1 - x - y)\mathbf{j} - x\mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k})\ \mathrm{d}y\, \mathrm{d}x \\
&= \int_0^1 \int_0^{1-x} -(x + y + 1 - x - y)\, \mathrm{d}y\, \mathrm{d}x \\
&= -\int_0^1 \int_0^{1-x} 1\, \mathrm{d}y\, \mathrm{d}x \\
&= -\frac{1}{2}
\end{aligned}$$

(d) (3 points) Compute $\int_C \mathbf{v} \cdot \mathrm{d}\mathbf{r}$, and verify that Stokes' theorem holds. (Integrate along each line segment separately to do this integral.)

ANS: Split the curve C into three parts C_1, C_2 and C_3, going clockwise around S. The first is from $(1,0,0)$ to $(0,1,0)$, with $(x, y, z) = (x, 0, 1-x) = \mathbf{r}(x)$ and x going from 0 to 1. On the curve, $\mathbf{v} = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k} = x(1-x)\mathbf{k}$ so that

$$\begin{aligned}
\int_{C_1} \mathbf{v} \cdot\, \mathrm{d}\mathbf{r} &= \int_0^1 x(1-x)\mathbf{k} \cdot (\mathbf{i} - \mathbf{k})\, \mathrm{d}x \\
&= \int_0^1 -x(1-x)\, \mathrm{d}x \\
&= \frac{1}{3} - \frac{1}{2} \\
&= -\frac{1}{6}
\end{aligned}$$

The integrals along the other sides turn out to be $-1/6$ as well, something you might intuit from symmetry. Specifically, C_2 can be written as $(x, y, z) = (1-y, y, 0) = \mathbf{r}(y)$ with y going from 0 to 1, and $\mathbf{v} = (1-y)y\mathbf{i}$ so

$$\begin{aligned}\int_{C_1} \mathbf{v} \cdot \, \mathrm{d}\mathbf{r} &= \int_0^1 y(1-y)\mathbf{i} \cdot (-\mathbf{i} + \mathbf{j}) \, \mathrm{d}y \\ &= \int_0^1 -y(1-y) \, \mathrm{d}y \\ &= -\frac{1}{6}\end{aligned}$$

and the computation for C_3 can be done in the same way. Hence

$$\int_C \mathbf{v} \cdot \, \mathrm{d}\mathbf{r} = -\frac{1}{6} - \frac{1}{6} - \frac{1}{6} = -\frac{1}{2} = \int_S (\nabla \times \mathbf{v}) \cdot \hat{\mathbf{n}} \, \mathrm{d}S$$

and Stokes' theorem holds.

2. (Vector calculus and Poisson's equation) Consider an infinitely long cylinder centred on the z-axis, containing electical charge with uniform charge density ρ_0. The electrical potential ϕ due to this cylinder satisfies

$$-\epsilon_0 \nabla^2 \phi = \rho_0.$$

Let

$$\mathbf{r}(x,y,z) = x\mathbf{i} + y\mathbf{j}, \qquad r(x,y,z) = \sqrt{x^2+y^2} = |\mathbf{r}(x,y,z)|.$$

so r is distance of the point (x,y,z) from the symmetry axis of the cylinder. Recall the product and chain rules for divergences and gradients,

$$\nabla f(g(x,y,z)) = \frac{\mathrm{d}f}{\mathrm{d}g}\nabla g, \qquad \nabla(fg) = g\nabla f + f\nabla g, \qquad \nabla\cdot(f\mathbf{v}) = f\nabla\cdot\mathbf{v} + \mathbf{v}\cdot\nabla f.$$

Assume that the symmetry of the cylinder means that ϕ can be written as $\phi = \phi(r)$.

(a) (2 points) Show that

$$\nabla r = \frac{\mathbf{r}}{r}$$

ANS: Do this by direct differentiation

$$\nabla r = \frac{\partial\sqrt{x^2+y^2}}{\partial x}\mathbf{i} + \frac{\partial\sqrt{x^2+y^2}}{\partial y}\mathbf{j} + \frac{\partial\sqrt{x^2+y^2}}{\partial z}\mathbf{k} \tag{1}$$

$$= \frac{x}{\sqrt{x^2+y^2}}\mathbf{i} + \frac{y}{\sqrt{x^2+y^2}}\mathbf{j} \tag{2}$$

$$= \frac{\mathbf{r}}{r} \tag{3}$$

(b) (1 point) Show that

$$\nabla\cdot\mathbf{r} = 2$$

ANS: Again, direct differentiation

$$\nabla\cdot\mathbf{r} = \nabla\cdot(x\mathbf{i} + y\mathbf{j}) = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} = 2.$$

