Exercise answers: EOSC 250

2. (Notes on differential equations) See Figure 1

3. (Notes on differential equations) See Figure 2

4. (Notes on differential equations) Decide whether the following differential equations can be solved by separation of variables, and if they can, find a general solution and sketch it. Comment on any unusual attributes the solutions may have.

1.
$$\frac{\mathrm{d}v}{\mathrm{d}t} = t + v$$
ANS: Cannot be solved by separation of variables

2.
$$\frac{\mathrm{d}v}{\mathrm{d}t} = t + vt + v + 1$$
ANS: Can be solved by separation of variables, writing
$$\frac{\mathrm{d}v}{\mathrm{d}t} = -(t + vt + v + 1) = -t(1 + v) - (v + 1) = -(t + 1)(v + 1)$$
so that
$$\frac{1}{1+v}\frac{\mathrm{d}v}{\mathrm{d}t} = -(t + 1)$$
or
$$\log(v + 1) = -\frac{(t + 1)^2}{2} + C$$
or
$$v = A\exp\left(-\frac{(t + 1)^2}{2}\right) - 1$$
where $A = \exp(C)$.

3.
$$\frac{\mathrm{d}v}{\mathrm{d}t} = -2tv$$
ANS: Can be solved,
$$\frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}t} = -2t$$

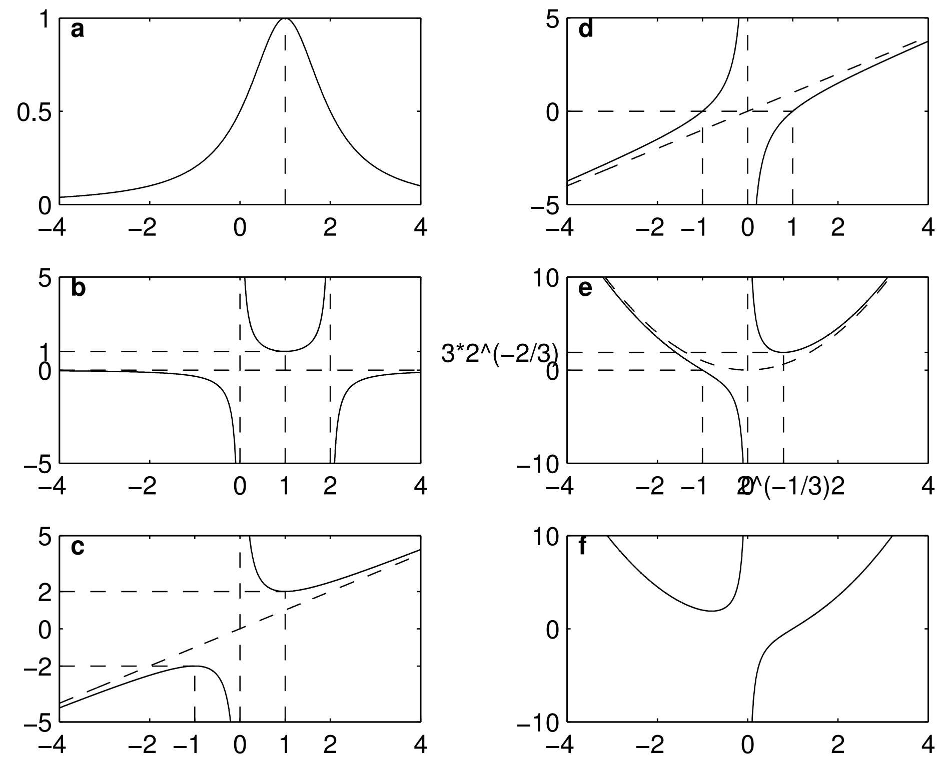

Figure 1: Sketches for the functions in problem 2. The panel labels a-f correspond to the numerical problem labels 1-6. In each case, a dashed line corresponds to an asymptote (including one marking a singularity) or it identifies a zero of the function or a stationary point. Note that the function in panel f is the reflection of the function in panel e about the vertical axis, which is why the asymptotes, singularities, stationary points and zeros are not identified explicitly

