Exercise answers: EOS 250

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4. (Notes on volume integrals)

    1. $f(x, y) = x^2 + y$ has contours at $y = c - x^2$, where c is the contour level. This is a family of parabolas, centred on the y axis, pointing down, shifted up by c.

    2. Suppose the contour level c is positive. $f(x, y) = x^2 + 2xy - y^2$ has contours at $(x + y)^2 = c + 2y^2$ where c is the contour level, or equally, where $x = -y \pm \sqrt{c + 2y^2}$. The contours have asymptotes that behave as $x = (-1 \pm \sqrt{2})y$ for $y \to \infty$, and $x = (-1 \mp \sqrt{2}$. For negative contour levels, we can find a formula for the contour by writing $(y - x)^2 = 2x^2 - c$, so $y = x \pm \sqrt{2x^2 - c}$. The asymptotes have the same form as for positive c. The contours turn out to be tilted hyperbolae. You can go further: for $c > 0$, the contours have turning points at $y = sqrtc/2$, $x = \sqrt{c/2}$ and $y = -\sqrt{c/2}$, $x = -\sqrt{c/2}$, and no contour with $c > 0$ passes inside the gap $-\sqrt{c/2} < x < \sqrt{c/2}$. Similarly, for $c < 0$, the contours have local minima at $y = -\sqrt{-c/2}$, $y = \sqrt{-c/2}$ and local maxima at $y = \sqrt{-c/2}$, $y = -\sqrt{-c/2}$, and no contour passes inside the gap between $-\sqrt{c/2} < y < \sqrt{c/2}$.

    3. $f(x, y) = xy + x^2$ has contours at $y = x + c/x$, where x is the contour level. An earlier homework dealt with plotting $y = x + c/x$: Asymptotes are $x = 0$ and $y = x$, with turning points at $x = \pm\sqrt{c}$, $y = 2\sqrt{c}$ for $c > 0$.

1. (Notes on surface integrals) See figure 2

3. (Notes on surface integrals)

    1. For $\mathbf{v} = x\mathbf{i} - y\mathbf{j}$, we get

    $$\frac{\mathrm{d}x}{\mathrm{d}t} = x, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -y$$

    so, separating variables for both, we get the parametric form for the streamlines

    $$x = x_0 \exp(t), \qquad y = y_0 \exp(-t)$$

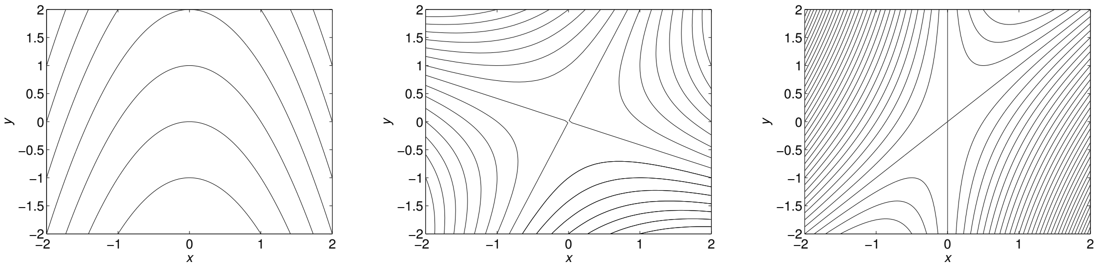


Figure 1: From left to right: contour plots of $f(x, y) = x^2 + y$, $f(x, y) = x^2 + 2xy - y^2$ and $f(x, y) = xy + x^2$.