(c) (1 point) Show that

$$\nabla\phi = \frac{\mathrm{d}\phi}{\mathrm{d}r}\frac{\mathbf{r}}{r}$$

ANS: Use the chain rule and the result for ∇r above

$$\nabla\phi(r(x,y,z)) = \frac{\mathrm{d}\phi}{\mathrm{d}r}\nabla r = \frac{\mathrm{d}\phi}{\mathrm{d}r}\frac{\mathbf{r}}{r}.$$

(d) (3 points) Show that

$$\nabla^2\phi = \nabla\cdot(\nabla\phi) = \frac{\mathrm{d}^2\phi}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\phi}{\mathrm{d}r} = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\phi}{\mathrm{d}r}\right).$$

ANS: Use the result for $\nabla\phi$ above and then use the product and chain rules

$$\begin{aligned}
\nabla\cdot(\nabla\phi) &= \nabla\cdot\left[\left(\frac{1}{r}\frac{\mathrm{d}\phi}{\mathrm{d}r}\right)\mathbf{r}\right] \\
&= \nabla\left(\frac{1}{r}\frac{\mathrm{d}\phi}{\mathrm{d}r}\right)\cdot\mathbf{r} + \frac{1}{r}\frac{\mathrm{d}\phi}{\mathrm{d}r}\nabla\cdot\mathbf{r} \\
&= \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r}\frac{\mathrm{d}\phi}{\mathrm{d}r}\right)\frac{\mathbf{r}}{r}\cdot\mathbf{r} + \frac{1}{r}\frac{\mathrm{d}\phi}{\mathrm{d}r}\times 2 \\
&= \left(-\frac{1}{r^2}\frac{\mathrm{d}\phi}{\mathrm{d}r} + \frac{1}{r}\frac{\mathrm{d}^2\phi}{\mathrm{d}r^2}\right)\frac{|\mathbf{r}|^2}{r} + \frac{2}{r}\frac{\mathrm{d}\phi}{\mathrm{d}r} \\
&= \left(-\frac{1}{r^2}\frac{\mathrm{d}\phi}{\mathrm{d}r} + \frac{1}{r}\frac{\mathrm{d}^2\phi}{\mathrm{d}r^2}\right)r + \frac{2}{r}\frac{\mathrm{d}\phi}{\mathrm{d}r} \\
&= \frac{1}{r}\frac{\mathrm{d}\phi}{\mathrm{d}r} + \frac{\mathrm{d}^2\phi}{\mathrm{d}r^2} \\
&= \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\phi}{\mathrm{d}r}\right)
\end{aligned}$$

where the last step uses the ordinary product rule in reverse

(e) (3 points) Find a general solution to Poisson's equation

$$-\epsilon_0\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\phi}{\mathrm{d}r}\right) = \rho_0$$

containing two constants of integration. Do not attempt to impose any boundary conditions.

ANS: Separate variables

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\phi}{\mathrm{d}r}\right) = -\frac{\rho_0}{\epsilon_0}r$$

Integrate

$$r\frac{\mathrm{d}\phi}{\mathrm{d}r} = -\frac{\rho_0}{2\epsilon_0}r^2 + C_1$$

Separate variables again

$$\frac{\mathrm{d}\phi}{\mathrm{d}r} = -\frac{\rho_0}{2\epsilon_0}r + \frac{C_1}{r}$$

Integrate again

$$\phi(r) = -\frac{\rho_0}{4\epsilon_0}r^2 + C_1\log(r) + C_2.$$

(f) (1 point) BONUS: If a point charge gives rise to an 'inverse square law' electrical field (meaning $|\mathbf{E}|$ is proportional to one over the square of distance from the point source), how does the electrical field strength vary with distance from a line of electrical charge?
ANS: The line source is what must gives rise to the singular term again (as in the same problem for a sphere). In the present case, we have

$$|\mathbf{E}| = \left|\frac{\mathrm{d}\phi}{\mathrm{d}r}\right|$$

behaving as C_1/r near the axis of symmetry, so the electrical field is proportional to $1/r$ (rather than $1/r^2$ for a point charge).