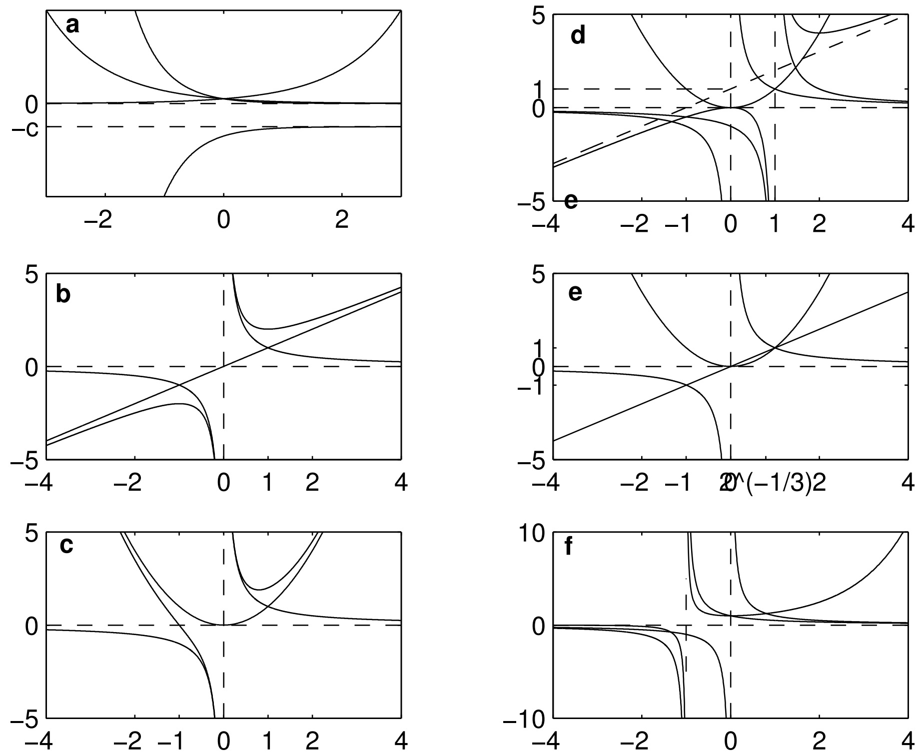

Figure 2: Sketches for the functions in problem 2. The panel labels a-f correspond to the numerical problem labels 1-6. In each case, a dashed line corresponds to an asymptote (including one marking a singularity) or it identifies an intercept of an asymptote (specifically, in panel e: the asymptotes of $x^2/(x-1)$ in panel d are $y = x + 1$ and x = 1)

or

$$\log(v) = -t^2 + C,$$

so

$$v = A\exp(-t^2)$$

4.

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 2tv + 1$$

ANS: Cannot be solved by separation of variables

5.

$$\frac{\mathrm{d}v}{\mathrm{d}t} = tv^2$$

ANS: Can be solved by separation of variables

$$\frac{1}{v^2}\frac{\mathrm{d}v}{\mathrm{d}t} = t$$

so

$$-\frac{1}{v} = t - C$$

and

$$v = \frac{1}{C - t}.$$

6.

$$\frac{\mathrm{d}v}{\mathrm{d}t} = t\exp(v) + t$$

(You may need the result that $1/(\exp(x) + 1) = \exp(-x)/(1 + \exp(-x))$) ANS: Can be solved by separation of variables, writing

$$\frac{\mathrm{d}v}{\mathrm{d}t} = (\exp(v) + 1)t,$$

or, dividing and using the identity given,

$$\frac{\exp(-v)}{1 + \exp(-v)}\frac{\mathrm{d}v}{\mathrm{d}t} = t$$

Integrating,

$$-\log[1 + \exp(-v)] = \frac{t^2}{2} + C$$

or

$$v = -\log\left[A\exp\left(-\frac{t^2}{2}\right) - 1\right]$$

where $A = \exp(-C)$

7.

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \exp(v+t).$$

ANS: This one is a bit tricky, but can be done: write

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \exp(v+t) = \exp(v)\exp(t)$$

so

$$\exp(-v)\frac{\mathrm{d}v}{\mathrm{d}t} = \exp(t),$$

and integrating

$$-\exp(-v) = \exp(t) - C$$

so

$$v = -\log\left[C - \exp(t)\right]$$

See figure 3 for sketches and comments in the caption.

7. (Notes on differential equations) Re-consider the differential equations in exercise 4. Determine which ones can be solved by integrating factors, and solve them. All can be solved by either separation of variables or by integrating factors. At least two can be solved by both methods.

1.