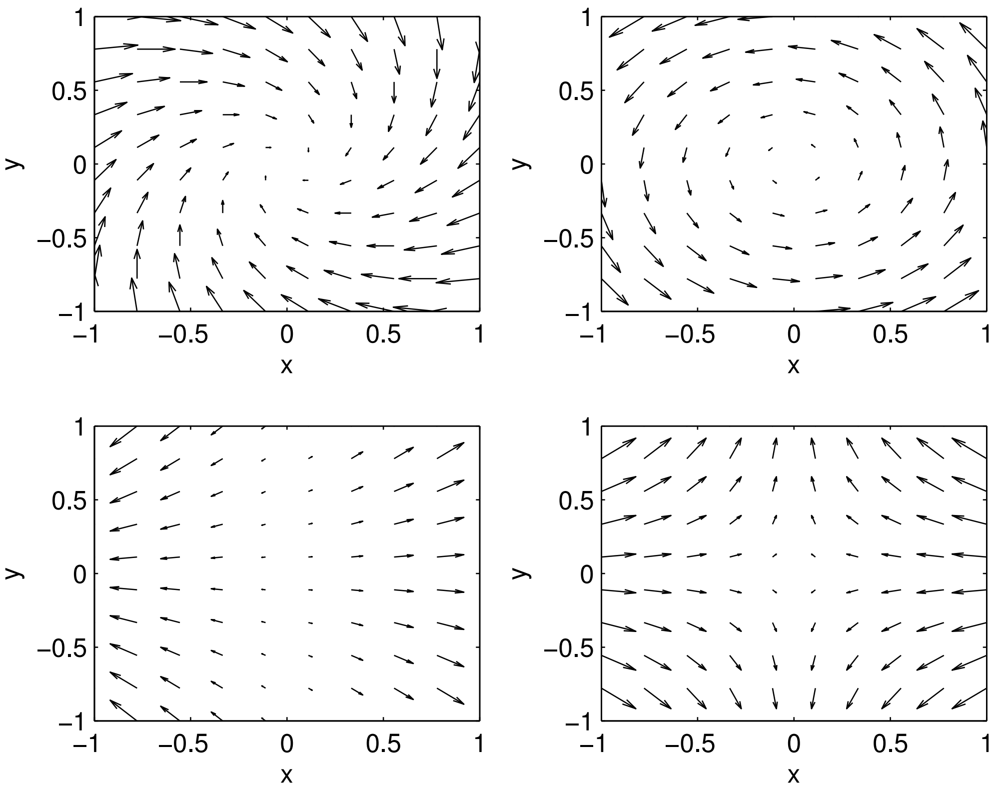


Figure 2: Clockwise, from top left, vector fields in parts 1–4 ($\mathbf{v} = (y - x)\mathbf{i} - (x + y)\mathbf{j}$, $\mathbf{v} = -y\mathbf{i} + x\mathbf{j}$, $\mathbf{v} = -x\mathbf{i} + y\mathbf{j}$, $\mathbf{v} = x\mathbf{i} + xy\mathbf{j}$.

Eliminating t by mutliplying both equations together,

$$xy = x_0 y_0$$

or

$$y = \frac{x_0 y_0}{x} = Cx$$

for constant C.

2. For $\mathbf{v} = -y\mathbf{i} + x\mathbf{j}$,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = x,$$

or, writing this as a differential equation for y in terms of x,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \Big/ \frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{x}{y}.$$

Separating variables and integrating gives

$$\int y \, \mathrm{d}y = -\int x \, \mathrm{d}x$$

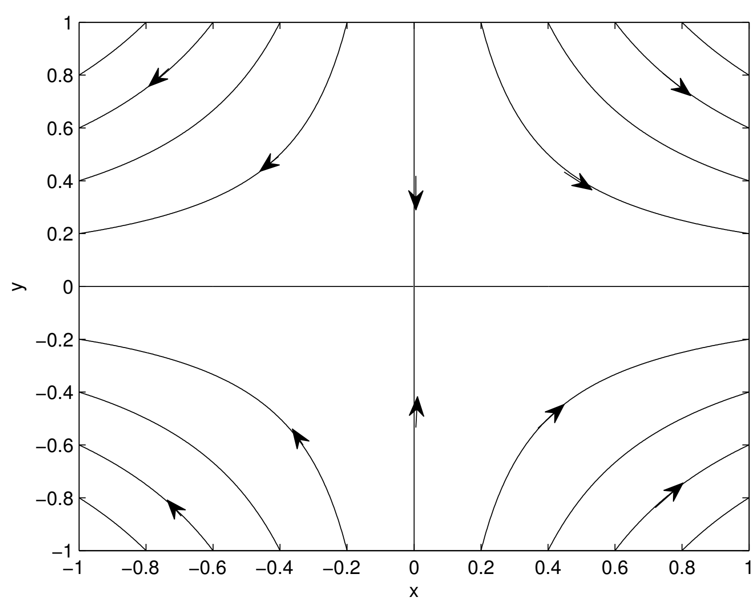
or

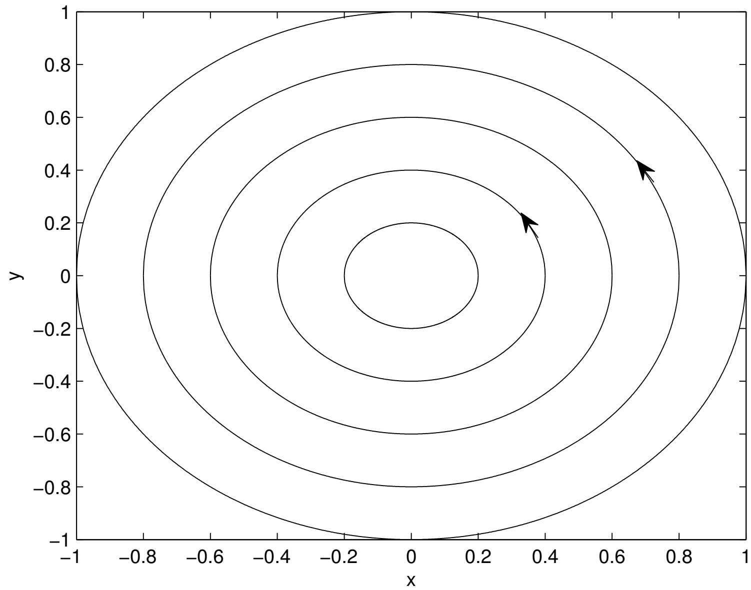
$$\frac{1}{2} y^2 = \frac{1}{2} x^2 + K$$

so

$$x^2 + y^2 = 2K$$

which is the equation for a circle of radius $\sqrt{2K}$





y
x
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1 −0.8 −0.6 −0.4 −0.2 0 0.2 0.4 0.6 0.8 1