3. (Conservation laws and the heat equation) Consider a sphere of radius R. Let the heat production rate density in the sphere be a, where a is constant. At the surface of the sphere, heat is received from space through incoming radiation at a fixed rate of q_0 per unit surface area. Treat q_0 as a constant, with dimensions of energy over time and area. Heat is also lost into space through radiation from the surface. Per unit surface area, the rate of heat loss is σT_s^4, where T_s is the surface temperature, and σ is a constant known as the Stefan-Boltzmann constant. Treat T_s as having the same value everywhere on the surface of the sphere.

   (a) (2 points) Assume the sphere is in a steady state. Without solving any differential equations, use a simple energy balance argument and algebra / geometry to compute the surface temperature of the sphere in terms of a, q_0, σ and R. Show that

   $$T_s = \left(\frac{q_0 + aR/3}{\sigma} \right)^{1/4}.$$

   ANS: The total heat produced in the sphere is volume $\times a = (4\pi R^3/3)a$, while the total amount of heat absorbed is surface area $\times q_0 = 4\pi R^2 q_0$, and the total amount of heat radiated into space is surface area $\times \sigma T_s^4 = 4\pi R^2 \sigma T_s^4$. Assuming they balance, we have

   $$(4\pi R^3/3)a + 4\pi R^2 q_0 = 4\pi R^2 \sigma T_s^4$$

   Rearranging gives the required result.

   (b) (8 points) Assume that the the temperature field in the sphere $T(r)$ depends only on the distance r of a point from the centre of the sphere, and that heat transport occurs purely be conduction. $T(r)$ satisfies

   $$-\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 k \frac{\mathrm{d}T}{\mathrm{d}r} \right) = a$$

   with boundary condition

   $$-k\frac{\mathrm{d}T}{\mathrm{d}r} = \sigma T^4 - q_0 \qquad \text{at } r = R. \tag{4}$$

   Solve for $T(r)$, being careful in how you derive the constants of integration. Clearly state any further assumptions you make. Show that your solution at $T(R)$ equals T_s computed in part a.
   ANS: Separate variables

   $$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 k \frac{\mathrm{d}T}{\mathrm{d}r} \right) = ar^2$$

Integrate

$$-r^2 k\frac{\mathrm{d}T}{\mathrm{d}r} = \frac{ar^3}{3} + C_1$$

Separate variables

$$-k\frac{\mathrm{d}T}{\mathrm{d}r} = \frac{ar}{3} + \frac{C_1}{r^2}$$

Integrate again

$$-kT = \frac{ar^2}{6} - \frac{C_1}{r} + C_2,$$

so that

$$T(r) = -\frac{ar^2}{6k} + \frac{C_1}{kr} - \frac{C_2}{k}.$$

Assume the heat flux $-k\,\mathrm{d}T/\,\mathrm{d}r$ is bounded at the origin, so there is no point heat source. This is only possible if $C_1 = 0$. At the surface $T = R$, we have

$$-k\frac{\mathrm{d}T}{\mathrm{d}r} = \frac{aR}{3} = \sigma\left(-\frac{aR^2}{6k} - \frac{C_2}{k}\right)^4 - q_0$$

Solve for $-C_2/k$. First, rearrange to give

$$\left(\frac{aR + 3q_0}{3\sigma}\right)^{1/4} = -\frac{aR^2}{6k} - \frac{C_2}{k} = T(R)$$

which gives $T(R) = T_s$ as required. Then

$$-\frac{C_2}{k} = \left(\frac{aR + 3q_0}{3\sigma}\right)^{1/4}\frac{aR^2}{6k}$$

and so

$$T(r) = \frac{a(R^2 - r^2)}{6k} + \left(\frac{aR + 3q_0}{3\sigma}\right)^{1/4} = \frac{a(R^2 - r^2)}{6k} + T_s.$$

(Note that you wouldn't get full marks for simply abandoning the boundary condition and imposing T_s at the surface; you are supposed to show that (4) indeed gives the same solution for temperature at the surface as a simple energy balance calculation would demand.)