$$\frac{\mathrm{d}v}{\mathrm{d}t} = t + v$$

ANS: This can be solved with an integrating factor $I = \exp(-t)$: Rearrange

$$\frac{\mathrm{d}v}{\mathrm{d}t} - v = t$$

and multiply by the integrating factor

$$\exp(-t)\left(\frac{\mathrm{d}v}{\mathrm{d}t} - v]\right) = t\exp(-t)$$

so

$$\frac{\mathrm{d}[\exp(-t)v]}{\mathrm{d}t} = t\exp(-t)$$

Integrating, using integrating by parts on the right

$$\exp(-t)v = -t\exp(-t) - \exp(-t) + C$$

and so

$$v = -(t+1) + C\exp(-t).$$

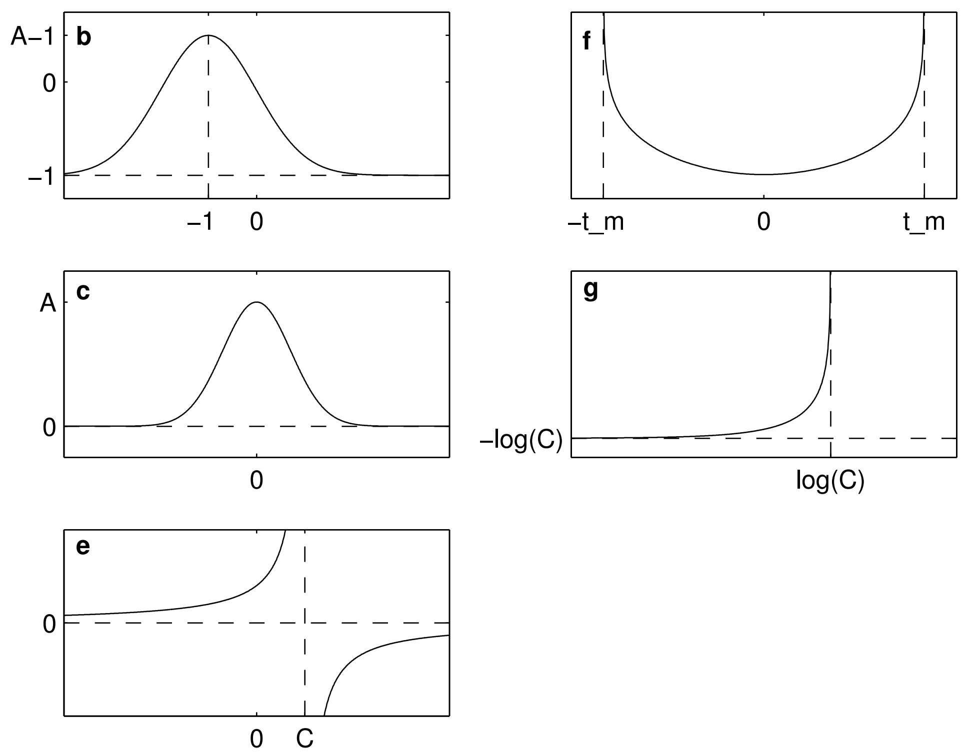

Figure 3: Sketched solutions for the parts of problem 4 solvable by separation of variables. 'b' corresponds to part 2, 'c' to part 3, 'e' to part 5, 'f' to part 6 and 'g' to part 7. Dashed lines are asymptotes or indicate the position of a local minimum or maximum, where the latter is not at $t = 0$. $t_m = \sqrt{2\log(A)}$ in panel f where $A > 1$ is necessary; the solution is only defined for $-t_m < t < t_m$. Zeros are at $t = \pm\sqrt{2\log(A/2)}$, provided $A > 2$. In panel b, zeros are at $t = -1 \pm \sqrt{2\log(A)}$ if $A > 1$. In panel g, there is a zero at $t = \log(C-1)$ provided $C > 1$, and the solution is only defined for $t < \log(C)$, C must be positive.

2.
$$\frac{\mathrm{d}v}{\mathrm{d}t} = -(t + vt + v + 1)$$

ANS: This can also be solved by integrating factors. In this case the integrating factor is $\exp((t+1)^2/2)$:

$$\exp\left(\frac{(t+1)^2}{2}\right)\left(\frac{\mathrm{d}v}{\mathrm{d}t} + (t+1)v\right) = -(t+1)\exp\left(\frac{(t+1)^2}{2}\right)$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\exp\left(\frac{(t+1)^2}{2}\right)v\right] = -(t+1)\exp\left(\frac{(t+1)^2}{2}\right)$$

Both sides can be integrated,

$$\exp\left(\frac{(t+1)^2}{2}\right)v = -\exp\left(\frac{(t+1)^2}{2}\right) + A$$

so

$$v = -1 + A\exp\left(-\frac{(t+1)^2}{2}\right).$$

This is the same answer as we obtained by separation of variables.

3.
$$\frac{\mathrm{d}v}{\mathrm{d}t} = -2tv$$

ANS: Same as above, the integrating factor is now $\exp(t^2)$.

4.
$$\frac{\mathrm{d}v}{\mathrm{d}t} = -2tv + 1$$

ANS: Again, we can use an integrating factor

$$\exp(t^2)\left(\frac{\mathrm{d}v}{\mathrm{d}t} + 2tv\right) = \exp(t^2)$$

We cannot do the integration in closed form this time, and end up left with

$$v = \exp(-t^2)\int_0^t \exp(t'^2)\,\mathrm{d}t' + C\exp(-t^2).$$

(It turns out the integral term is a a multiple of a function called erfcx that is widely implemented in computational software like MATLAB, but that is not the point here)

5.
$$\frac{\mathrm{d}v}{\mathrm{d}t} = tv^2$$

ANS: The problem is nonlinear and cannot be solved using integrating factors

6.

$$\frac{\mathrm{d}v}{\mathrm{d}t} = t\exp(v) + t$$

(You may need the result that $1/(\exp(x) + 1) = \exp(-x)/(1 + \exp(-x)))$
ANS: This problem is also nonlinear and cannot be solved using integrating factors

7.

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \exp(v + t).$$

ANS: Ditto.