(c) (1 point) BONUS: For a planet, q_0 is usually not constant over the surface of the planet. A constant q_0 would imply for instance incoming sunlight everywhere falling *vertically* on the planetary surface. This is not the case: for the Earth, less light is received per unit area of surface near the poles because sunlight does not reach the surface vertically. Assume instead that parallel rays of light reach the planetary surface from one side, and that these rays of light carry energy at a rate q_s per unit area perpendicular to the rays. Again using basic geometry and algebra, show why

$$\sigma T_s^4 = \frac{aR}{3} + \frac{q_s}{4}.$$

ANS: Incoming rays of light are parallel, and are effectively intercepted by a 'disk' equivalent to a cross-section of the planet placed at right angles to the rays of light. The area of that disk is πR^2, so the amount of heat absorbed is $\pi R^2 q_s$. Substituting that in the calculation in part a gives

$$\frac{4}{3}\pi R^3 a + \pi R^2 q_s = 4\pi R^2 \sigma T_s^4$$

and the required solution follows

4. (Differential equations) This question is about modelling population size. Let $n(t)$ be the number of living individuals in a population at time t (which could be bacteria in a petri dish or humans in a society). Assume that in a given time interval δt, a constant *fraction* $\lambda \delta t$ of individuals that are alive will reproduce successfully. Assume also that the fraction of individuals that will die in a given time interval δt increases with population size: individuals are more likely to die in a crowded population, where competition for resources becomes more intense. Assume that the *fraction* of initially living individuals that die in the interval is $\mu n(t)\delta t$.

   (a) (2 points) Explain carefully why

   $$\delta n = (\lambda n - \mu n^2)\delta t.$$

   Be sure to explain why it is *not true* that

   $$\delta n = (\lambda - \mu n)\delta t.$$

   Why does δt have to be small? Be succinct and precise.
   In a short interval δt, the number of individuals born is equal to the fraction of individuals that reproduce times the number of individuals at the start of the interval, so $\lambda \delta t \times n$. (In particular, the number of births is *not* $\lambda \delta t$: the birth rate scales with the size of the population and is not simply a constant!) Similarly, the number of individuals that die is equal to the fraction of individuals that die times the number of individuals at the start of the interval, so $\mu n \delta t \times n$. Both calculations only make sense because we can treat n as constant over the interval, so δt must be small. The increase in population size is the number of births minus the number of deaths, so

   $$\delta n = \lambda n \delta t - \mu n^2 \delta t,$$

   which is the required result.

   (b) (1 point) The rate of change in the population size is therefore

   $$\frac{\mathrm{d}n}{\mathrm{d}t} = \lambda n - \mu n^2. \tag{5}$$

   What are the possible steady states for the population? Assume λ and μ are constants.
   ANS: Steady states are defined by $\mathrm{d}n/\,\mathrm{d}t = 0$, so $n = 0$ or $n = \lambda/\mu$.

   (c) (1 point) As a preliminary step in solving equation (5), demonstrate the identity

   $$\frac{1}{n(a-n)} = \frac{1}{a}\left(\frac{1}{n} + \frac{1}{a-n}\right).$$

You can either show that the expression on the right equals the expression on the left, or vice versa.
ANS: Right-to-left is by far the easiest:

$$\begin{aligned}
\frac{1}{a}\left(\frac{1}{n}+\frac{1}{a-n}\right) &= \frac{1}{a}\left(\frac{a-n}{n(a-n)}+\frac{n}{n(a-n)}\right) \\
&= \frac{1}{a}\frac{a-n+n}{n(a-n)} \\
&= \frac{1}{n(a-n)}.
\end{aligned}$$

(d) (6 points) Solve (5) for $n(t)$, assuming that $n(0) = n_0$ is given.
Use separation of variables:

$$\frac{\mathrm{d}n}{\mathrm{d}t} = \lambda n - \mu n^2 = \mu n\left(\frac{\lambda}{\mu} - n\right)$$

so

$$\frac{1}{n[(\lambda/\mu) - n]}\frac{\mathrm{d}n}{\mathrm{d}t} = \mu$$

Integrate both sides with respect to t

$$\int \frac{1}{n[(\lambda/\mu) - n]}\,\mathrm{d}n = \mu t + C$$

Use the result from the previous part

$$\begin{aligned}
\int \frac{1}{n[(\lambda/\mu) - n]}\,\mathrm{d}n &= \frac{1}{\lambda/\mu}\int \frac{1}{n} + \frac{(\lambda/\mu) - n}{\mathrm{d}}n \\
&= \frac{\mu}{\lambda}\left[\log(n) - \log\left(\frac{\lambda}{\mu} - n\right)\right] \\
&= \mu t + C
\end{aligned}$$

Rearrange

$$\log\left(\frac{n}{(\lambda/mu) - n}\right) = \lambda t + C'$$

or

$$n[1 + K\exp(\lambda t)] = \frac{\lambda}{\mu}K\exp(\lambda t)$$

where $K = \exp(C') = \exp(\lambda C/mu)$. At $t = 0$, $n = n_0$ so

$$n_0(1 + K) = \frac{\lambda}{\mu}K,$$

and

$$K = \frac{\mu n_0}{\lambda - \mu n_0}$$

Hence

$$n(t) = \frac{\lambda n_0 \exp(\lambda t)}{\lambda + \mu n_0 \left[\exp(\lambda t) - 1\right]} = \frac{\lambda n_0}{(\lambda - \mu n_0)\exp(-\lambda t) + \mu n_0}$$

(e) (1 point) BONUS: One of the steady states you have identified in part b is 'unstable'. Which one, and (physically) why?
An 'unstable' equilibrium is one from which the solution moves away over time if you start close to it. Looking at the solution above, it is clear that as $t \to \infty$, the solution will evolve towards

$$\lim_{t \to \infty} \frac{\lambda n_0}{\mu n_0} = \frac{\lambda}{\mu}.$$

unless $n_0 = 0$ The equilibrium $n = \lambda/\mu$ is clearly stable. The other equilibrium at $n = 0$ is clearly not stable: the solution moves away from it. Physically, this happens because a small population, so long as it is not zero in size, will have births in it but virtually no deaths (as the fraction of individuals that die per unit time is proportional to the population size, so almost zero when n is close to zero). Populations tend to grow exponentially when there are no resource restrictions on them. (Sound familiar?)

EOSC 250 - Geophysical Fields and Fluxes
Equation Summary

$$A = \pi r^2$$

$$A = 2\pi r h$$

$$A = 4\pi r^2$$

$$A = \frac{1}{2} \times \text{base} \times \text{height}$$

$$V = \pi r^2 h$$

$$V = \frac{4}{3}\pi r^3$$

$$V = \frac{1}{3} \times \text{base} \times \text{height}$$

$$M = \int_V \rho \, \mathrm{d}V$$

$$E = \int_V e \, \mathrm{d}V$$

$$\hat{\mathbf{n}} = \pm \frac{\mathbf{k} - \frac{\partial h}{\partial x}\mathbf{i} - \frac{\partial h}{\partial y}\mathbf{j}}{\sqrt{1 + \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2}}$$

$$\mathrm{d}S = \sqrt{1 + \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2} \, \mathrm{d}x \, \mathrm{d}y$$

$$\int \mathbf{q} \cdot \hat{\mathbf{n}} \, \mathrm{d}S = \int\int q_z - q_x \frac{\partial h}{\partial x} - q_y \frac{\partial h}{\partial y} \, \mathrm{d}y \, \mathrm{d}x$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V \rho \, \mathrm{d}V = -\int_S \rho \mathbf{v} \cdot \hat{\mathbf{n}} \, \mathrm{d}S$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V e \, \mathrm{d}V = -\int_S e\mathbf{v} \cdot \hat{\mathbf{n}} \, \mathrm{d}S - \int_S \mathbf{q}_c \cdot \hat{\mathbf{n}} \, \mathrm{d}S + \int_V a \, \mathrm{d}V$$

$$\nabla \cdot \mathbf{q} = \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}, \qquad \mathbf{q} = q_x\mathbf{i} + q_y\mathbf{j} + q_z\mathbf{k}$$

$$\int_S \mathbf{q} \cdot \hat{\mathbf{n}} \, \mathrm{d}S = \int_V \nabla \cdot \mathbf{q} \, \mathrm{d}V$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial e}{\partial t} + \nabla \cdot (e\mathbf{v}) + \nabla \cdot \mathbf{q}_c = a$$

$$\mathbf{q}_c = -k\nabla T, \qquad e = \rho c T$$

$$\nabla T = \mathbf{i}\frac{\partial T}{\partial x} + \mathbf{j}\frac{\partial T}{\partial y} + \mathbf{k}\frac{\partial T}{\partial z}$$

$$\rho c\frac{\partial T}{\partial t} + \rho c\mathbf{v} \cdot \nabla T - \nabla \cdot (k\nabla T) = a$$

$$-\nabla \cdot (k\nabla T) = a$$

$$\nabla \cdot \nabla T = \nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

$$-k\nabla^2 T = a$$

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(k\frac{\mathrm{d}T}{\mathrm{d}x}\right) = a(x),$$

$$-\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(rk\frac{\mathrm{d}T}{\mathrm{d}r}\right) = a(r)$$

$$-\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2k\frac{\mathrm{d}T}{\mathrm{d}r}\right) = a(r)$$

$$\frac{\mathrm{d}q}{\mathrm{d}x} = a(x) \qquad q(x) = -k\frac{\mathrm{d}T}{\mathrm{d}x}$$

$$\frac{1}{r}\frac{\mathrm{d}(rq)}{\mathrm{d}r} = a(r), \qquad q(r) = -k\frac{\mathrm{d}T}{\mathrm{d}r}$$

$$\frac{1}{r^2}\frac{\mathrm{d}(r^2q)}{\mathrm{d}r} = a(r) \qquad q(r) = -k\frac{\mathrm{d}T}{\mathrm{d}r}$$

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}, \qquad r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}, \qquad \hat{\mathbf{r}} = \frac{\mathbf{r}}{r}.$$

$$\nabla T(r) = \frac{\mathrm{d}T}{\mathrm{d}r}\hat{\mathbf{r}}$$

$$\nabla \cdot [q(r)\hat{\mathbf{r}}] = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left[r^2 q(r)\right].$$

$$\mathbf{q}(\mathbf{r}) = \frac{Q_0}{4\pi r^2}\hat{\mathbf{r}}.$$

$$T(\mathbf{r}) = \frac{Q_0}{4\pi k r}$$

$$\mathbf{q}(\mathbf{r}) = \frac{Q_0}{4\pi|\mathbf{r} - \mathbf{r}_0|^2}\frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|}$$

$$T(\mathbf{r}) = T_\infty + \frac{Q_0}{4\pi k|\mathbf{r} - \mathbf{r}_0|}$$

$$\mathbf{q}(\mathbf{r}) = \sum_i \frac{Q_i}{4\pi|\mathbf{r}-\mathbf{r}_i|^2}\frac{(\mathbf{r}-\mathbf{r}_i)}{|\mathbf{r}-\mathbf{r}_i|}$$

$$T(\mathbf{r}) = T_\infty + \sum_i \frac{Q_i}{4\pi k|\mathbf{r}-\mathbf{r}_i|}$$

$$\int_C \mathbf{f}\cdot \mathrm{d}\mathbf{r} = \int_{t_1}^{t_2} \left[f_x(x(t),y(t),z(t))\frac{\mathrm{d}x}{\mathrm{d}t} + f_y(x(t),y(t),z(t))\frac{\mathrm{d}y}{\mathrm{d}t} + f_z(x(t),y(t),z(t))\frac{\mathrm{d}z}{\mathrm{d}t}\right] \mathrm{d}t$$

$$\nabla\times\mathbf{f} = \left(\frac{\partial f_z}{\partial y} + \frac{\partial f_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right)\mathbf{k}$$

$$\nabla\times\mathbf{f} = \begin{vmatrix} \mathbf{j} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix}$$

$$\int_S (\nabla\times\mathbf{f})\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \int_C \mathbf{f}\cdot\,\mathrm{d}\mathbf{r}$$

$$\phi = -\int_C \mathbf{f}\cdot\,\mathrm{d}\mathbf{r}$$

$$\phi(\mathbf{r}_B) - \phi(\mathbf{r}_A) = \int_C \nabla\phi\cdot\,\mathrm{d}\mathbf{r}$$

$$\mathbf{f} = -\nabla\phi$$

$$\mathbf{f} = -\frac{Gm}{r^2}\hat{\mathbf{r}}$$

$$\phi = -\frac{Gm}{r}$$

$$\nabla^2\phi = 4\pi G\rho, \qquad \mathbf{g} = -\nabla\phi$$

$$-\epsilon\nabla^2\phi = \rho_c, \qquad \mathbf{E} = -\nabla\phi$$

$$\nabla(fg) = (\nabla f)g + f(\nabla g)$$

$$\nabla f(g) = \frac{\mathrm{d}f}{\mathrm{d}g}\nabla g$$

$$\nabla\cdot(\phi\mathbf{f}) = \phi\nabla\cdot\mathbf{f} + \mathbf{f}\cdot\nabla\phi$$

$$\nabla\times(\phi\mathbf{f}) = (\nabla\phi)\times\mathbf{f} + \phi\nabla\times\mathbf{f}$$

$$\nabla\times\nabla\phi = \mathbf{0}$$

$$\nabla\cdot(\nabla\times\mathbf{f}) = 0$$